Degree Master of Science in Mathematical Modelling and Scientific Computing

Numerical Linear Algebra & Finite Element Methods

Thursday, 17th April 2008, 2:00 p.m. – 4:00 p.m.

Candidates may attempt as many questions as they wish but must attempt at least one of questions 5 and 6. The best four solutions,including one from questions 5 and 6, will count. Solutions to questions 1–4, and 5–6 should be handed in separately.

Please start the answer to each question on a new page.

All questions will carry equal marks.

Do not turn over until told that you may do so.

Question 1

Define the Sobolev space $\mathrm{H}^1(0,1)$ and the Sobolev norm $\|\cdot\|_{\mathrm{H}^1(0,1)}$.
What is meant by saying that u is a weak solution in $\mathrm{H}^1(0,1)$ of the boundary value problem

$$-u'' + \mathrm{e}^x u = f(x), \quad x \in (0,1); \qquad u(0) - u'(0) = 0, \quad u(1) + u'(1) = 0,$$

where $f \in \mathrm{L}^2(0,1)$?
Show that the bilinear form associated with the weak formulation of this problem is coercive on $\mathrm{H}^1(0,1)$.

Consider the continuous piecewise linear basis functions φ_i, $i = 0, 1, \dots, N$, defined by $\varphi_i(x) = (1 - |x - x_i|/h)_+$ on the uniform mesh of size $h = 1/N$, $N \geq 2$, with mesh-points $x_i = ih$, $i = 0, 1, \dots, N$. Using the basis functions φ_i, $i = 0, 1, \dots, N$, define the finite element approximation of the boundary value problem and show that it has a unique solution u_h.

Expand u_h in terms of the basis functions φ_i, $i = 0, 1, \dots, N$, by writing

$$u_h(x) = \sum_{i=0}^{N} U_i \varphi_i(x)$$

where $\mathbf{U} = (U_0, U_1, \dots, U_N)^T \in \mathbb{R}^{N+1}$, to obtain a system of linear algebraic equations for the vector of unknowns $\mathbf{U}$. Show that the matrix $\mathcal{A}$ of this linear system is symmetric (*i.e.*, $\mathcal{A}^T = \mathcal{A}$) and positive definite (*i.e.*, $\mathbf{V}^T \mathcal{A} \mathbf{V} > 0$ for all $\mathbf{V} \in \mathbb{R}^{N+1}$, $\mathbf{V} \neq \mathbf{0}$).
Show also that $\|u - u_h\|_{\mathrm{H}^1(0,1)} = \mathcal{O}(h)$ as $h \to 0$.
[Any bound on the error between u and its finite element interpolant $\mathcal{I}_h u$ may be used without proof, but must be stated carefully.]

Question 2

Suppose that Ω is a bounded polygonal domain in $\mathbb{R}^2$ with boundary Γ, oriented in the anticlockwise direction. Consider the quadratic functional $J : v \in \mathrm{H}^1(\Omega) \mapsto J(v) \in \mathbb{R}$ defined by

$$J(v) = \frac{1}{2}\int_\Omega (|\nabla v|^2 + v^2)\,\mathrm{d}x - \int_\Gamma v\,\mathrm{d}s.$$

(a) Show that if $u \in \mathrm{H}^1(\Omega)$ is such that $J(u) \leq J(v)$ for all $v \in \mathrm{H}^1(\Omega)$, then there exist a bilinear functional $a(\cdot,\cdot)$ defined on $\mathrm{H}^1(\Omega) \times \mathrm{H}^1(\Omega)$ and a linear functional $\ell(\cdot)$ defined on $\mathrm{H}^1(\Omega)$ such that

$$a(u, v) = \ell(v) \qquad \forall v \in \mathrm{H}^1(\Omega). \tag{P}$$

(b) Show that (P) is the weak formulation of the elliptic boundary-value problem

$$-\nabla^2 u + u = 0 \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} = 1 \quad \text{on } \Gamma,$$

where $\frac{\partial u}{\partial n} = \nabla u \cdot \mathbf{n}$ and $\mathbf{n}$ denotes the unit outward normal vector to Γ.

