Degree Master of Science in Mathematical Modelling and Scientific Computing

Mathematical Methods I

Thursday, 12th January 2012, 9:30 a.m.- 11:30 a.m.

Candidates should submit answers to a maximum of four questions that include an answer to at least one question in each section.

Please start the answer to each question on a new page.

All questions will carry equal marks.

Do not turn over until told that you may do so.

Section A — Mathematical Methods

Question 1

(a) Find the general *real* solution for the differential equation:

$$Ly(x) \equiv \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}(x) - 3\frac{\mathrm{d}y}{\mathrm{d}x}(x) + \frac{13}{4}y(x) = 0. \tag{1}$$

[5 marks]

(b) For the boundary value problem

$$Ly(x) = f(x) \quad \text{on } 0 < x < \pi, \qquad y(0) = 0, \quad \frac{\mathrm{d}y}{\mathrm{d}x}(\pi) + y(\pi) = 0, \tag{2}$$

(for Ly as in (1), and a given function $f(x)$), give two equivalent definitions of the Green's function $g(x,\xi)$: (i) one using the delta function $\delta(x)$; (ii) the other using only classical functions and appropriate conditions at $x = \xi$.

[8 marks]

(c) Explicitly determine the Green's function $g(x,\xi)$ for the boundary value problem (2) and use it to write down the solution $y(x)$ for a general, continuous function $f(x)$.

[12 marks]

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Question 2

The self-adjoint integral operator L is defined by

$$Ly(x) = \int_0^\pi k(x,t)y(t)\mathrm{d}t$$

with

$$k(x,t) = \begin{cases} \sin(t/2)\cos(x/2) & \text{for } x \leqslant t, \\ \sin(x/2)\cos(t/2) & \text{for } x > t. \end{cases}$$

(a) Show that

$$\lambda_n = \frac{2}{1-4n^2} \qquad \text{and} \qquad y_n(x) = \cos(nx)$$

are eigenvalues and eigenfunctions of the operator L, respectively, for any integer $n \geqslant 0$ (i.e. show that $Ly_n = \lambda_n y_n$).

[10 marks]

(b) Suppose that $y(x)$ satisfies

$$Ly(x) + \frac{2}{3}y(x) = f(x).$$

For each of the cases
(i) $f(x) = \sin(x)$;
(ii) $f(x) = \cos(2x)\cos(x)$
state, with reasons, whether the problem has a solution; if so, state whether it is unique (but you do not need to find it).
[Hint: Here, and in (c) below, you may assume, without proof, that all eigenvalues/functions are as listed in question (a)].

[9 marks]

(c) Now suppose instead that $Ly + \alpha y = \sin^2(x)$. Determine for which real α there is a unique solution $y(x)$ (if there is, you do not need to find it).

[6 marks]

You may find one or more of the following identities helpful (use without proof):

$$\sin(\alpha)\sin(\beta) = \frac{1}{2}\cos(\alpha-\beta) - \frac{1}{2}\cos(\alpha+\beta),$$
$$\cos(\alpha)\cos(\beta) = \frac{1}{2}\cos(\alpha-\beta) + \frac{1}{2}\cos(\alpha+\beta),$$
$$\sin(\alpha)\cos(\beta) = \frac{1}{2}\sin(\alpha-\beta) + \frac{1}{2}\sin(\alpha+\beta).$$

Question 3

We consider here, for integers $n \geqslant 0$, the **polynomial solutions** $y(x) = H_n(x)$ of *Hermite's equation* $(' = d/dx)$

$$y'' - 2xy' + 2ny = 0.$$

(a) Write Hermite's equation in Sturm-Liouville form

$$Ly + 2nry = 0, \qquad \text{where } Ly \equiv \left(py'\right)',$$

giving explicit expressions for the functions $p(x)$ and $r(x)$.
Show, via integration by parts, that the orthogonality relation

$$\int_{-\infty}^{\infty} H_n(x)H_m(x)\,\mathrm{e}^{-x^2}\mathrm{d}x = 0 \qquad \text{if } m \neq n$$

holds for general integers $n \geqslant 0$, $m \geqslant 0$.
(You do not need to find the value of the integral for $m = n$.)

