Degree Master of Science in Mathematical Modelling and Scientific Computing

Numerical Solution of Differential Equations & Numerical Linear Algebra

Thursday, 10th January 2008, 2:00 p.m. – 4:00 p.m.

Candidates may attempt as many questions as they wish but must attempt at least one of questions 5 and 6.
The best four solutions, including one from questions 5 and 6, will count.
Solutions to questions 1–4, and 5–6 should be handed in separately.

Please start the answer to each question on a new page.

All questions will carry equal marks.

Do not turn over until told that you may do so.

Question 1

Let y be the solution of the initial-value problem

$$y'(x) = \tan^{-1}(1 + 3y^4(x)), \qquad y(0) = 0, \qquad (*)$$

where $\tan^{-1}$ denotes arc tan.

(i) Show that $|y'(x)| < \frac{1}{2}\pi$ and $|y''(x)| < \frac{3}{4}\pi$ for all $x \in \mathbb{R}$.
Let $f(t) := \tan^{-1}(1 + 3t^4)$. Show that f satisfies the Lipschitz condition

$$|f(t_1) - f(t_2)| \leq \frac{3}{2}|t_1 - t_2| \qquad \forall t_1, t_2 \in \mathbb{R}.$$

[You may use, without proof, that $4|t|^3 \leq 1 + 3t^4$ for all $t \in \mathbb{R}$.]

(ii) State the implicit Euler scheme for the numerical solution of the initial-value problem $(*)$ on the uniform mesh $\{x_n \,:\, x_n := nh,\, n = 0, 1, \ldots\}$ of spacing $h > 0$ over the interval $[0, \infty) \subset \mathbb{R}$.
Consider the function $t \mapsto g(t) := t - hf(t)$, with f as in part (i). Show that:

(a) for each $h > 0$, $\lim_{t\to\pm\infty} g(t) = \pm\infty$; and
(b) if $h \in (0, \frac{2}{3})$, then $g'(t) > 0$ for all $t \in \mathbb{R}$.

Hence deduce that, for each $h \in (0, \frac{2}{3})$, there *exists* a *unique* sequence $(y_n)_{n=0}^{\infty} \subset \mathbb{R}$ of Euler approximations with $y_0 := 0$.

(iii) Let T_n denote the truncation error of the method at the mesh point $x = x_n$. Show that

$$\max_{n\geq 0} |T_n| < \frac{3}{8}\pi\, h.$$

Show further that, for $h \in (0, \frac{2}{3})$,

$$|y(x_n) - y_n| \leq \frac{\pi}{4}\left\{\left(1 + \frac{3}{2}h\right)^n - 1\right\} \cdot h, \qquad n = 0, 1, 2, \ldots.$$

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Question 2

Consider the initial-value problem

$$y' = f(x, y), \qquad y(0) = y_0,$$

where f is a smooth function of two variables and $y_0 \in \mathbb{R}$, and the linear three-step method

$$y_{n+3} + \alpha y_{n+1} + \beta y_n = hf(x_{n+2}, y_{n+2})$$

on a uniform mesh $\{x_n : x_n = nh,\ n = 0, 1, \ldots\}$ of spacing $h > 0$, where α and β are fixed real numbers.

(i) Show that there exists a unique choice of α and β such that the three-step method is consistent, i.e., its order of accuracy as at least 1. Find these values of α and β and show that the order of accuracy of the method cannot exceed 1.

(ii) Show further that, for the values of α and β found in part (i), the method is *not* zero-stable. Use Dahlquist's Theorem, which you should carefully state, to deduce that the sequence of numerical solutions generated by the method does not, in general, converge to the analytical solution y in the limit of $h \to 0$.

(iii) By considering the general solution of the recurrence relation $y_{n+3} + \alpha y_{n+1} + \beta y_n = 0$ (corresponding to $f(x, y) \equiv 0$), with α and β as in part (i), describe quantitatively what zero-unstable solutions will look like for small values of h.

Question 3

Consider the finite difference mesh $\mathcal{M} := \{(x_j, t_m) \,:\, j = 0, 1, \ldots, J,\ m = 0, 1, \ldots, M\}$, where $x_j := j\Delta x$ and $t_m := m\Delta t$, with $\Delta x := 1/J$, $\Delta t := T/M$, $J \geq 2$, $M \geq 1$, and $T > 0$.

(i) Formulate the θ-scheme on $\mathcal{M}$, with $\theta \in [0, 1]$ (and with $\theta = 1$ corresponding to the implicit Euler scheme) for the numerical solution of the initial-boundary-value problem

$$u_t = \kappa u_{xx}, \quad x \in (0, 1), \quad t \in (0, T];$$

$$u(0, t) = A(t), \quad u(1, t) = B(t), \quad t \in (0, T]; \qquad u(x, 0) = u_0(x), \quad x \in [0, 1],$$

where A, B are continuous real-valued functions defined on $[0, T]$ and u_0 is a continuous real-valued function defined on $[0, 1]$ with $u_0(0) = A(0)$ and $u_0(1) = B(0)$.

