JMAT 7304

Degree Master of Science in Mathematical Modelling and Scientific Computing

Numerical Linear Algebra & Finite Element Methods

TRINITY TERM 2012
Friday 20th April 2012, 9.30 a.m. – 11:30 a.m.

Candidates should submit answers to a maximum of four questions that include an answer to at least one question in each section.

Please start the answer to each question on a new page.

All questions will carry equal marks.

Do not turn over until told that you may do so.

Part A — Numerical Linear Algebra

Question 1

Let $A \in \mathbb{R}^{n\times n}$ be symmetric and positive definite and $\mathbf{b} \in \mathbb{R}^n$. Let $A = L + D + L^T$ with a strictly lower triangular matrix L and a diagonal matrix D. Given $\mathbf{x}^{(0)} \in \mathbb{R}^n$, the Gauss-Seidel iteration is defined by

$$(D+L)\mathbf{x}^{(j+1)} = \mathbf{b} - L^T\mathbf{x}^{(j)}.$$

(a) Define $L_1 = D^{-1/2}LD^{-1/2}$ and $G = D^{1/2}(D+L)^{-1}L^TD^{-1/2}$. Show that

$$G = (I + L_1)^{-1}L_1^T.$$

[7 marks]

Define $\mathbf{e}^{(j)} = \mathbf{x} - \mathbf{x}^{(j)}$, where $\mathbf{x}$ is the solution of $A\mathbf{x} = \mathbf{b}$. Show that for any vector norm and induced matrix norm,

$$\|\mathbf{e}^{(j+1)}\| \le \|(D+L)^{-1}L^T\|^{j+1}\|\mathbf{e}^{(0)}\|.$$

[7 marks]

(b) Let λ be the largest eigenvalue of G in magnitude and $\mathbf{z}$ be the corresponding eigenvector such that

$$G\mathbf{z} = \lambda\mathbf{z}, \qquad \overline{\mathbf{z}}^T\mathbf{z} = 1.$$

If $\overline{\mathbf{z}}^TL_1^T\mathbf{z} = a + ib$ with $a, b \in \mathbb{R}$, show that

$$|\lambda|^2 = \frac{a^2+b^2}{1+2a+a^2+b^2}.$$

[7 marks]

(c) Given that $1+2a > 0$, what can you deduce about the eigenvalues of $(D+L)^{-1}L^T$ and the convergence of the Gauss-Seidel iteration for positive definite matrices. Explain your answer. **[4 marks]**

Question 2

Throughout this question $A \in \mathbb{R}^{n\times n}$ is a symmetric, positive definite matrix.

(a) Define what is meant by saying that $\{\mathbf{p}_1, \dots, \mathbf{p}_n\} \subset \mathbb{R}^n$ are A-conjugate. Show that a basis consisting of orthogonal eigenvectors of A consists of A-conjugate vectors. **[5 marks]**

(b) Define $f(\mathbf{x}) := \frac{1}{2}\mathbf{x}^T A\mathbf{x} + \mathbf{b}^T\mathbf{x} + c$ with $\mathbf{b} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Show that there is a linear function $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$f(\mathbf{x}+\mathbf{p}) = f(\mathbf{x}) + \frac{1}{2}\mathbf{p}^T A\mathbf{p} + \mathbf{g}(\mathbf{x})^T\mathbf{p}$$

for all $\mathbf{x}, \mathbf{p} \in \mathbb{R}^n$. Show also that there are quadratic polynomials $F_k \in \Pi_2$ for $1 \le k \le n$ such that

$$f\left(\mathbf{x} + \sum_{k=1}^{n} t_k\mathbf{p}_k\right) = f(\mathbf{x}) + \sum_{k=1}^{n} F_k(t_k),$$

where $t_1, \dots, t_n \in \mathbb{R}$ and $\{\mathbf{p}_1, \dots, \mathbf{p}_n\}$ are A-conjugate. **[10 marks]**