[8 marks]

(b) Let

$$G(x,h) \equiv \mathrm{e}^{2hx-h^2}. \tag{3}$$

(i) Show that

$$I_n(h) \equiv \int_{-\infty}^{\infty} G(x,h)H_n(x)\mathrm{e}^{-x^2}\mathrm{d}x.$$

satisfies

$$\frac{\mathrm{d}}{\mathrm{d}h}I_n(h) = \frac{n}{h}I_n(h).$$

[6 marks]

(ii) Obtain

$$\left.\frac{\partial^{2n}G(x,h)}{\partial x^n \partial h^n}\right|_{x=0,\ h=0}$$

by direct calculation from (3).

[5 marks]

(iii) Show that the coefficents $c_n(h)$ of the expansion for $G(x,h)$ in terms of the Hermite polynomials $H_n(x)$ are given by

$$c_n = \frac{h^n}{n!}$$

[You may use, without proof, that the n-th Hermite polynomial $H_n(x)$ has degree n and leading coefficient 2^n.]

[6 marks]

Question 4

(a) For a general explicit linear second order ordinary differential equation (ODE)

$$y''(x) + p_1(x)y'(x) + p_0(x)y(x) = 0, \tag{4}$$

what does it mean that a $x = x_0 \in \mathbb{C}$ is an ordinary, regular singular or irregular singular point of the ODE, respectively? Furthermore, explain how the classification is carried out for $x_0 = \infty$, stating the required change of variables and the resulting ODE explicitly.

[9 marks]

(b) You are given the following ODE for $y(x)$,

$$xy''(x) + r(x)y(x) = 0.$$

For each of the following two choices for $r(x)$,

(i) $r(x) = 1$,

(ii) $r(x) = \sin(x)$.

classify the points $x_0 = 0$ and $x_0 = \infty$. If x_0 is a regular singular (and not an ordinary) point, determine the *leading* term in the series expansion about x_0 of two linearly independent solutions for $y(x)$.

[11 marks]

(c) Consider now the general explicit linear second order ODE in (4).
You are given that $x_0 = 0$ and $x_0 = \infty$ are regular singular points and all other $x \in \mathbb{C}$ are ordinary points of this ODE, and also that

$$\lim_{x \to 0} x^2 p_0(x) = -9 \qquad \text{and} \qquad \lim_{x \to 0} x p_1(x) = 1.$$

From this information, determine the functions $p_0(x)$ and $p_1(x)$, giving reasons why they are uniquely determined.

[Hint: Use Liouville's theorem (without proof): Every function that is bounded and analytic everywhere in $\mathbb{C}$ must necessarily be constant.]

[5 marks]

Section B — Further Mathematical Methods

Question 5

Suppose the function u minimises the functional

$$J[u] = \int_a^b F(x, u, u')\,\mathrm{d}x,$$

over all $u \in C^2[a, b]$, subject to $u(a) = c$, $u(b) = d$, where F is continuously differentiable. Show that u satisfies the Euler equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial u'}\right) - \frac{\partial F}{\partial u} = 0.$$

[8 marks]

Define the *Hamiltonian*, H, and show that if $F = F(u, u')$ then H is constant when u satisfies Euler's equation.

[4 marks]

Explain briefly why the area of the surface $x^2 + y^2 = u(z)^2$ lying between the planes $z = a$ and $z = b$, where u is a given positive function, is

$$J[u] = 2\pi \int_a^b u \sqrt{1 + \left(\frac{\mathrm{d}u}{\mathrm{d}z}\right)^2}\,\mathrm{d}z.$$

[3 marks]

Find the surface of minimal area which is bounded by the two circles $z = -1$, $x^2 + y^2 = \cosh^2(1)$ and $z = 1$, $x^2 + y^2 = \cosh^2(1)$.

[10 marks]

Question 6

Find and classify the critical points of the system

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= -x - \varepsilon y(x^2 - 1).\end{aligned}$$

[6 marks]

Now suppose that ε is small and positive. Use the Poincaré-Lindstedt method to find the periodic orbits when ε is small. Show that the frequency of the oscillation is $\omega = 1 + O(\varepsilon^2)$. For what values of the amplitude does a periodic orbit exist?

[13 marks]

Sketch the phase plane for both $0 < \varepsilon < 2$ and $-2 < \varepsilon < 0$. When is the limit cycle stable?

[6 marks]