EXTENDED ESSAYS:
Option BE or OE in Part B of the Final Honour School of Mathematics

DISSERTATIONS:
Option CD or OD in Part C of the Final Honour School of Mathematics

SOME IDEAS FOR PROJECTS

June 2, 2011

Contents

1 Logic, Set Theory and Combinatorics **5**

1.1 Model Theory — Strongly minimal structures 5

1.2 Model Theory — The model theory of cyclic groups. 5

1.3 Model theory of the real numbers 6

1.4 O-minimal structures 6

1.5 Local equivalents of the Axiom of Choice 7

1.6 Theories of the real numbers 7

1.7 Combinatorics — Graphs of large chromatic number 7

2 Geometry and Topology **8**

2.1 Minimal Surfaces 8

2.2 Characterisation of certain topological spaces related to $\mathbb{R}$ . . 8

2.3 Clifford algebras 9

2.4 Knot theory 9

2.5 Flexible polyhedra 9
2.6 Dimension theory 9
2.7 Transversality 9
2.8 Classification of complex algebraic surfaces 9
2.9 Thom's Cobordism Theorem 10
2.10 Classification of unoriented (1 + 1)-dimensional topological field theories 11
2.11 Mirror symmetry and algebraic geometry 12
2.12 Platonic solids in four dimensional space 12
2.13 Topological field theories with boundaries 12
2.14 Pits, passes and peaks 13
2.15 Number systems and vector fields 13
2.16 Topology and partial differential equations 13
2.17 Infinite groups as geometric objects 13

3 Algebra and Number Theory **14**
3.1 Converses of Lagrange's Theorem 14
3.2 Simple groups of odd order 14
3.3 Alhazen's Problem 14
3.4 Reduction of quadratic forms 15
3.5 Costing algorithms in algebra and number theory 15
3.6 Euclidean and Principal Ideal Domains 15
3.7 Primality testing 15
3.8 Approximation of Irrational numbers by Rationals 15
3.9 Unique Factorization in Quadratic Number Fields 16
3.10 Waring's problem 17
3.11 Modular functions and modular forms 18

4 Analysis **18**
4.1 Abelian locally compact groups 18
4.2 Kazhdan's property (T) 19
4.3 Analysis in a rational world 19
4.4 Analysis of holomorphic functions with special values 20
4.5 Integration 20
4.6 Measure Theory 20
4.7 Univalent functions 20
4.8 Distributions 21
4.9 Special functions 21
4.10 Riemann surfaces 21
4.11 The Schwarzian derivative 21
4.12 Chaos in nonlinear ordinary differential equations 21

5 Numerical Analysis **22**
5.1 Stiff ordinary differential equations 22
5.2 Numerical approximation of singular integrals 22
5.3 Newton's method for nonlinear systems 22
5.4 Finite element methods for singularly perturbed problems . . 23
5.5 A posteriori error analysis of finite element methods 23
5.6 Fast iterative methods for systems of linear equations 23

6 Applied mathematics **24**
6.1 Mathematics and the environment 24
6.2 Mathematical physiology 24
6.3 Mathematical models in finance 24
6.4 Fluid dynamics 24
6.5 Mechanics of Solids 25
6.6 Mathematical biology 25
6.7 Mathematical Genealogy and the Evolution of Research Groups 25
6.8 Nonlinear Schrödinger Equations with Lattice and Superlattice Potentials 26
6.9 Community Structure in Multislice Networks 27

7 Theoretical physics **28**
7.1 Extremum principles in theoretical physics 28
7.2 Hamilton–Jacobi theory 28
7.3 Concepts of quantum mechanics 28
7.4 Symmetries in quantum mechanics 28
7.5 Quantum mechanics of scattering 29
7.6 Waveguides 29
7.7 The microwave background 29
7.8 Tests of general relativity 29
7.9 Hot big bang versus steady state in cosmology 29
7.10 Appearance of moving objects in special relativity 29
7.11 Study of a special metric 30
7.12 SO(3), SU(2), Euler angles and angular momentum 30
7.13 Momentum space in quantum mechanics 30
7.14 Translation of some well known theorem in euclidean space to Minkowski space 30
7.15 Crystal symmetry, non-periodic tiling and quasi-crystals . . . 30