(c) Suppose that Ω is the unit square $(0,1) \times (0,1)$, and let $\mathcal{T}_h$ be a triangulation of Ω constructed from a uniform square grid of spacing $h = 1/N$ by subdividing each grid-square by the diagonal of negative slope. Formulate the piecewise linear finite element approximation (P_h) of problem (P) on the triangulation $\mathcal{T}_h$. Show that (P_h) has a unique solution u_h and that $J(u) \leq J(u_h) \leq J(v_h)$ for any continuous piecewise linear function v_h defined on the triangulation $\mathcal{T}_h$.

Question 3

(a) Let $\psi \in \mathrm{L}^2(0,1)$ and let $a(\cdot,\cdot)$ be the bilinear form on $\mathrm{H}^1(0,1) \times \mathrm{H}^1(0,1)$ defined by

$$a(w,v) = \int_0^1 (w'v' + wv)\,\mathrm{d}x.$$

Suppose, further, that $z \in \mathrm{H}^1(0,1)$ is such that

$$a(w,z) = \int_0^1 w \cdot \psi \,\mathrm{d}x \qquad \forall w \in \mathrm{H}^1(0,1).$$

Show that $z \in \mathrm{H}^2(0,1)$ and $\|z''\|_{\mathrm{L}^2(0,1)} \leq \|\psi\|_{\mathrm{L}^2(0,1)}$.

(b) Suppose that $f \in \mathrm{L}^2(0,1)$ and let $u \in \mathrm{H}^1(0,1)$ be the weak solution of the problem

$$a(u,v) = \int_0^1 f \cdot v \,\mathrm{d}x \qquad \forall v \in \mathrm{H}^1(0,1).$$

Let, further, u_h denote the piecewise linear finite element approximation to u on the subdivision $\mathcal{S}_h = \{[x_{i-1}, x_i] \,:\, i = 1, 2, \ldots, N\}$, where $x_i - x_{i-1} = h_i$, $i = 1, 2, \ldots, N$.

Show that

$$\int_0^1 (u - u_h)\psi \,\mathrm{d}x = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} R(u_h) \cdot (z - I_h z)\,\mathrm{d}x,$$

where $I_h z$ is the continuous piecewise linear finite element interpolant of z on the subdivision $\mathcal{S}_h$, and $R(u_h)$ is the *residual* that you should carefully define in terms of f and u_h.

(c) Show that

$$\int_0^1 (u - u_h)\psi \,\mathrm{d}x \leq \frac{1}{\pi^2} \left(\sum_{i=1}^{N} \|R(u_h)\|^2_{\mathrm{L}^2(x_{i-1},x_i)} h_i^4 \right)^{1/2} \|\psi\|_{\mathrm{L}^2(0,1)},$$

and deduce the *a posteriori* error bound

$$\|u - u_h\|_{\mathrm{L}^2(0,1)} \leq \frac{1}{\pi^2} \left(\sum_{i=1}^{N} \|R(u_h)\|^2_{\mathrm{L}^2(x_{i-1},x_i)} h_i^4 \right)^{1/2}.$$

Discuss, briefly, how this *a posteriori* error bound could be implemented into an adaptive mesh refinement algorithm to compute, for a prescribed tolerance $\mathtt{TOL} > 0$, an approximation u_h to u such that $\|u - u_h\|_{\mathrm{L}^2(0,1)} \leq \mathtt{TOL}$.

Question 4

Let $u(x,t)$ denote the solution to the initial boundary value problem

$$
\begin{aligned}
p(x)\frac{\partial u}{\partial t} + u = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1, \quad 0 < t \le T, \\
u(0,t) = 0, \quad u(1,t) = 0, \qquad 0 \le t \le T, \\
u(x,0) = u_0(x), \qquad 0 < x < 1,
\end{aligned}
$$

where $T > 0$, $u_0 \in \mathrm{L}^2(0,1)$, and p is a continuous function defined on the closed interval $[0,1]$ of the real line, such that $0 < c_0 \le p(x) \le c_1$ for all $x \in [0,1]$.

Construct a finite element method for the numerical solution of this problem, based on the backward Euler scheme with time step $\Delta t = T/M$, $M \ge 2$, and a piecewise linear approximation in x on a uniform subdivision of spacing $h = 1/N$, $N \ge 2$, of the interval $[0,1]$, denoting by u_h^m the finite element approximation to $u(\cdot, t^m)$ where $t^m = m\Delta t$, $0 \le m \le M$.

