Degree Master of Science in Mathematical Modelling and Scientific Computing

Numerical Solution of Differential Equations & Numerical Linear Algebra

Thursday, 14th January 2010, 2:00 p.m. – 4:00 p.m.

Candidates should submit answers to a maximum of four questions that include an answer to at least one question in each section.

Please start the answer to each question on a new page.

All questions will carry equal marks.

Do not turn over until told that you may do so.

Section A —Numerical Solution of Differential Equations

Question 1

Consider the initial-value problem $y' = f(y)$, $y(0) = 1$, where $f(y) = 1/(1+y^2)$. [You may assume that this problem has a unique solution $x \mapsto y(x)$, defined for all $x \in \mathbb{R}$, such that the functions $x \mapsto y'(x)$ and $x \mapsto y''(x)$ are defined and continuous on the whole of $\mathbb{R}$.]

(a) Show that $|y''(x)| \leq 1$ for all $x \in \mathbb{R}$. Show further that the function f satisfies the following Lipschitz condition:

$$|f(u) - f(v)| \leq |u - v|.$$

[5 marks]

(b) The implicit Euler approximation y_n to $y(x_n)$ on the mesh $\{x_n : x_n = nh, n = 0, 1, \ldots\}$ of uniform spacing $h \in (0, 1)$ is obtained from the formula

$$\frac{y_n - y_{n-1}}{h} = f(y_n), \qquad n = 1, 2, \ldots; \quad y_0 = 1.$$

Let $g(y) := y - hf(y)$. Show that the function $y \mapsto g(y)$ is strictly monotonic increasing and $\lim_{y\to\pm\infty} g(y) = \pm\infty$.

By rewriting Euler's method as $g(y_n) = y_{n-1}$, deduce that, given $y_{n-1} \in \mathbb{R}$, the Euler approximation y_n is uniquely defined in $\mathbb{R}$. **[8 marks]**

(c) Show that the truncation error T_n of the implicit Euler method applied to the initial-value problem under consideration satisfies

$$|T_n| \leq \frac{1}{2}h, \qquad n = 1, 2, \ldots.$$

Show further that

$$|y(x_n) - y_n| \leq \frac{1}{1-h}\,|y(x_{n-1}) - y_{n-1}| + \frac{h}{1-h}|T_n|, \qquad n = 1, 2, \ldots,$$

and deduce that

$$|y(x_n) - y_n| \leq \frac{h}{2}\left[\left(1 + \frac{h}{1-h}\right)^n - 1\right], \qquad n = 1, 2, \ldots.$$

Hence show that for any tolerance $\mathtt{TOL} > 0$ there exists $h_0 = h_0(\mathtt{TOL}) \in (0, 1)$ such that if $h \leq h_0$, then $|y(x_n) - y_n| \leq \mathtt{TOL}$ for all $x_n \in [0, 1]$. **[12 marks]**

Question 2

(a) State the general form of a linear k-step method for the numerical solution of the initial-value problem $y' = f(x, y)$, $y(x_0) = y_0$ on the mesh $\{x_j : x_j = x_0 + jh\}$ of uniform spacing $h > 0$. **[2 marks]**

(b) Define the *truncation error* of a linear k-step method. What is meant by saying that a linear k-step method is *consistent*? **[4 marks]**

(c) What is meant by saying that a linear k-step method is *zero-stable*? Formulate an equivalent characterisation of zero-stability in terms of the roots of a certain polynomial of degree k.

[4 marks]

(d) Consider the two-parameter family of linear two-step methods defined by

$$y_{n+2} - (1 + a)\, y_{n+1} + a\, y_n = b\, h\, f(x_{n+2}, y_{n+2}), \qquad n = 0, 1, \ldots,$$

where a and b are real numbers, $a \neq 0$, $b \neq 0$. Determine the set of all values of the parameters a and b such that the method is zero-stable. **[4 marks]**

(e) Show that there exists a unique choice of the parameters a and b such that the method is second-order accurate. Is the method convergent for these values of a and b? Justify your answer by using Dahlquist's Theorem, which you should carefully state. **[5 + 6 marks]**

Question 3

Consider the initial-boundary-value problem

$$p(x)\,\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1, \quad 0 < t \le T,$$

$$u(x,0) = u_0(x), \quad 0 < x < 1, \qquad u(0,t) = A, \quad u(1,t) = B, \quad t \in [0,T],$$

where A and B are given real numbers, T is a fixed positive real number, p is a continuous real-valued function of $x \in [0,1]$, $p(x) \ge c_0 > 0$ for all $x \in [0,1]$, and u_0 is a real-valued function, defined and continuous on the closed real interval $[0,1]$ such that $u_0(0) = A$ and $u_0(1) = B$.

(a) Formulate the explicit Euler scheme for the numerical solution of this initial-boundary-value problem on a mesh with uniform spacings $\Delta x = 1/J$ and $\Delta t = T/M$ in the x and t co-ordinate directions, respectively, where J and M are positive integers. **[8 marks]**

(b) Define the truncation error of the scheme, and show that it is bounded in absolute value by $C_1 \Delta t + C_2 (\Delta x)^2$, where C_1 and C_2 are positive constants that you should specify.