8 Probability and Statistics **30**
8.1 Mathematical models in evolution 30
8.2 Duality and Random Walks 31

8.3 The Coupling Method . . . 31
8.4 Applied Probability . . . 31
8.5 Operational Research . . . 31

9 History of Mathematics **32**

10 General **33**

11 Titles of Previous Projects **34**
11.1 Extended Essays . . . 34
11.2 Dissertations . . . 35

1 Logic, Set Theory and Combinatorics

1.1 Model Theory — Strongly minimal structures

Part C half-unit project.

Strongly minimal structures and their combinatorial geometries are im- portant notions of modern model theory. These notion are also central for understanding principles of a model-theoretic classification of classical mathematical structures. There is an extensive discussion of these in graduate textbooks.
The aims of the project are:

1. to present and prove basic facts about strongly minimal structures;
2. to present classical examples of strongly minimal structures and to discuss the existence of non-classical once;
3. to discuss generalisations of strongly minimal structures.

1.2 Model Theory — The model theory of cyclic groups.

Part C half-unit project.

The aim of this project is to understand the model-theoretic limit of a sequence of finite cyclic groups and relate it to the theory of the infinite cyclic group.
The main intended result would be the following.

Theorem Given a positive sentence φ in the standard language of Abelian groups, $\mathbb{Z}/n\mathbb{Z} \vDash \varphi$ for all $n \in \mathbb{N}$, if and only if $\mathbb{Z} \vDash \varphi$.

The proof of the right-to-left implication is relatively easy and is based on one of the basic facts of model theory, the preservation of positive formulas under homomorphisms.

The converse requires much more work and reading. Textbook material on the theory of abelian groups and the paper *The elementary theory of abelian* *groups* Ann. Math. Logic 4 (1972), 115–171. by P.Eklof and E.Fisher is recommended.

1.3 Model theory of the real numbers

CD or CD/2

According to Gödel's theorem and its developments, the first order theory of the natural numbers is undecidable: there is no decision procedure for deciding whether statements are true or not. However, some other natural mathematical structures are decidable, including the theory of the complex numbers as an algebraically closed field of characteristic zero, and the theory of the real numbers as an ordered field. How are such results established and can they be extended to more complicated structures?

D. Marker, *Model theory: an introduction,* Graduate Texts in Mathematics **217**, Springer-Verlag, 2002.

P.J. Cohen, Decision procedures for real and p-adic fields, *Comm. Pure Appl. Math.* **22** (1969), 131–151.

1.4 O-minimal structures

CD or CD/2

An o-minimal structure is a model-theoretic structure $\mathcal{M}$ whose underlying domain M possesses a dense linear order such that the definable subsets of M are "as simple as possible" : they are just the finite unions of points and open intervals. This simple requirement has very strong consequences. For example, definable functions in such a structure are continuous (in the order topology) except at finitely many points of their domain. Further, if M is a field, then definable functions are differentiable except at finitely many points in their domain. A good deal of real (and complex) analysis can be developed in this setting, where the underlying field may be very different from the real or complex numbers.

L. van den Dries, *Tame topology and o-minimal structures,* LMS Lecture Note Series **248**, CUP, 1998.

1.5 Local equivalents of the Axiom of Choice

The famous equivalence of the Axiom of Choice and the Well-Order Principle can be proved 'locally': a set X has a choice function if and only if X is well orderable. When we come to examine the equivalence of the Axiom of Choice with other assertions of Set Theory we often find that mismatches appear in the local versions. In Zorn's Lemma, for example, if X has a choice function then in every inductive partial ordering on X there are maximal elements; on the other hand, the usual argument requires that there should be maximal elements in every inductive partial ordering of the power set of X^2 to yield that there is a choice function on X. Investigate such 'gaps' in local equivalents of the Axiom of choice.