(ii) Show that U_j^m, the approximation to $u(x_j, t_m)$ computed from the θ-scheme, is bounded above by $U_{\max}$, where

$$U_{\max} := \max\{\max_{0 \leq m \leq M} A(t_m),\ \max_{0 \leq m \leq M} B(t_m),\ \max_{0 \leq j \leq J} u_0(x_j)\},$$

provided that the stability condition

$$\mu(1 - \theta) \leq \frac{1}{2}$$

is satisfied with $\mu := \frac{\kappa \Delta t}{(\Delta x)^2}$.

(iii) Write down the recurrence relation satisfied by the error at the mesh points

$$e_j^m := u(x_j, t_m) - U_j^m.$$

Assuming that the initial and boundary conditions for the θ-scheme are exact, and Δx and Δt are such that $\mu(1 - \theta) \leq \frac{1}{2}$, derive a bound on

$$E^m := \max_{0 \leq j \leq J} |e_j^m|, \quad 0 \leq m \leq M,$$

in terms of

$$T^m := \max_{1 \leq j \leq J-1} |T_j^m|, \quad 0 \leq m \leq M,$$

where T_j^m is the truncation error of the θ-scheme at the mesh point (x_j, t_m).

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Question 4

Suppose that $u_0 : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function and $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function such that f' is monotonic increasing.

(i) Formulate the upwind finite difference scheme for the numerical solution of the nonlinear hyperbolic partial differential equation

$$u_t + (f(u))_x = 0, \qquad x \in \mathbb{R}, \quad t > 0,$$

subject to the initial condition $u(x,0) = u_0(x)$ for $x \in \mathbb{R}$, on a uniform finite difference mesh of spacing Δx in the x-direction and spacing Δt in the t-direction.

(ii) Let $\mathbb{Z}$ denote the set of all integers and let U_j^m denote the upwind finite difference approximation to u at the mesh point (x_j, t_m), where $x_j = j\Delta x$, $j \in \mathbb{Z}$ and $t_m = m\Delta t$, $m = 0, 1, 2, \ldots$. Show by induction that, if

$$\left(\max_{x \in \mathbb{R}} |f'(u_0(x))| \right) \frac{\Delta t}{\Delta x} \leq 1,$$

then the following inequalities hold:

(a)

$$\left(\max_{j \in \mathbb{Z}} |f'(U_j^m)| \right) \frac{\Delta t}{\Delta x} \leq 1 \qquad \text{for all } m = 0, 1, 2, \ldots \;;$$

(b)

$$\max_{j \in \mathbb{Z}} |U_j^{m+1}| \leq \max_{j \in \mathbb{Z}} |U_j^m| \qquad \text{for all } m = 0, 1, 2, \ldots \,.$$

Question 5

(i) For a square matrix $A = \{a_{i,j}, i, j = 1, \ldots, n\}$ and a vector $b = \{b_i, i = 1, \ldots, n\}$ write down an algorithm for Gaussian Elimination (without pivoting). To what matrix factorization is this essentially equivalent?

(ii) By performing Gauss Elimination for

$$A = \begin{bmatrix} 2 & -2 & 3 & -3 \\ 1 & 1 & 1 & 0 \\ 1 & -1 & 4 & 3 \\ 1 & -1 & 2 & -1 \end{bmatrix}$$

(i.e. ignoring b) calculate the factors in this matrix factorization for A.

If a given matrix A was symmetric and positive definite what is the corresponding matrix factorization which explicitly accounts for this structure? (Do not give details of any algorithm for the computation of this factorization.)

(iii) If $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, m > n$, what matrix factorization is usually used to solve the problem

$$(\star) \qquad \min_{x \in \mathbb{R}^n} \|Ax - b\|_2?$$

Indicate how x is found if such a matrix factorization has already been computed.

(iv) Show that if b is changed by a small amount $\delta b \in \mathbb{R}^m$, x solves $(\star)$ and $(x + \delta x) \in \mathbb{R}^n$ solves

$$\min \|A(x + \delta x) - (b + \delta b)\|_2,$$

then $\|\delta x\|_2 \leq \|(A^T A)^{-1}\|_2 \|A\|_2 \|\delta b\|_2$.

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Question 6

The five-point finite difference approximation of the Laplacian with Dirichlet boundary conditions using a uniform grid with spacing h on a unit square gives rise to the linear system of equations

$$4U_{j,k} - U_{j+1,k} - U_{j-1,k} - U_{j,k+1} - U_{j,k-1} = b_{j,k}, \quad j,k = 1,\ldots,n, \quad (\star)$$

where $U_{0,k}$ and $U_{n+1,k}$ are known for each k, $U_{j,0}$ and $U_{j,n+1}$ are known for each j and $b_{j,k}$ is known for all j and k. A simple smoothing iteration:

$$MU^{(\ell)} = NU^{(\ell-1)} + b$$

where $A = M - N$ is the coefficient matrix in $(\star)$ is used in a multigrid iteration with only 2 levels of gridding (ie. a 2-grid iteration).

(i) Explain what is meant by a *prolongation operator*, P, a *restriction operator*, R and a *coarse grid operator*, $\overline{A}$ in this context and give examples of each.

(ii) Write down (as a procedure) the steps in the 2-grid algorithm with pre- and post-smoothing.

Show that the error e_i in the i^{th} 2-grid iteration satisfies

$$e_i = (M^{-1}N)^{\mu}(A^{-1} - P\overline{A}^{-1}R)A(M^{-1}N)^{\nu}e_{i-1}$$

when ν pre-smoothing steps and μ post-smoothing steps are employed.