(c) Define $K_i = \operatorname{span}\{\mathbf{p}_1, \dots, \mathbf{p}_i\}$, where $\{\mathbf{p}_1, \dots, \mathbf{p}_n\}$ are A-conjugate and $1 \le i \le n$. Let $\mathbf{x}_1 \in \mathbb{R}^n$ be given. Any $\mathbf{x} \in \mathbb{R}^n$ can be written in the form

$$\mathbf{x} = \mathbf{x}_1 + \sum_{k=1}^{n} \alpha_k\mathbf{p}_k.$$

Determine the coefficients α_k in terms of A, $\mathbf{p}_k$ and $\mathbf{e}_1 = \mathbf{x} - \mathbf{x}_1$. Given that $\|\mathbf{y}\|_A := (\mathbf{y}^T A\mathbf{y})^{1/2}$, $\mathbf{y} \in \mathbb{R}^n$ defines a norm on $\mathbb{R}^n$, show that

$$\mathbf{x}_{i+1} := \mathbf{x}_1 + \sum_{k=1}^{i} \alpha_k\mathbf{p}_k$$

is the best approximation to $\mathbf{x}$ from $\mathbf{x}_1 + K_i$ in this norm, i.e. it satisfies

$$\|\mathbf{x} - \mathbf{x}_{i+1}\|_A \le \|\mathbf{x} - \mathbf{y}\|_A$$

for all $\mathbf{y} \in \mathbf{x}_1 + K_i$. **[10 marks]**

Section B — Finite Element Methods

Question 3

Suppose that (a, b) is a nonempty bounded open interval of the real line.

(a) Define the Sobolev space $\mathrm{H}^1(a, b)$ and the Sobolev norm $\|\cdot\|_{\mathrm{H}^1(a,b)}$.

What is meant by saying that u is a weak solution in $\mathrm{H}^1(a, b)$ of the boundary-value problem

$$-u'' + (\cosh x)u = f(x), \quad x \in (a, b); \qquad u'(a) = 0, \quad u'(b) = 0,$$

where $f \in \mathrm{L}^2(a, b)$?

[3 marks]

By using the Lax–Milgram theorem, which you should carefully state, show that this boundary-value problem has a unique weak solution u in $\mathrm{H}^1(a, b)$.

[6 marks]

(b) Consider the piecewise linear finite element basis functions φ_i, $i = 0, 1, \ldots, N$, defined by $\varphi_i(x) := (1 - |x - x_i|/h)_+$, $x \in [a, b]$, on the uniform mesh of size $h = (b - a)/N$, $N \geq 2$, with mesh-points $x_i = a + ih$, $i = 0, 1, \ldots, N$.

Show that the basis functions φ_i, $i = 0, 1, \ldots, N$ are linearly independent. Hence deduce that the finite element space $\mathrm{V}_h := \mathrm{span}\{\varphi_0, \varphi_1, \ldots, \varphi_N\}$ is an $(N + 1)$-dimensional linear subspace of $\mathrm{H}^1(a, b)$.

State the finite element approximation of the boundary-value problem using the basis functions φ_i, $i = 0, 1, \ldots, N$, and show that it has a unique solution $u_h \in \mathrm{V}_h$.

[6 marks]

(c) Expand u_h in terms of the basis functions φ_i, $i = 0, 1, \ldots, N$, by writing

$$u_h(x) = \sum_{i=0}^{N} U_i \varphi_i(x), \qquad x \in [a, b],$$

where $\mathbf{U} := (U_0, U_1, \ldots, U_N)^{\mathrm{T}} \in \mathbb{R}^{N+1}$, to obtain a system of linear algebraic equations for the vector of unknowns $\mathbf{U}$. Show that the matrix $\mathcal{A}$ of this linear system is symmetric (i.e., $\mathcal{A}^{\mathrm{T}} = \mathcal{A}$).

[4 marks]

(d) Show that $u'' \in \mathrm{L}^2(a, b)$. Show further that there exists a positive constant C, independent of h such that $\|u - u_h\|_{\mathrm{H}^1(a,b)} \leq Ch\|u''\|_{\mathrm{L}^2(a,b)}$.

