JMAT 7302

M.Sc. in Mathematical Modelling and Scientific Computing
MSc in Applied and Computational Mathematics

Paper B (Numerical Analysis)

Friday 20 April, 2001, 9.30 a.m. - 12.30 p.m.

Candidates may attempt as many questions as they wish. All questions will carry equal marks.

Do not turn this page until you are told that you may do so

1. Numerical Linear Algebra

(a) If $A \in \mathbb{R}^{m\times n}$, what is the *Singular Value Decomposition* (SVD) of A? If the singular values $\{\sigma_i\}$ satisfy $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0$, $\sigma_j = 0$, $j > r$, show that $\mathrm{Rank}(A) = r$. Show also that $\|A\|_2 = \sigma_1$ (proving any results that you need).

Suppose that the SVD of A has been computed and is available, and $b \in \mathbb{R}^m$ is given.

If $m = n = r$, how might the linear system $Ax = b$ be solved without further factorisation of A?

If $m > n = r$, how might the least squares problem

$$\min_{x\in\mathbb{R}^n} \|Ax - b\|_2$$

be solved without further factorising A?

(b) What is the *Gauss–Seidel iteration* for the solution of $Ax = b$ with $b \in \mathbb{R}^n$, and $A \in \mathbb{R}^{n\times n}$ nonsingular and having entries $\{a_{i,j}\}$ with $a_{i,i} \neq 0$?

State, but do not prove, a necessary and sufficient condition for the convergence of this iteration.

If

$$|a_{i,i}| > \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{i,j}| \qquad \text{for all } i \in \{1,\ldots,n\}\,,$$

prove that the Gauss–Seidel iteration converges.

2. Numerical Solution of Ordinary Differential Equations

State the general form of a linear k-step method, $k \geq 1$, for the numerical solution of the initial-value problem $y' = f(x,y)$, $y(x_0) = y_0$, on the mesh $\{x_n : x_n = x_0 + nh,\ n = 0,1,2,\ldots\}$ of uniform spacing $h > 0$. What does it mean to say that the method is *absolutely stable*?

(a) Find the interval of absolute stability of the explicit Euler method.

(b) Find the interval of absolute stability of the implicit Simpson-rule method

$$y_{n+2} - y_n = \frac{1}{3}h\left(f_{n+2} + 4f_{n+1} + f_n\right)\,,$$

where f_ℓ denotes $f(x_\ell, y_\ell)$, $\ell \geq 0$.

(c) Let m be a positive integer. State the Predictor-Corrector method $P(EC)^mE$ where the predictor P is the explicit Euler method and the corrector C is the implicit Simpson-rule method.

(d) Find the interval of absolute stability of the $P(EC)^1E$ method from the family $P(EC)^mE$ defined under c).

3. Numerical Solution of Partial Differential Equations

Consider the equation

$$u_t = u_{xx} - u_{xxxx}\,, \qquad 0 < x < 1\,, \quad t > 0\,,$$

with initial condition $u(x,t) = u_0(x)$ and boundary conditions

$$u(0) = u(1) = u_x(1) = 0\,, \qquad u_x(0) = 1\,.$$

Consider solving this problem numerically by a finite difference method applied on a regular grid $x_j = j\Delta x$, $j = 0, 1, \ldots, N$, with $\Delta x = 1/N$ for some integer $N \geq 4$.

(a) Ignoring boundary conditions for the moment, write down the explicit finite difference model obtained by approximating the time derivative by the usual forward difference, the second space derivative by the usual three-point centred difference, and the fourth space derivative by the square of the latter, i.e., the second difference of the second difference.

(b) Complete your model by proposing a set of appropriate numerical boundary conditions. There is some flexibility in the choice, but make sure that you give the correct number of boundary conditions and that they are consistent with the problem. (Continue to use the grid x_0, $x_1, \cdots x_N$; do not introduce additional grid points).

(c) For a successful computation, the time step Δt will have to satisfy a certain stability restriction. Work out this restriction by Fourier analysis. For this calculation you may ignore the boundary conditions.

(d) To get around the stability restriction, we could use an implicit formula of Crank-Nicolson type, i.e., with equal weightings at time steps n and $n+1$. Write down this finite difference model, and for the case $N = 8$, write down explicitly the 5×5 matrix problem that must be solved to step from U^n to U^{n+1}.

4. Suppose that the function u satisfies

$$\nabla^2 u = f\,, \qquad (x,y) \in \Omega = (0,1) \times (0,1)\,,$$

and that u obeys a certain boundary condition on $\partial\Omega$.

Let $h = 1/N$ with N an integer, $N \geq 2$. An approximation $U_{r,s}$ to $u_{r,s} = u(rh, sh)$, $r, s = 0, \ldots, N$, is defined on a uniform mesh of size h by

$$L_h U_{r,s} \equiv \frac{1}{h^2}(U_{r+1,s} + U_{r-1,s} + U_{r,s+1} + U_{r,s-1} - 4U_{r,s}) = f_{r,s}\,, \qquad r, s = 1, \ldots, N-1\,,$$

where $f_{r,s} = f(rh, sh)$ for $r, s = 0, \ldots, N$.

