Degree Master of Science in Mathematical Modelling and Scientific Computing

Mathematical Methods I

Thursday, 10th January 2013, 9:30 a.m.- 11:30 a.m.

Candidates should submit answers to a maximum of four questions that include an answer to at least one question in each section.

Please start the answer to each question on a new page.

All questions will carry equal marks.

Do not turn over until told that you may do so.

Section A — Mathematical Methods

Question 1

(a) Find the general solution for the differential equation:

$$Ly(x) \equiv \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}(x) - \frac{\mathrm{d}y}{\mathrm{d}x}(x) - 2y(x) = 0. \tag{1}$$

[5 marks]

(b) For the boundary-value problem

$$Ly(x) = f(x) \quad \text{on} \quad -1 < x < 1, \quad y(-1) = 0, \quad \frac{\mathrm{d}y}{\mathrm{d}x}(1) + y(1) = 0, \tag{2}$$

(for Ly as in (1), and a given function $f(x)$), give two equivalent definitions of the Green's function $g(x,\xi)$: (i) one using the delta distribution $\delta(x)$; (ii) the other using only classical functions and appropriate conditions at $x = \xi$.
Explicitly determine the Green's function $g(x,\xi)$ for the boundary value problem (2).

[12 marks]

(c) Let $C_0^\infty(\mathbb{R})$ denote the set of all "test" functions that have compact support and derivatives of arbitrary order. Give the definition of the derivative of a distribution.

Use this definition to explicitly determine the *second* derivative of the distribution $T : C_0^\infty(\mathbb{R}) \to \mathbb{R}$, given by

$$\langle T, \phi \rangle = \int_{-\infty}^{\infty} f(x)\phi(x)\,\mathrm{d}x, \quad \phi \in C_0^\infty(\mathbb{R}),$$

where

$$f(x) = \begin{cases} -x & \text{for } x < 1/2 \\ x - 1 & \text{for } x \geqslant 1/2, \end{cases}$$

and express the result in terms of the δ distribution.
(You do not need to show that this T is a distribution.) **[8 marks]**

Question 2

The self-adjoint integral operator M is defined by

$$My(x) = \int_{-1/2}^{1/2} k(x,t)y(t)\mathrm{d}t$$

with

$$k(x,t) = \begin{cases} -\frac{1}{12} + \frac{1}{2}(t-x) - \frac{1}{2}(x^2+t^2) & \text{for } x \leqslant t, \\ -\frac{1}{12} + \frac{1}{2}(x-t) - \frac{1}{2}(x^2+t^2) & \text{for } x > t. \end{cases}$$

(a) The operator M has the eigenvalues and eigenfunctions

$$\lambda_0 = 0, \qquad y_0(x) = 1, \tag{i}$$
$$\lambda_n = -1/(n^2\pi^2), \qquad y_n(x) = \sin(n\pi x), \qquad \text{for odd integer } n > 0, \tag{ii}$$
$$\lambda_n = -1/(n^2\pi^2), \qquad y_n(x) = \cos(n\pi x), \qquad \text{for even integer } n > 0. \tag{iii}$$

Show that this is true for the cases (i) and (ii). (You are *not* required to show this for the case (iii).)
[For the following items, you may use, without proof, that the eigenvalues listed in (i)-(iii) *are indeed* *all* *eigenvalues and eigenfunctions of M.]* **[10 marks]**

(b) Suppose that $y(x)$ satisfies $\pi^2 My + y = f(x)$. For each of the cases (i) $f(x) = \sin(2\pi x)$, (ii) $f(x) = \cos(\pi x)$; state, with reasons, whether the problem has a solution; if so, state whether it is unique (but you do not need to find it).
[Hint: What are the eigenvalues and eigenfunctions of the operator $Ky \equiv \pi^2 My + y$?]. **[9 marks]**

(c) Suppose, instead, that $\alpha\pi^2 My + y = f(x)$ for some given function $f(x)$. Determine for which real $\alpha \neq 0$ there is a unique solution $y(x)$. (You do not need to find it.) **[6 marks]**

You may find one or more of the following identities helpful (use without proof):

$$\sin(\alpha)\sin(\beta) = \frac{1}{2}\cos(\alpha-\beta) - \frac{1}{2}\cos(\alpha+\beta),$$
$$\cos(\alpha)\cos(\beta) = \frac{1}{2}\cos(\alpha-\beta) + \frac{1}{2}\cos(\alpha+\beta),$$
$$\sin(\alpha)\cos(\beta) = \frac{1}{2}\sin(\alpha-\beta) + \frac{1}{2}\sin(\alpha+\beta).$$

Question 3

Consider the equation

$$xy'' + (2-x)y' + ny = 0 \tag{3}$$

($' = d/dx$) for integers $n \geqslant 0$.

