JMAT 7304

Degree Master of Science in Mathematical Modelling and Scientific Computing

Numerical Linear Algebra & Finite Element Methods

TRINITY TERM 2010

Thursday, 22nd April 2010, 2:00 p.m. – 4:00 p.m.

Candidates should submit answers to a maximum of four questions that include an answer to at least one question in each section.

Please start the answer to each question on a new page.

All questions will carry equal marks.

Do not turn over until told that you may do so.

Part A — Numerical Linear Algebra

Question 1

The Chebyshev polynomials $\{T_k(x)\}$ are a family of polynomials defined by the conditions $T_0(x) = 1$, $T_1(x) = x$, and

$$\tfrac{1}{2}T_k(x) + \tfrac{1}{2}T_{k-2}(x) = xT_{k-1}(x), \qquad k \geq 2.$$

For this question you do not need to know any more about them than this.

(a) Write down the first five Chebyshev polynomials $T_0(x), \ldots, T_4(x)$. (2 marks)

(b) Consider monomial and Chebyshev representations of a polynomial p of degree at most n:

$$p(x) = \sum_{k=0}^{n} c_k x^k, \qquad p(x) = \sum_{k=0}^{n} a_k T_k(x),$$

and define the "monomial coefficient vector" and "Chebyshev coefficient vector"

$$c = (c_0, \ldots, c_n)^T, \quad a = (a_0, \ldots, a_n)^T.$$

Write down a matrix A such that if a is a Chebyshev coefficient vector of p, then $c = Aa$ is a monomial coefficient vector of p. It is enough to show A for the case $n = 4$ to make it clear you know what is going on. (3 marks)

(c) Prove that A is nonsingular. Prove that for each monomial coefficient vector there exists a unique Chebyshev coefficient vector. (4 marks)

(d) Let p be a monic polynomial of degree n with $c_n = 1$ in the monomial representation. For any number x, define

$$v_x = (1, x, \ldots, x^{n-1})^T,$$

and let G be the *companion matrix* defined by this pattern shown for $n = 5$:

$$G = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -c_0 & -c_1 & -c_2 & -c_3 & -c_4 \end{pmatrix}.$$

Show that if x is a root of p, then x is an eigenvalue of G with eigenvector v_x. (4 marks)

(e) Prove that the geometric multiplicity of any eigenvalue of G is 1. (Hint: show that for any λ, $G - \lambda I$ has rank at least $n - 1$.) (4 marks)

(f) Let p be a polynomial of degree n with $a_n = 1/2$ in the Chebyshev representation. For any number x, define

$$w_x = (1, T_1(x), \ldots, T_{n-1}(x))^T.$$

Determine a matrix H containing the Chebyshev coefficients $a_0, \ldots, a_{n-1}$ such that if x is a root of p, then x is an eigenvalue of H with eigenvector w_x. (8 marks)

Question 2

Let A be an $m \times m$ real symmetric matrix. The classical algorithm for computing its eigenvalues first orthogonally tridiagonalizes A: $Q^T AQ = T$ for some orthogonal matrix Q and tridiagonal symmetric matrix T. Then it finds eigenvalues of T by the QR algorithm or divide and conquer.

(a) Why is the preliminary tridiagonalization important? For full marks, give two reasons. (5 marks)

(b) Describe in words the standard tridiagonalization process based on Householder reflectors. Do not give details of how each Householder reflector is constructed, but make it clear exactly what it achieves in terms of structure of the matrix. Why is the operation count $O(m^3)$? (6 marks)

(c) Now suppose A is pentadiagonal: $a_{ij} = 0$ for $|i - j| > 2$. Describe the standard tridiagonalization process as applied to this matrix. Use the following notation:

x for a nonzero
blank for a zero
X for a nonzero created or modified at the most recent step
0 for a zero introduced at the most recent step.

Sketch a 7×7 pentadiagonal matrix with x's. Now using the above notation carefully, show exactly what the matrix looks like after the first Householder reflector is applied, and the second, and the third, and so on. Continue in this way to the end of the process, making it clear exactly how many Householder reflectors are involved all together for this 7×7 matrix and exactly what structure is present after each. (10 marks)

(d) Explain why the operation count of the process just described is $O(m^3)$ flops (for the $m \times m$ case), despite the pentadiagonal structure of A. (There is an alternative algorithm that reduces this figure to $O(m^2)$, which you are not expected to derive.) (4 marks)

Section B — Finite Element Methods

Question 3

If $\Omega \subset \mathbb{R}^2$ is a bounded region with polygonal boundary $\partial\Omega$, define the function spaces $L_2(\Omega)$ and $\mathcal{H}^1(\Omega)$?
(2 + 2 marks)

Derive a weak form (W) which requires only that the weak solution and test functions are members of $\mathcal{H}^1(\Omega)$ for the boundary value problem

$$-\nabla \cdot (\alpha \nabla u) + \beta\ (1,1) \cdot \nabla u + \gamma u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega \qquad \text{(P)}$$

where each of the known functions $\alpha = \alpha(x,y), \beta = \beta(x,y), \gamma = \gamma(x,y)$ and $f = f(x,y)$ are in $L_2(\Omega)$. In precisely what space is any weak solution to be found? **(5 + 1 marks)**