[Any bound on the error between u and its continuous piecewise linear finite element interpolant $\mathcal{I}_h u$ may be used without proof, but must be stated carefully.] **[6 marks]**

Question 4

Suppose that Ω is the open square $(-1,1) \times (-1,1)$ in $\mathbb{R}^2$ whose closure $\overline{\Omega}$ has been subdivided into M closed triangles so that any pair of triangles intersect along a complete edge, at a vertex, or not at all.

(a) State the piecewise linear finite element approximation, on the given triangulation of $\overline{\Omega}$, of the partial differential equation

$$-\Delta u + \frac{\partial u}{\partial x} = 4 \qquad \text{in } \Omega,$$

subject to the homogeneous Dirichlet boundary condition $u = 0$ on $\partial\Omega$.

[7 marks]

(b) Consider a triangle K in the triangulation of $\overline{\Omega}$ whose vertices P_i, $i = 1,2,3$, numbered in an anti-clockwise direction, have position vectors $\mathbf{r}_i = (x_i, y_i)$, $i = 1,2,3$. Suppose, further, that u_h and v_h are linear functions defined on K such that $u_h(P_i) = U_i^K$ and $v_h(P_i) = V_i^K$, $i = 1,2,3$.

Show that

$$\int_K \left(\nabla u_h \cdot \nabla v_h + \frac{\partial u_h}{\partial x} v_h \right) \mathrm{d}x \, \mathrm{d}y = [U_1^K, U_2^K, U_3^K] \, \mathcal{S}_k \begin{bmatrix} V_1^K \\ V_2^K \\ V_3^K \end{bmatrix},$$

where $k \in \{1, 2, \ldots, M\}$ is the number of the triangle K in a global element numbering, and $\mathcal{S}_k$ is the associated 3×3 element stiffness matrix whose (i,j)-entry is given in terms of the nodal basis functions ψ_i, $i = 1,2,3$, of element K by the formula

$$(\mathcal{S}_k)_{i,j} = \int_K \left(\nabla \psi_i \cdot \nabla \psi_j + \frac{\partial \psi_i}{\partial x} \psi_j \right) \mathrm{d}x \, \mathrm{d}y.$$

Show further that

$$\psi_1(x,y) = a_1(x - x_1) + b_1(y - y_1) + 1,$$

where $a_1 = (y_2 - y_3)/(2|K|)$, $b_1 = (x_3 - x_2)/(2|K|)$, and

$$|K| := \frac{1}{2} \begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix}$$

is the area of K. Use a cyclic permutation of the indices to give similar formulae for $\psi_2(x,y)$ and $\psi_3(x,y)$. Compute the $(1,1)$ entry, $(\mathcal{S}_k)_{11}$, of the matrix $\mathcal{S}_k$.

[9 marks]

(c) Now suppose that $\overline{\Omega}$ has been divided into four squares with a uniform mesh of spacing $h = 1$ in the x and y directions, and that each of the four squares has been further subdivided into two right-angle triangles with the diagonal of negative slope. Let u_h denote the continuous piecewise linear finite element approximation u_h to u on this triangulation. Show that $u_h(0,0) = 1$.

[9 marks]

Question 5

Suppose that $\Omega = (0,1)^2$ and $f \in \mathrm{L}^2(\Omega)$. Consider the quadratic energy-functional $J : \mathrm{H}^1(\Omega) \to \mathbb{R}$ defined by

$$J(v) = \frac{1}{2}a(v,v) - \ell(v),$$

where

$$a(w,v) = \int_\Omega \left[2^x \frac{\partial w}{\partial x}\frac{\partial v}{\partial x} + 2^y \frac{\partial w}{\partial y}\frac{\partial v}{\partial y} + wv\right] \mathrm{d}x\,\mathrm{d}y \quad \text{and} \quad \ell(v) = \int_\Omega fv\,\mathrm{d}x\,\mathrm{d}y.$$

(a) Show that u is a minimizer of J over $\mathrm{H}^1(\Omega)$ (i.e., $J(u) \leq J(v)$ for all $v \in \mathrm{H}^1(\Omega)$) if, and only if,

$$a(u,v) = \ell(v) \qquad \text{for all } v \in \mathrm{H}^1(\Omega). \tag{1}$$

[10 marks]

(b) State the elliptic boundary-value problem whose weak formulation (1) is.