[You may assume that u has as many bounded and continuous partial derivatives with respect to x and t as are required by your proof.] **[8 marks]**

(c) Now, suppose that $\Delta t \le \frac{1}{2} c_0 (\Delta x)^2$. Show that

$$\max_{0 \le j \le J} |U_j^m| \le \max_{0 \le j \le J} |U_j^0|$$

for all m, $1 \le m \le M$. **[9 marks]**

Question 4

Consider the initial-value problem for the scalar nonlinear hyperbolic equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} g(u(x,t)) = 0, \qquad -\infty < x < \infty, \quad 0 < t \leq T,$$

subject to the initial condition $u(x,0) = u_0(x)$. Here T is a positive real number, u_0 is a real-valued, bounded, monotonic increasing and continuously differentiable function of $x \in (-\infty, \infty)$, and g is a real-valued, twice continuously differentiable function on $\mathbb{R}$, whose second derivative is nonnegative on $\mathbb{R}$.

(a) By using the chain rule, or otherwise, verify that the continuously differentiable function u, defined implicitly by the equation

$$u(x,t) = u_0\left(x - t g'(u(x,t))\right),$$

is a solution to the initial-value problem. **[12 marks]**

(b) Suppose that $g(u) = \exp(u)$ and $u_0(x) = \tanh x$. Formulate the first-order upwind scheme for the numerical solution of the initial-value problem on a uniform mesh of size $\Delta x > 0$ in the x-direction and size $\Delta t = T/M$ in the t-direction, where $M \geq 1$, denoting by U_j^m the approximation to $u(j\Delta x, m\Delta t)$.

Suppose that

$$\frac{\Delta t}{\Delta x} \max_{x \in \mathbb{R}} \exp(u_0(x)) \leq 1.$$

Show by induction that

$$\max_{j \in \mathbb{Z}} |U_j^m| \leq \max_{j \in \mathbb{Z}} |U_j^{m-1}| \qquad \text{and} \qquad \frac{\Delta t}{\Delta x} \max_{j \in \mathbb{Z}} \exp(U_j^m) \leq \frac{\Delta t}{\Delta x} \max_{j \in \mathbb{Z}} \exp(U_j^{m-1})$$

for all m, $1 \leq m \leq M$. Hence deduce that

$$\max_{0 \leq m \leq M} \max_{j \in \mathbb{Z}} |U_j^m| \leq 1.$$

[4+4+4+1 marks]

Section B — Numerical Linear Algebra

Question 5

(a) Let $\|\cdot\|$ be a function that maps $\mathbb{R}^m$ to $\mathbb{R}$. What are the conditions for $\|\cdot\|$ to be a norm on $\mathbb{R}^m$?

[4 marks]

(b) The most familiar family of norms on $\mathbb{R}^m$ are the p-norms, where p is a number in the range $1 \leq p \leq \infty$. Define these norms. (You will probably need to define $\|\cdot\|_\infty$ separately from $\|\cdot\|_p$ for $p < \infty$.) Also, in the case $m = 2$, draw sketches of the unit ball with respect the norms $\|\cdot\|_{10}$ and $\|\cdot\|_{1.1}$. Your sketches do not have to be precision images, but they should make it clear what the shapes look like and they should label exactly what the intercepts are with the two axes. **[6 marks]**

(c) Now consider the case $0 < p < 1$. The definition of part (b) still makes sense, but show by a precise numerical example that the function it defines is no longer a norm. In addition, draw a sketch of the unit ball in a case $0 < p < 1$ and indicate how this sketch confirms that the function is not a norm.

[6 marks]

(d) Now let p approach 0 and consider the "0-norm" that arises in a natural way from this limit. State precisely what this "0-norm" function is. For example, with $m = 5$, what are the "0-norms" of the vectors $(1, 2, 3, 4, 5)^T$ and $(1, 0, 1, 0, 1)$? Then, show again by an explicit example that this function is also not a norm.

[6 marks]

(e) In applications, the "0-norm" is nevertheless sometimes very important. Comment on why a "minimal 0-norm" solution of a linear algebra problem might be of interest in some applications.

[3 marks]

Question 6

(a) Let $A \in \mathbb{R}^{m \times m}$ be nonsingular and consider the linear system of equations $Ax = b$ where $b \in \mathbb{R}^m$. One way to solve this problem is by Gaussian elimination. In the limit $m \to \infty$, the number of floating-point operations required by Gaussian elimination is asymptotic to am^k for certain constants a and k. What are these constants? For full marks do not just write down the answer but also derive it from a specification of how Gaussian elimination works. You may ignore pivoting. **[7 marks]**

(b) An alternative algorithm for solving $Ax = b$ is QR factorization by Householder reflectors. What is the asymptotic operation count in this case? Again, for full marks do not just write down the answer but derive it. **[8 marks]**

(c) Yet another algorithm would be form the *normal equations* associated with $Ax = b$ and solve these by the conjugate gradient iteration. Write down the normal equations, and calculate how many floating point operations, to leading order, are required to compute the necessary matrix product (exploiting symmetry).

[8 marks]

(d) For general matrices A (without a preconditioner), the conjugate gradients method of (c) will not converge in less than m steps. In exact arithmetic, it would normally converge in exactly m steps. Assuming this is what happens (which in fact is an unrealistic assumption because of the effect of rounding errors), determine the total operation count for solving $Ax = b$ by conjugate gradients (both forming the normal equations, then solving them). You may assume that the work per step of the conjugate gradient iteration is dominated by one matrix-vector product. **[5 marks]**