(a) Classify the point $x = 0$ as ordinary, regular singular or irregular singular point. If it is a regular singular point, determine the indicial equation and the indicial exponents.

[5 marks]

(b) Find the series expansion $Q_n(x) = \sum_{k=0}^{\infty} a_k x^k$ for the solution of (3) that satisfies $Q_n(0) = 1$, giving the coefficients a_k in closed form. Show in particular that $Q_n(x)$ is a polynomial of degree n.

[10 marks]

(c) Using the *Rodrigues' formula (without proof)*,

$$Q_n(x) = \frac{x^{-1}e^x}{(n+1)!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^{n+1}e^{-x}\right),$$

or otherwise, show that the $Q_n(x)$ satisfy the orthogonality relation

$$\int_0^\infty Q_n(x)Q_m(x)x\mathrm{e}^{-x}\mathrm{d}x = 0 \qquad \text{for } m \neq n.$$

Determine the value of the integral in the case where $m = n$. **[10 marks]**

Question 4

(a) For a general explicit linear second-order ordinary differential equation (ODE)

$$y''(x) + p_1(x)y'(x) + p_0(x)y(x) = 0, \tag{4}$$

what does it mean that an $x = x_0 \in \mathbb{C}$ is an ordinary, regular singular or irregular singular point of the ODE, respectively? Furthermore, explain how the classification is carried out for $x_0 = \infty$, stating the required change of variables and the resulting ODE explicitly.

[9 marks]

(b) You are given the following ODE for $y(x)$,

$$x^2y''(x) + q(x)y'(x) + r(x)y(x) = 0.$$

For each of the following two choices for $q(x)$ and $r(x)$,

(i) $q(x) = \sin(x^2)$ and $r(x) = \cos(x) - 1$;

(ii) $q(x) = x^2 + x$ and $r(x) = -1$,

classify the points $x_0 = 0$ and $x_0 = \infty$.

[9 marks]

(c) If $x_0 = 0$ is a regular singular (and not an ordinary point) in any of the examples in (b), determine the *leading* term of two linearly independent solutions. Furthermore, in this case, state (with reasons) if there is a solution for the initial condition

$$y(x_0) = 1,$$

or for the initial condition

$$y'(x_0) = 1,$$

and if so, give its leading term.

[7 marks]

Section B — Further Mathematical Methods

Question 5

Suppose the function $u(t)$ is the optimal control, which minimises the cost functional

$$C[x,u] = \int_0^T h(t,x,u)\,\mathrm{d}t$$

over all controls $u(t) \in C^1[0,T]$ satisfying the control problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(t,x,u), \qquad x(0) = a, \quad x(T) = b,$$

where $\partial f/\partial u \neq 0$.
Show that u satisfies the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial h}{\partial u}\bigg/\frac{\partial f}{\partial u}\right) = \frac{\partial h}{\partial x} - \frac{\partial f}{\partial x}\left(\frac{\partial h}{\partial u}\bigg/\frac{\partial f}{\partial u}\right).$$

[12 marks]

A process obeys the control problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u - x, \qquad x(0) = 0, \quad x(1) = 1,$$

and it is desired to minimise the integral

$$C[x,u] = \int_0^1 (u(t)^2 + u(t)x(t) + x(t)^2)\,\mathrm{d}t.$$

Show that

$$u = \frac{\sinh\sqrt{3}\,t + \sqrt{3}\,\cosh\sqrt{3}\,t}{\sinh\sqrt{3}},$$

and find the corresponding behaviour of x. **[6 marks]**

Now suppose that the requirement $x(0) = 0$ is removed, leaving only the condition $x(1) = 1$. Show that the natural boundary condition at $t = 0$ is

$$2\dot{x}(0) + 3x(0) = 0.$$

[2 marks]

Show that the optimal solution for x is now

$$x = \frac{2\cosh\sqrt{3}t - \sqrt{3}\,\sinh\sqrt{3}t}{2\cosh\sqrt{3} - \sqrt{3}\,\sinh\sqrt{3}},$$

and find the corresponding control u. **[3 + 2 marks]**

Question 6

Find the critical points of the plane autonomous system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -\epsilon(\lambda - x^2)y - x + \epsilon x^3,$$

where $\epsilon > 0$. How does the type and stability of the critical point at the origin depend on λ? What type of bifurcation occurs when $\lambda = 0$? **[3 + 7 +1 marks]**

Use the Poincaré–Lindstedt method to find the periodic orbits when ϵ is small and $\lambda > 0$. Show that the frequency of the oscillation is

$$\omega = 1 - \frac{3\epsilon\lambda}{2} + O(\epsilon^2),$$

and find the corresponding value of the amplitude. **[6 + 8 marks]**