Degree Master of Science in Mathematical Modelling and Scientific Computing

Mathematical Methods I

Thursday, 13th January 2011, 9:30 a.m.- 11:30 a.m.

Candidates should submit answers to a maximum of four questions that include an answer to at least one question in each section.

Please start the answer to each question on a new page.

All questions will carry equal marks.

Do not turn over until told that you may do so.

Section A — Mathematical Methods

Question 1

1. Find the general *real* solution for the differential equation:

$$Ly(x) \equiv \frac{d^2y}{dx^2}(x) - 2\frac{dy}{dx}(x) + 5y(x) = 0. \tag{1}$$

[5 marks]

2. For the boundary value problem

$$Ly(x) = f(x) \quad \text{on } 0 < x < \pi, \qquad y(0) = 0, \quad \frac{dy}{dx}(\pi) = 0, \tag{2}$$

(for Ly as in (1)), give two equivalent definitions of the Green's function $g(x,\xi)$: (i) one using the delta function $\delta(x)$; (ii) the other using only classical functions and appropriate conditions at $x = \xi$.

[8 marks]

3. Explicitly determine the Green's function $g(x,\xi)$ for the boundary value problem (2) and use it to write down the solution $y(x)$ for a general, continuous function $f(x)$. **[12 marks]**

Question 2

We consider here the solutions $y(x)$ of *Chebyshev's equation*

$$(1-x^2)\frac{d^2y}{dx^2}(x) - x\frac{dy}{dx}(x) + n^2y = 0 \tag{3}$$

for integer $n \geqslant 0$.

1. By using the substitution $x = \cos z$, or otherwise, show that

   $$T_n(x) = \cos(n \arccos(x)), \qquad V_n(x) = \sin(n \arccos(x)), \tag{4}$$

   are solutions of (3) for $n \geqslant 0$; they are linearly independent solutions for $n \geqslant 1$. **[6 marks]**

2. Show by induction over $n \geqslant 0$ that the $T_n(x)$ are polynomials of degree n. **[7 marks]**

3. Write (3) in an equivalent Sturm-Liouville form

   $$\frac{d}{dx}\left[p(x)\frac{dy}{dx}(x)\right] + q(x)y(x) + n^2r(x)y(x) = 0 \tag{5}$$

   giving explicit expressions for the coefficient functions $p(x)$, $q(x)$, $r(x)$. **[5 marks]**

4. Determine the value of

   $$I_{mn} = \int_{-1}^{1} T_m(x)T_n(x)\,(1-x^2)^{-1/2}dx.$$

   for general integers $n \geqslant 0$, $m \geqslant 0$. **[7 marks]**

Question 3

The differential operator L is defined by

$$Ly = -\frac{d}{dx}\left(e^x \frac{dy}{dx}\right) - \frac{1}{4}e^x y.$$

1. Determine the eigenvalues λ_n of the problem

$$Ly_n = \lambda_n e^x y_n, \quad 0 < x < 1,$$

with boundary conditions

$$y(0) = 0, \quad \frac{dy}{dx} + \frac{1}{2}y = 0 \quad \text{at } x = 1. \tag{6}$$

[10 marks]

2. Find the corresponding unnormalized y_n and also a weight function $r(x)$ with respect to which the y_n are orthogonal. Hence, select a suitable normalization for the y_n. **[9 marks]**

3. Solve the equation

$$Ly = -e^{x/2}, \quad 0 < x < 1,$$

subject to the boundary conditions (6), giving the solution as an expansion in terms of the eigenfunctions y_n. **[6 marks]**

Question 4

We consider here the ordinary differential equation (ODE)

$$x(1-x)\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + \lambda y = 0 \tag{7}$$

for real λ.

1. (i) Classify all points $x \in \mathbb{C} \cup \{\infty\}$ as ordinary, regular singular or irregular singular points of the ODE, giving reasons.
   (ii) Show that for the singular point $x = 0$, the only indicial exponent is $\alpha = 0$. For $x = 1$, find the indicial equation and the indicial exponents. **[10 marks]**

2. (i) Show that for a power series solution of the ODE (7) about $x = 0$,

   $$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

   the coefficients need to satisfy the recurrence relation

   $$a_{n+1} = \frac{n^2 - \lambda}{(n+1)^2} a_n, \qquad n \geqslant 0.$$

   State the a_n in closed form.
   (ii) State the form of a second solution that is linearly independent of the solution in (b). (Give the most specific answer possible without calculating any additional coefficients.) **[10 marks]**

3. Find the values of λ for which there is a polynomial solution of degree N. Evaluate the first four polynomials (i.e. for $N = 0, 1, 2, 3$) normalized in such a way that $P_N(0) = 1$. **[5 marks]**

Section B — Further Mathematical Methods

Question 5

The function $y(x)$ minimises the functional

$$J = \int_0^{\alpha} F(x, y(x),\, y'(x))\mathrm{d}x$$

where F has continuous second partial derivatives and $y(x)$ is a function which is continuously differentiable and such that $y(0) = 0$ and $(\alpha, y(\alpha))$ lies on a given curve $y = g(x)$.

1. Show that $y(x)$ satisfies the equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$$

with $y(0) = 0$ and $F + (g' - y')F_{y'} = 0$ at $x = \alpha$. **[12 marks]**

2. Hence show that the shortest path from the origin to any curve is a straight line and confirm that this line will be perpendicular to the curve at the point of intersection. **[8 marks]**

3. Find the shortest distance from the origin to the curve $y = \sqrt{1 + x}$. **[5 marks]**

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Question 6

A small bead moves on a smooth circular wire of radius a which lies in a vertical plane and rotates with constant angular velocity ω about a fixed vertical diameter. The radius to the bead makes an angle θ with the downward vertical. It can be shown that $\theta(t)$ satisfies the equation

$$a\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -g\sin\theta + a\omega^2\cos\theta\sin\theta.$$

1. By nondimensionalising the equation suitably show that it depends on the single nondimensional parameter $\alpha = a\omega^2/g$. **[2 marks]**

2. Find the positions of equilibrium of the bead relative to the wire and determine their stability for $\alpha < 1$ and $\alpha > 1$. **[16 marks]**

3. Sketch the $(\theta, \dot{\theta})$ phase plane (with $-\pi \leqslant \theta \leqslant \pi$) for $\alpha < 1$ and $\alpha > 1$ and hence sketch the bifurcation diagram in the (α, θ) plane. **[7 marks]**