1.6 Theories of the real numbers

The system of real numbers may be defined as a complete linearly ordered field. What this is may be defined in many ways. In particular, many different versions of the completeness axiom have been proposed and used. Collect, compare and contrast these various theories.

1.7 Combinatorics — Graphs of large chromatic number

BE Extended Essay.

It is easy to see that the chromatic number of a graph is at least as large as its clique number (the number of vertices in a largest complete subgraph). A graph is *perfect* if its chromatic number equals its clique number, and the same holds for all its induced subgraphs. An essay on this topic could investigate the theory of perfect graphs, including the theorem of Lovász that a graph is perfect if and only if its complement is perfect. An essay should certainly discuss the Strong Perfect Graph Theorem of Chudnovsky, Robertson, Seymour and Thomas, which gives a structural characterization of perfect graphs, although there would not be space to include a proof.

Alternatively, the essay might look at what subgraphs must appear in graphs with large chromatic number. Relevant topics would include graphs with large girth and large chromatic number, and χ-bounded classes and the Gyárfás-Sumner Conjecture.

2 Geometry and Topology

2.1 Minimal Surfaces

Part C half-unit project.
Prerequisites: B3a.
Reading:

1. for background, Do Carmo, Differential Geometry of Curves and Surfaces, Prentice Hall, 1976.
2. M.Berger and B. Gostiaux, Differential Geometry: Manifolds,Curves and Surfaces,Springer-Verlag, 1988.
3. U. Dierkes, S. Hildebrandt, A. Künster, O. Wohlrag, Minimal Surfaces I, Springer-Verlag, 1991.
4. R. Osserman, A Survey of Minimal Surfaces, Dover, 1986.

The essay should introduce the reader to the subject of Minimal Surfaces and include a careful treatment of at least one special topic, for example, isothermal coordinates, Bernstein's Theorem, symmetry, other representations or interesting examples.

[The difficulty will come in confining the dissertation, using consistent notation and chosing suitable material for an introductory account.]

2.2 Characterisation of certain topological spaces related to $\mathbb{R}$

The real line $\mathbb{R}$ can be characterised as a path-connected Hausdorff topological space X with the property that for any point x the space $X - \{x\}$ is disconnected and has precisely two components. Start from a proof of this and generalise. What happens if $X - \{x\}$ has precisely three (or precisely n) components for each $x \in X$? What spaces arise if $X - \{x\}$ remains path-connected, but $X - \{x, y\}$ has n components whenever x, y are distinct points of X?

2.3 Clifford algebras

These algebras have been used very fruitfully and are an important tool for topologists, geometers and physicists. Give an account of the theory of Clifford algebras associated to finite dimensional spaces.

2.4 Knot theory

Investigate the place of knots in mathematics. In particular, explain the importance of the discovery of the Jones polynomial in the early eighties.

2.5 Flexible polyhedra

Cauchy proved that no closed convex polyhedron, with rigid faces but hinged at the edges, can be flexible. However, examples of closed non-convex flexible polyhedra are known. Discuss these results.

2.6 Dimension theory

How can the *dimension* of a (compact, metric) space be defined in such a way that the n-cell I^n has dimension n? How does this notion of topological dimension relate to 'fractal dimension' as popularised by Mandelbrot?

2.7 Transversality

'In general' a curve and a surface in 3-space will intersect in a discrete set of points. Investigate how these "in general" statements can be made precise, and how they can be used to prove things.

2.8 Classification of complex algebraic surfaces

BE Essay (ambitious) or CD
Prerequisites B3b Algebraic curves, and preferably C3.1a Algebraic geometry

The classification of complex algebraic surfaces is one of the highlights of 20th century algebraic geometry: on the one hand, complete and understandable; on the other hand, a source of inspiration for much further research. The purpose of the essay/dissertation would be to introduce the notions necessary to state the result in full (canonical class, Kodaira dimension, ruled surfaces, blowups), and then perhaps study one of the classes in a little more detail (such rational, abelian, K3 or general type surfaces).