If $\beta = 0$ for $(x,y) \in \Omega$, derive also a minimization form, (M), for the problem (P) clearly stating any required properties of the functions α and γ. If any such properties are satisfied, prove that $u \in \mathcal{H}^1(\Omega)$ satisfies (M) if it satisfies (W). **(3 + 3 marks)**

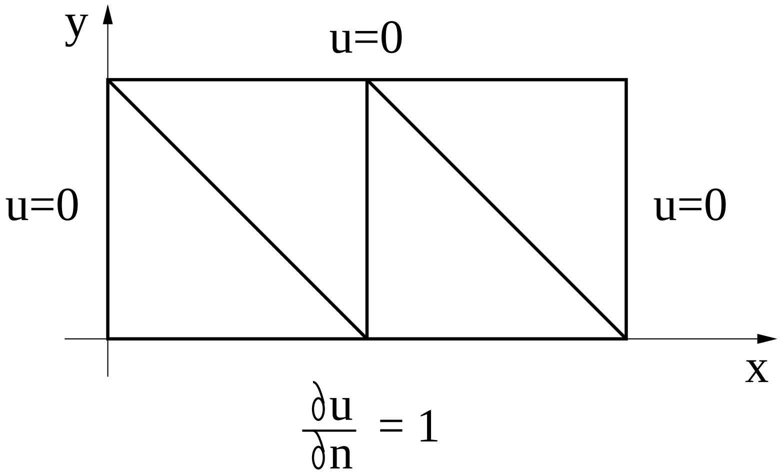
Is there a minimization form if $\beta \neq 0$? **(2 marks)**

Suppose now that β is a non-zero constant and a Galerkin finite element solution is to be sought in a finite dimensional subspace of $\mathcal{H}^1(\Omega)$ with basis $\{\phi_1, \phi_2, \ldots, \phi_n\}$. What matrix arises just from the single term $\beta\ (1,1) \cdot \nabla u$?

Prove that this matrix is skew-symmetric (i.e.the transpose is the negative of the matrix itself). If β is a non-constant function, what conditions must it satisfy in order that the corresponding matrix is skew-symmetric?
(3 + 3 + 1 marks)

Question 4

Calculate the P1 Galerkin finite element solution for the differential equation $-\nabla^2 u = y$ on the domain $(0,2)\times(0,1)$ and with boundary conditions as illustrated using the four elements shown. All of the four elements are right-angled isosceles triangles with equal sides of length 1.



(17 marks)

Suppose now that the boundary condition on the part of the boundary with $x = 2$ is changed so that $\frac{\partial u}{\partial n} = 0$ whilst the boundary conditions on the other three sides remain as shown. Write down the Galerkin finite element equations for this problem (you do not need to solve them). **(8 marks)**

Question 5

Proving any result that you need, verify that

$$\langle u, v\rangle = \int_a^b u'v'\mathrm{d}x$$

defines an inner product on

$$\mathcal{H}^1_{E_0}(a,b) = \left\{w \in \mathcal{H}^1(a,b); w(a) = 0 = w(b)\right\}$$

whenever a and b are bounded numbers. Of what relevance is this for the boundary value problem

$$\text{(BVP)} \qquad -u'' = f, \quad x \in (a,b), \quad u(a) = 0 = u(b)?$$

(7 + 2 marks)

If $S \subset \mathcal{H}^1_{E_0}(a,b)$ is a finite dimensional vector space, what is Galerkin's method for this problem? Prove that the Galerkin solution is the best approximation to the solution of (BVP) in a sense that you should make precise. **(2 + 5 marks)**

If now S is the vector space of continuous piecewise linear functions (P1) on a partition

$$a = x_0 < x_1 < x_2 < \ldots < x_n < x_{n+1} = b$$

and $u_h \in S$ is the Galerkin solution, prove that

$$\|(u - u_h)'\|_{L_2(a,b)} \leq h\|u''\|_{L_2(a,b)}$$

where u solves (BVP) and $h = \max\{|x_i - x_{i-1}|, i = 1, 2, \ldots, n+1\}$. You should carefully state any assumptions that you make. **(7 marks)**

What argument allows $O(h^2)$ convergence in $L_2(a,b)$ to be established? (You do not need to give the argument.)

(2 marks)

Question 6

The domain $\Omega = (0,1) \times (0,1)$ with boundary $\partial\Omega$ is divided into squares of side length $h = 1/n$ where n is a positive integer greater than 1. Each square is then divided into two P1 (continuous piecewise linear) triangular elements by the diagonal of positive slope. Calculate the element stiffness matrix for representative elements. **(10 marks)**

What is the form of the global coefficient matrix for the problem

$$-\nabla^2 u = f \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega$$

approximated by the Galerkin finite element method with P1 elements as above? Identify the kernel of this matrix. **(7 + 2 marks)**

Suppose now that the differential equation problem is changed to

$$-\nabla^2 u + xu = f \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

Is the corresponding coefficient matrix which arises from P1 Galerkin finite element approximation (i) symmetric, (ii) invertible? Give your reasons.

(3 + 3 marks)