Degree Master of Science in Mathematical Modelling and Scientific Computing

Mathematical Methods I

Thursday, 11th January 2007, 9:30 a.m. – 11:30 a.m.

Candidates may attempt as many questions as they wish. The best four solutions will count.
Solutions to questions 1, 2–5, and 6 should be handed in separately.

Please start the answer to each question on a new page.

All questions will carry equal marks.

Do not turn over until told that you may do so.

Question 1

In a model for the laying of an undersea cable, whose height above the seabed $y = 0$ is given by $y = h(x)$, the angle between between the tangent to the cable and the horizontal, θ, satisfies the equation

$$\epsilon^2 \frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} - \sin\theta + (F_0 + s)\cos\theta = 0, \qquad 0 < s < L,$$

where s is arclength measured from the origin, taken at the point where the cable leaves the seabed. Here $0 < \epsilon \ll 1$, and F_0, L are unknown constants. The boundary conditions are

$$\theta = 0, \quad \frac{\mathrm{d}\theta}{\mathrm{d}s} = 0 \quad \text{at } s = 0, \qquad \theta = \theta_1 \quad \text{(given) at } s = L.$$

Assuming that there is a solution with a boundary layer at $s = 0$, in which $s = O(\epsilon)$, find the leading order terms in inner and outer expansions for $\theta(s)$ (rescaling θ and F_0 as necessary for the latter) and show that $F_0 = -\epsilon$. Show that the inner limit of the leading order outer solution agrees with the outer limit of the leading order inner solution.

Verify from the outer solution that, when $x = O(1)$, $h(x) = \cosh x - 1 + O(\epsilon)$.

Question 2

The temperature T of a piece of coal is described by the equation

$$c\dot{T} = -k(T - T_0) + A\exp\left(-\frac{E}{RT}\right),$$

where T_0 is the ambient atmospheric temperature. By scaling the variables suitably, show that the equations can be written in the dimensionless form

$$\dot{\theta} = -\theta + \mu\exp\left(\frac{\theta}{1 + \varepsilon\theta}\right),$$

and define the parameters μ and ε.

Show that multiple steady states are possible if $\varepsilon < \frac{1}{4}$, and if this is the case, that there are three solutions for $\mu_- < \mu < \mu_+$, where

$$\mu_\pm = \theta_\pm e^{-\sqrt{\theta_\pm}},$$

and

$$\theta_\pm = \frac{1}{2\varepsilon^2}\left[1 - 2\varepsilon \mp (1 - 4\varepsilon)^{1/2}\right].$$

Hence plot the steady state solutions as a function of μ when $\varepsilon < \frac{1}{4}$, and show that for small ε, $\theta \approx \mu$ for small μ. Comment on the stability of the solutions.

Question 3

The function $v(x)$ satisfies the equation

$$v'' + v - v^3 = 0.$$

By finding a first integral, or otherwise, show that oscillatory solutions are possible in which v oscillates between values v_- and $v_+ > v_-$, where

$$-1 < v_- < 0 < v_+ < 1.$$

Show also that there are three constant solutions.

Now suppose that $u(x,t)$ satisfies

$$u_t = u_{xx} + u - u^3,$$

with

$$u_x = 0 \quad \text{at} \quad x = 0, l.$$

By linearising the system, show that if $u - v = U(x)e^{\sigma t}$, then U satisfies the Sturm-Liouville system

$$U'' + [s(x) - \sigma]U = 0, \quad U'(0) = U'(l) = 0,$$

and deduce that the steady states $v = \pm 1$ are stable.

By consideration of a suitable variational principle, show that $v = 0$ is unstable if $l > \pi$. [*Any results that you use concerning variational principles need not be proved, so long as they are clearly stated.*]

Question 4

Explain the Neumann iterative method for the solution of the Fredholm integral equation

$$\phi(x) = f(x) + \lambda \int_0^1 K(x,y)\phi(y)\,dy,$$

and give the form of the solution as an infinite power series in λ.

Find the solution of the integral equation

$$\phi(x) = 1 + \lambda \int_0^1 \phi(y)\cos\alpha y\,dy,$$

and show that the solution is uniquely defined except when λ takes one particular value λ_1, which you should determine.

Find the solution of the integral equation

$$\phi(x) = f(x) + \lambda \int_0^1 \phi(y)\cos\alpha y\,dy,$$

providing $\lambda \neq \lambda_1$. If $\lambda = \lambda_1$, show that no solution exists unless a certain integral condition is satisfied, and find the general solution when it *is* satisfied.

Question 5

The function $u(x,t)$ satisfies the equation

$$u_t + u^2 u_x = \varepsilon(1+u^2)u_{xx},$$

with

$$u(x,0) = \max[1-|x|, 0].$$

Solve the equation when $\varepsilon = 0$, and hence show that a shock will form when $x = x_c$ and $t = t_c$, and find the values of x_c and t_c.

If ε is small and positive, derive an approximate equation describing the shock structure, and hence derive an expression for the shock speed c in terms of the values u_- and u_+ behind and ahead of the shock.

In the particular case (for large time) where $u_+ = 0$ and u_- is small, show that

$$c \approx \tfrac{1}{3}u_-^2.$$

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Question 6

(i) Show that, if D is a closed region with smooth boundary ∂D, the Euler-Lagrange equations for the minimisations of

$$\int\int_D \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] dx\, dy$$

and

$$\int\int_D \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]^2 dx\, dy$$

are $\nabla^2 u = 0$ and $\nabla^4 u = 0$, respectively, and that the natural boundary conditions on ∂D are

$$\frac{\partial u}{\partial n} = 0$$

and

$$\nabla^2 u = \frac{\partial}{\partial n}\nabla^2 u = 0,$$

respectively.

(ii) By varying $y(x)$ to $y(x) + \varepsilon\eta_1(x) + \varepsilon\eta_2(x)$, show that when $\int_0^1 f(x, y, y')\, dx$ is minimised subject to $\int_0^1 g(x, y, y')\, dx = 0$, then there is a constant λ such that

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} - \lambda\left[\frac{d}{dx}\left(\frac{\partial g}{\partial y'}\right) - \frac{\partial g}{\partial y}\right] = 0.$$

(iii) A process $x(t)$ is governed by

$$\frac{dx}{dt} = f[t, x(t), u(t)], \quad x(0) = x(1) = 0,$$

where $u(t)$ is a control which is to be chosen to minimise the cost $\int_0^1 h[t, x(t), u(t)]\, dt$. Assuming $\frac{\partial f}{\partial u} \neq 0$, show that

$$\frac{d}{dt}\left(\frac{\partial h/\partial u}{\partial f/\partial u}\right) = \frac{\partial h}{\partial x} - \left(\frac{\partial h/\partial u}{\partial f/\partial u}\right)\frac{\partial f}{\partial x}.$$

Deduce that, if $p = \frac{\partial h/\partial u}{\partial f/\partial u}$, then

$$\frac{dx}{dt} = -\frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = \frac{\partial H}{\partial x},$$

where $H(x, p, t)$ is defined to be $-h + pf$.