References:

Arnaud Beauville, Complex algebraic surfaces, CUP

Miles Reid, Chapters on algebraic surfaces,

`http://arxiv.org/abs/alg-geom/9602006`

2.9 Thom's Cobordism Theorem

CD or CD/2

One of the most studied objects in mathematics are manifolds, and one of the central problems in geometry and topology is to classify compact smooth manifolds. Though in dimensions zero, one and two this has been achieved and the recent proof of the Poincare conjecture gives us an answer in dimension three, this is still a wide open problem in higher dimensions.

However, in the mid 50s Thom was able to completely describe compact smooth manifolds in any dimension up to the cobordism relation under which we identify two manifolds M_0, M_1 of the same dimension that form the boundary of a manifold W of dimension one higher: $\partial W = M_0 \sqcup M_1$.

The aim of the project is to describe the proof of Thom's theorem. This will require some preliminary work on vector bundles and homotopy groups as well as transversality theory. Possible extensions of the project could include the computation of the rational cobordism ring for oriented manifolds and a treatment of the L-genus and the signature theorem. This would require the study of the Thom isomorphism theorem and some theory of characteristic classes.

Main reference:

J.W. Milnor, J.D. Stasheff, Characteristic classes, Annals of Mathematics Studies, Study 76, PUP, 1974

Further references:

R. Bott, L.W. Tu, Differential Forms in Algebraic Topology, Graduate Texts in Mathematics 82, Springer, 1982

M.W. Hirsch, Differential Topology, Graduate Texts in Mathematics 33, Springer, 1976

R. Stong, Notes on Cobordism Theory, PUP, Princeton, 1968

2.10 Classification of unoriented (1+1)-dimensional topological field theories

Mathematical models for quantum field theories have been a source of new mathematical structures and the inspiration for many powerful invariants of smooth manifolds. Topological field theories, where the partition function of the theory is invariant of the metric on spacetime, have been a particularly fruitful area for investigation. A central question has been understanding the classification of topological field theories and the resulting invariants of manifolds.

The aim of this project is to describe the classification of unoriented $(1+1)$-dimensional topological field theories. The classification of oriented $(1+1)$-dimensional topological field theories in terms of Frobenius algebras is due originally to Dijkgraaf, though it remained a folk theorem until Abrams published a proof. The project will require first understanding the categorical notions involved in the definition of topological field theories, and then learning the classification of oriented (1+1)-dimensional field theories. Next would begin a corresponding investigation of field theories for unoriented manifolds. A possible extension of the project would be to compare the resulting classification of unoriented $(1+1)$-dimensional theories with the classification of unoriented local 2-dimensional field theories.

References

1. J. Kock, Frobenius Algebras and 2D Topological Quantum Field Theories.
2. L Abrams, Two-dimensional topological quantum field theories and Frobenius algebras.
3. M. Atiyah, Topological quantum field theories.

4. S. MacLane, Categories for the Working Mathematician.
5. J. Lurie, On the classification of topological field theories.

2.11 Mirror symmetry and algebraic geometry

BE Essay (ambitious) or CD
Prerequisites B3b Algebraic curves, and preferably C3.1a Algebraic geometry

The phenomenon of mirror symmetry, in its many manifestations and ramifications, remains one of the most high profile connections between theoretical physics (specifically superstring theory) and pure mathematics (algebraic geometry). The essay/dissertation would introduce some aspect of this relationship: there are options ranging from exploring the computation of degree d rational curves on the quintic threefold by Candelas et al (for the computationally minded) through an introduction to the toric technology (for the combinatorially minded) all the way to a study of homological mirror symmetry (for the categorically minded).

References:

David Cox and Sheldon Katz: Mirror symmetry and algebraic geometry, AMS

Sheldon Katz: Enumerative Geometry and String Theory, AMS

2.12 Platonic solids in four dimensional space

It is well known that there are exactly five Platonic solids. Present a proof of this result. Then analyse the existence of regular “solids” in four-dimensional space.