Show that, for $0 \le m \le M-1$,

$$
(pu_h^{m+1}, u_h^{m+1}) + 2\Delta t(1+\pi^2)\|u_h^{m+1}\|^2_{\mathrm{L}^2(0,1)} \le (pu_h^m, u_h^m),
$$

where $(\cdot,\cdot)$ denotes the inner product of $\mathrm{L}^2(0,1)$ and $\|\cdot\|_{\mathrm{L}^2(0,1)}$ is the L^2-norm on the interval $(0,1)$.

Hence deduce that the method is unconditionally stable in the L^2-norm in the sense that, for any Δt, independent of the choice of h,

$$
\|u_h^m\|^2_{\mathrm{L}^2(0,1)} \le \frac{c_1}{c_0}\left(1 + \frac{2\Delta t(1+\pi^2)}{c_1}\right)^{-m} \|u_0\|^2_{\mathrm{L}^2(0,1)}, \qquad 1 \le m \le M.
$$

Show that, for each m, $0 \le m \le M-1$, u_h^{m+1} can be obtained from u_h^m by solving a system of linear algebraic equations with a symmetric tridiagonal matrix $\mathcal{A}$ whose entries you should define in terms of the standard piecewise linear basis functions φ_i, $i = 1, \ldots, N-1$. Assuming that $p(x) \equiv 1$, compute the diagonal entries of the matrix $\mathcal{A}$.

Question 5

Let Π_k denote the set of real polynomials of degree less than or equal to k.

(a) If the Chebyshev polynomials $T_k \in \Pi_k, k = 0, 1, \ldots$ are defined for argument $t \in [-1, 1]$ by $T_0 = 1$ and for $k = 1, 2, \ldots$ by

$$T_k(t) = \frac{1}{2^{k-1}} \cos k\theta, \ t = \cos\theta, \quad 0 \leq \theta \leq \pi$$

show that for $m = 2, 3, \ldots$

$$T_{m+1}(t) = t\,T_m(t) - \frac{1}{4}T_{m-1}(t).$$

(Please note the scaling of the Chebyshev polynomials here which is different to that which is in most common usage.)

(b) If $S = X^T \Lambda X$ is a diagonalisation of the symmetric matrix $S \in \mathbb{R}^{n\times n}$ so that $\Lambda \in \mathbb{R}^{n\times n}$ is a diagonal matrix of the eigenvalues and X is orthogonal, show for any polynomial $p \in \Pi_\ell$ that $p(S) = X^T p(\Lambda) X$ and deduce that

$$\|p(S)\|_2 = \|p(\Lambda)\|_2 = \max_j \ |p(\lambda_j)|$$

where $\{\lambda_1, \ldots, \lambda_n\}$ are the eigenvalues of S.

(c) For a symmetric matrix $A \in \mathbb{R}^{n\times n}$ if a splitting $A = I - S$ is used and iterates for the solution of $Ax = b$ are defined by a simple iteration

$$x_k = Sx_{k-1} + b$$

show that

$$x - x_k = S^k(x - x_0).$$

For *any* given polynomial $p_k \in \Pi_k$ satisfying $p_k(1) = 1$, how can the iterates $\{x_k\}$ be linearly combined to give a sequence $\{y_k\}$ so that we have

$$x - y_k = p_k(S)(x - x_0)?$$

(d) Why is the choice that p_k is an appropriately shifted and scaled version of T_k a good choice? Quote but do not prove any results you require. Explain why with this choice the sequence $\{y_k\}$ can converge even when the sequence $\{x_k\}$ diverges.

Question 6

Let $A \in \mathbb{R}^{n \times n}$ be symmetric and positive definite.

(a) If for given $b \in \mathbb{R}^n$, $x \in \mathbb{R}^n$ satisfies $Ax = b$ and the functional $\phi : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\phi(y) = \frac{1}{2} y^T A y - y^T b$$

show that

$$\phi(y) = \frac{1}{2}\|x - y\|_A^2 - \frac{1}{2}\|x\|_A^2$$

where $\|z\|_A^2 = z^T A z$ for any $z \in \mathbb{R}^n$. Show further that $\phi(x + y)$ is minimal when $y = 0$.

(b) What is the Steepest Descent Method for the solution of the linear system of equations $Ax = b$? For the iterates $\{x_k\}$ generated by the Steepest Descent Method and the corresponding residuals $\{r_k\}$ with $r_k = b - Ax_k$ show that

$$r_k \in r_0 + \text{span}\{Ar_0, \ldots, A^k r_0\}.$$

(c) If

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad x_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

what is the angle between the first two descent directions?