Define the truncation error $T_{r,s}$ of this method and, assuming that u is sufficiently smooth, show that

$$|T_{r,s}| \leq K_1 h^2\,, \qquad r, s = 1, \ldots, N-1\,,$$

for some $K_1 > 0$.

If both $u = 0$ and $U_{r,s} = 0$ on $\partial\Omega$ show that the error $e_{r,s} = u_{r,s} - U_{r,s}$ satisfies

$$|e_{r,s}| \leq \frac{1}{8} K_1 h^2\,.$$

[*Hint: You may use without proof that if a function $\psi(x,y)$ with $\psi \geq 0$ on $\partial\Omega$ satisfies* $L_h \psi_{r,s} \geq 0$ for $r, s = 1, \ldots, N-1$, *then the values $\psi_{r,s} = \psi(rh, sh)$ attain a maximum on the boundary, $\partial\Omega$, where either r or s equals 0 or N.*]

Now, suppose that the boundary condition $u = 0$ on $x = 1$ is replaced by

$$\frac{\partial u}{\partial x}(1, y) = g(y)\,, \qquad 0 \leq y \leq 1\,.$$

Using a local Taylor expansion in x about $x = 1$, show that an approximation to this boundary condition can be incorporated implicitly into L_h, and that the truncation error on $x = 1$ then satisfies

$$|T_{N,s}| \leq K_2 h\,, \qquad s = 1, \ldots, N-1\,.$$

Show that there exists a constant $K_3 > 0$ such that

$$|e_{r,s}| \leq K_3 h^2\,, \qquad r = 1, \ldots, N, \;\; s = 1, \ldots, N-1\,.$$

5. Finite Element Methods for Partial Differential Equations

Consider the boundary-value problem

$$-(xu')' + u = f(x)\,, \quad x \in (1,2)\,, \qquad u(1) = u'(1)\,, \quad u'(2) = -u(2)\,,$$

where $f \in L_2(1,2)$. Define a symmetric bilinear functional $a(\cdot,\cdot)$ on $H^1(1,2) \times H^1(1,2)$ and a linear functional $\ell(\cdot)$ on $H^1(1,2)$ such that the weak formulation of the boundary value problem has the form

$$\text{find } u \in H^1(1,2) \text{ such that } a(u,v) = \ell(v) \text{ for all } v \in H^1(1,2)\,.$$

Apply the Lax-Milgram theorem to show the existence of a unique weak solution to the boundary value problem.

[*You may assume that* $\max_{x\in[1,2]} |v(x)| \le \|v\|_{L_2(1,2)} + \|v'\|_{L_2(1,2)}$ *for any* $v \in H^1(1,2)$.]

Formulate the piecewise linear finite element method for the numerical solution of the boundary value problem on a subdivision $\mathcal{T}^h$ of the interval $[1,2]$ of uniform spacing $h = 1/N$, $N \ge 2$. Show that the finite element method has a unique solution u_h.

State explicitly the energy norm $\|\cdot\|_a$ associated with $a(\cdot,\cdot)$, and show that the method has the following best-approximation property:

$$\|u - u_h\|_a = \min_{v \in V^h} \|u - v\|_a\,,$$

where V^h is the $(N+1)$-dimensional linear subspace of $H^1(1,2)$ consisting of all continuous piecewise linear functions defined on $\mathcal{T}^h$.

6. Consider the following initial-boundary value problem for the heat equation:

$$\begin{aligned} u_t = u_{xx} - u\,, \quad x \in (0,1)\,, \quad t \in (0,1]\,, \\ u_x(0,t) = 0\,,\ u_x(1,t) = 0\,, \quad t \in (0,1]\,, \\ u(x,0) = u_0(x)\,, \end{aligned}$$

where u_0 is a given function in $L_2(0,1)$.

Using piecewise linear finite elements in the x-direction, formulate the implicit Euler method for the numerical solution of this problem on a uniform mesh of spacing $h = 1/N$ in the x-direction and spacing $\Delta t = 1/K$ in the t-direction; $N \geq 2$, $K \geq 1$.

Show that the method is unconditionally stable in the $L_2(0,1)$ norm.

On denoting by $u_h^k(x)$ the finite element approximation to $u(x, k\Delta t)$, $0 \leq k \leq K$, and expanding $u_h^k(x)$ in terms of the standard piecewise linear finite element basis functions $\varphi_i(x)$, $i = 0, \ldots, N$, as

$$u_h^k(x) = \sum_{i=0}^{N} U_i^k \varphi_i(x)\,,$$

show that the vector $U^k = (U_0^k, \ldots, U_N^k)^{\mathrm{T}}$ may be determined by solving a set of linear algebraic equations of the form

$$\begin{aligned} M\frac{U^k - U^{k-1}}{\Delta t} + (S+M)U^k &= 0\,, \qquad 1 \leq k \leq K\,, \\ MU^0 &= G^0\,, \end{aligned}$$

where M and S are $(N+1)\times(N+1)$ matrices whose entries you should specify, and G^0 is a column vector of size $N+1$ whose entries should be expressed in terms of u_0.