2.13 Topological field theories with boundaries

Topological field theories in dimension $1+1$ for circles (= closed strings) are well understood. Indeed, any such theory determines a finite-dimensional vector space with a multiplication and inner product. Investigate topological field theories with boundaries, that is when the objects are not just circles but also intervals (= open strings). Though this project is motivated by ideas from mathematical physics, the methods are purely algebraic and combinatorial.

2.14 Pits, passes and peaks

The idea of a minimum, a saddle point or a maximum of a smooth (real-valued) function of real variables can be extended to smooth functions defined, for example, on a compact surface. For a suitable smooth function defined on a surface S, the numbers of minima, saddle points and maxima can be related to the topology of the surface. (This is touched on briefly at the end of Section 11 in the Mathematical Institute lecture notes *Geometry of Surfaces* by Graeme Segal.) The same theme has far-reaching extensions, forming the subject called Morse theory.

2.15 Number systems and vector fields

Multiplication by i in the complex numbers gives rise to a 'vector field' on the unit circle S^1 in $\mathbb{C}$: from a point z in S^1 we can draw the unit vector parallel to the vector from 0 to iz. This generalizes to higher dimensions in several ways, for example involving the quaternion number system. The connection allows information to flow both ways between algebra and topology.

2.16 Topology and partial differential equations

Let U be an open subset of $\mathbb{R}^3$. One knows that every irrotational vector field on U is the gradient of a potential, *provided that U is simply connected*: if the simple connectivity is omitted, the statement becomes false. Investigate the generalizations of this fact to higher-dimensional manifolds, and use them to demonstrate some results in topology (e.g. Brouwer's theorem that any map from the n-cell to itself must have a fixed point).

2.17 Infinite groups as geometric objects

Any finitely generated group can be made into a metric space in a way that is unique up to 'large scale equivalence'. Thus, geometric methods may be applied in group theory. Find out about this. You might want to refer particularly to the notions of *ends* of groups, or of *word-hyperbolicity.*

3 Algebra and Number Theory

3.1 Converses of Lagrange's Theorem

Lagrange's Theorem states that if G is a finite group and H is a subgroup then $|H|$ divides $|G|$. In general, if m is a divisor of $|G|$ there need not be a subgroup of order m. Investigate those finite groups G with the property that for every divisor m of their order there exists at least one subgroup of order m.

3.2 Simple groups of odd order

The Feit–Thompson Theorem published in 1963 tells us that the only finite simple groups of odd order are the cyclic groups Z_p for odd prime numbers p. The proof is very long and sophisticated. Nevertheless, with quite elementary methods considerable progress can be made towards proving the theorem for small odd numbers, or for odd numbers with special arithmetic properties.

3.3 Alhazen's Problem

Alhazen's Problem asks for the point P on a given spherical mirror at which a ray of light is reflected from a source at A to an observer at B. See John D. Smith 'The remarkable Ibn al-Haytham', *Math. Gazette,* 76 (1992), 189–198 for a good account of the problem. It has been proved [Peter M. Neumann, 'Reflections on reflection in a spherical mirror', *Amer. Math. Monthly*, 105 (1998)] that there is in general no ruler-and-compass construction for P. Also, Michael Drexler & Martin J. Gander, in their paper 'Circular Billiard' in *SIAM Review* 1999, investigate for which configurations there are four reflection points, for which only two. Several further questions of a similar nature suggest themselves, and should be of a suitable standard for an undergraduate project.

3.4 Reduction of quadratic forms

One of the very useful theorems of algebra states that any real quadratic form $\sum a_{i,j}x_ix_j$ can be changed to the form $y_1^2+\cdots+y_p^2-y_{p+1}^2-\cdots-y_{p+q}^2$ by suitable non-singular linear change of variables, and that moreover the numbers p, q are independent of the particular method used ('Sylvester's Law of Inertia'). This theorem depends heavily on the fact that the coefficients come from $\mathbb{R}$. How far can analogous normal form theorems be proved for quadratic forms over $\mathbb{C}$, over $\mathbb{Q}$, or over other fields F, such as finite fields?