[5 marks]

(c) Consider a triangulation of $\overline{\Omega}$, which has been obtained from a square mesh of spacing $h = 1/N$, $N \geq 2$, in both co-ordinate directions by subdividing each mesh-square into two triangles with the diagonal of negative slope. Denote by V_h the finite-dimensional subspace of $\mathrm{H}^1(\Omega)$ consisting of all continuous piecewise linear functions defined on this triangulation.

Show that there exists a unique element u_h in V_h such that $J(u_h) \leq J(v_h)$ for all $v_h \in \mathrm{V}_h$.

[5 marks]

Show further that

$$\|u - u_h\|_{\mathrm{H}^1(\Omega)} \leq \sqrt{2} \min_{v_h \in \mathrm{V}_h} \|u - v_h\|_{\mathrm{H}^1(\Omega)}.$$

[5 marks]

Question 6

Let $u = u(x,t)$ denote the solution to the initial-boundary-value problem

$$\begin{aligned} \frac{\partial u}{\partial t} + u = \frac{\partial^2 u}{\partial x^2}, \qquad & 0 < x < 1, \quad 0 < t \le T, \\ \frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(1,t) = 0, \qquad & 0 \le t \le T, \\ u(x,0) = u_0(x), \qquad & 0 < x < 1, \end{aligned}$$

where $T > 0$, $u_0 \in \mathrm{L}^2(0,1)$.

(a) Construct a finite element method for the numerical solution of this problem, based on the backward Euler scheme with time step $\Delta t = T/M$, $M \ge 2$, and a piecewise linear approximation in x on a uniform subdivision of spacing $h = 1/N$, $N \ge 2$, of the interval $[0,1]$, denoting by u_h^m the finite element approximation to $u(\cdot, t^m)$ where $t^m = m\Delta t$, $0 \le m \le M$.

[9 marks]

(b) Show that, for $0 \le m \le M-1$,

$$\frac{1}{2\Delta t}\left(\|u_h^{m+1}\|^2_{\mathrm{L}^2(0,1)} - \|u_h^m\|^2_{\mathrm{L}^2(0,1)}\right) + \frac{1}{2\Delta t}\left\|u_h^{m+1} - u_h^m\right\|^2_{\mathrm{L}_2(0,1)} + \left\|u_h^{m+1}\right\|^2_{\mathrm{H}^1(0,1)} = 0,$$

where $\|\cdot\|_{\mathrm{L}^2(0,1)}$ is the L^2-norm on the interval $(0,1)$, and $\|\cdot\|_{\mathrm{H}^1(0,1)}$ is the norm of the Sobolev space $\mathrm{H}^1(0,1)$.

Hence deduce that the method is unconditionally stable in the L^2-norm in the sense that, for any Δt, independent of the choice of h,

$$\|u_h^m\|_{\mathrm{L}^2(0,1)} \le \|u_h^0\|_{\mathrm{L}^2(0,1)}, \qquad 1 \le m \le M.$$

[9 marks]

(c) Show that, for each m, $0 \le m \le M-1$, u_h^{m+1} can be obtained from u_h^m by solving a system of linear algebraic equations with a symmetric matrix $\mathcal{A}$ whose entries you should define in terms of the standard piecewise linear basis functions φ_i, $0 \le i \le N$. Show further that the matrix $\mathcal{A}$ is positive definite (i.e., $\mathbf{V}^{\mathrm{T}}\mathcal{A}\mathbf{V} > 0$ for all $\mathbf{V} \in \mathbb{R}^{N+1}$, $\mathbf{V} \ne \mathbf{0}$).

[7 marks]