3.5 Costing algorithms in algebra and number theory

Much of algebra and number theory is constructive insofar as we have good algorithms. Thus, for example, the Euclidean Algorithm can be used to find highest common factors; echelonisation can be used to solve linear equations or find determinants and inverses of square matrices; completing the square can be used to reduce a real quadratic form to its normal form. How costly are such computations? That is, how many field operations do they require?

3.6 Euclidean and Principal Ideal Domains

We know that every euclidean ring is a principal ideal domain. Investigate principal ideal domains that are not euclidean.

3.7 Primality testing

Fermat's Little Theorem states that if p is prime then $a^{(p-1)} \equiv 1 \pmod p$ whenever $1 \leq a < p$. The converse is not true in general. Nevertheless, the theorem is used for testing large numbers for primality. How is this done?

3.8 Approximation of Irrational numbers by Rationals

BE Extended Essay

If α is a real irrational number then one can prove that there are infinitely many rational numbers a/q such that

$$\left|\alpha - \frac{a}{q}\right| \leq \frac{3}{q^2}.$$

A number of questions naturally arise. Can one replace the constant 3 by anything smaller? Can one replace q^2 by q^3? What happens for specific irrationals like $\sqrt{2}$ or π?

The essay will discuss such issues. It will give an exposition of the theorems of Liouville, Thue, and Apéry, and describe applications to transcendence problems and Diophantine equations. If time permits a complete proof of the finiteness of integer solutions of the Mordell equation $y^3 = x^2 + k$ will be given. The latter will use results form the section B 'Algebraic Number Theory course".

References

G.H. Hardy and E.M. Wright, An introduction to the theory of numbers, Sixth edition. (Oxford University Press, Oxford, 2008). [Chapter XI in particular]

L.J. Mordell, Diophantine equations, Pure and Applied Mathematics, Vol. 30 (Academic Press, London-New York 1969).

A. van der Poorten, A proof that Euler missed ... Apéry's proof of the irrationality of $\zeta(3)$, Math. Intelligencer 1 (1978/79), no. 4, 195–203.

3.9 Unique Factorization in Quadratic Number Fields

CD/2 or CD Dissertation

This dissertation will be an expository account of unique factorization and the Euclidean property for quadratic number fields. The number of complex quadratic field with unique factrization is known to be finite, as is the number of real or imaginary fields for which the norm function is Euclidean. In each case the precise list of fields has been determined. The question of which real quadratic fields have unique factorization is still open, as is the question of Euclidean functions other than the usual norm.

All these problems will be discussed, along with some of the methods used to handle them. Some of these will be relatively elementary, but there are also methods from the geometry of numbers, analytic number theory and transcendental number theory.

References

G.H. Hardy and E.M. Wright, *An introduction to the theory of numbers, Sixth edition*, (Oxford University Press, Oxford, 2008). [Chapter XIV in particular]

D. Goldfeld, Gauss's class number problem for imaginary quadratic fields, *Bull. Amer. Math. Soc. (N.S.)* 13 (1985), no. 1, 23–37.

P. Samuel, About Euclidean ring, *J. Algebra*, 19 (1971), 282–301.

M. Harper, $\mathbb{Z}[\sqrt{14}]$ is Euclidean, *Canad. J. Math.*, 56 (2004), no. 1, 55–70.

H. Cohen and H. Lenstra, Heuristics on class groups of number fields, *Number theory, Noordwijkerhout 1983 (Noordwijkerhout, 1983)*, 33–62, Lecture Notes in Math., 1068, (Springer, Berlin, 1984).

3.10 Waring's problem

CD/2 or CD Dissertation

Waring stated in 1770 that every positive integer was a sum of at most 4 squares, 9 cubes, 19 fourth powers, and so on. Much work has been done which has eventually given an almost complete justification for this claim. However if one asks how many cubes, say, are required to represent all sufficiently large integers, the problem becomes much harder.

The project will look at elementary approaches to the problem, and then give an exposition of Hilbert's existence argument for solutions. Subsequent work uses the Hardy-Littlewood method, which is far more analytic in flavour. The dissertation will look at the principles behind this beautiful approach, and explore some of the more recent work using this method. If time permits the project will look at the numerical work needed to verify Waring's claim for the large but finite number of cases not covered by other methods.

References

G.H. Hardy and E.M. Wright, *An introduction to the theory of numbers, Sixth edition*, (Oxford University Press, Oxford, 2008). [Chapter XXI in particular]

M.B. Nathanson, *Additive number theory. The classical bases*, Graduate Texts in Mathematics, 164. (Springer-Verlag, New York, 1996).

R.C. Vaughan, *The Hardy-Littlewood method*, Second edition. Cambridge Tracts in Mathematics, 125. (Cambridge University Press, Cambridge, 1997).

3.11 Modular functions and modular forms

BE, CD or CD/2

Let $\mathbb{H} = \{z \in \mathbb{C} : \text{Im}z > 0\}$ be the complex upper half plane. The special linear group $\text{SL}_2(\mathbb{R})$ of 2×2 matrices of determinant 1 acts on $\mathbb{H}$ by fractional linear transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \mathbb{H} \to \mathbb{H}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}.$$

Modular functions are holomorphic functions $f : \mathbb{H} \to \mathbb{C}$ that are invariant under $\text{SL}_2(\mathbb{Z})$, i.e. $f(\gamma z) = f(z) \forall \gamma \in \text{SL}_2(\mathbb{Z})$, and satisfy a growth condition, while modular functions satisfy simple transformation formulae under $\text{SL}_2(\mathbb{Z})$. These functions thus generalise periodic functions on the real line. They have remarkable properties. Some key properties were discovered by Ramanujan and verified later (sometimes much later) by others. Even more remarkable properties – such as connections to Fermat's Last Theorem – were discovered more recently.

G. H. Hardy, *Ramanujan: Twelve lectures on his life and work,* CUP, 1940, reprinted Chelsea Publishing Company, 1978.

J. H. Brunier, G. van der Geer, G. Harder, D. Zagier, *The 1-2-3 of modular forms,* Springer-Verlag, 2008.

F. Diamond and J. Shurman, *A first course in modular forms,* Graduate Texts in Mathematics **228**, Springer-Verlag, 2005.

4 Analysis

4.1 Abelian locally compact groups

Part C half unit or whole unit project.

Abstract: The Fourier transform defined on the Hilbert space of square-integrable functions on the real-line can be readily extended to all locally compact abelian groups by the introduction of the dual group. An essay could look at what locally compact groups are, what the Haar measure is, how to define the Fourier transform, and what Pontryagin duality is. Ideas from functional analysis, group theory and measure (integration) theory are all involved.

4.2 Kazhdan's property (T)

Part C half unit or whole unit project.

Kazhdan's property (T) was introduced in a short paper, defined in terms of the Fell topology on the space of representations of a group. However, we can state the property, informally, in an easy way: a group has property (T) if whenever, given a (unitary) representation, there is a unit vector which is only perturbed slightly by the group action, then there is actually a unit vector fixed by the group action. Such groups hence have very "rigid" actions. Recently property (T) has found its way into a diverse collection of applications, from number theory, measure theory, and operator algebras, to graph theory and computer science. An essay could either explore some of these applications: for example, the applications to graph theory and computer science can be stated using a minimal amount of representation theory. Alternatively, one could look at the theory of representing topological groups, and how classical representation theory can be used to show that some concrete groups have property (T). A longer essay could touch upon both.

4.3 Analysis in a rational world

The basic concepts of real analysis make perfectly good sense for functions $f : \mathbb{Q} \to \mathbb{Q}$ and in a world where only rational numbers are contemplated. Thus, for example, continuity and differentiability are defined in exactly the same way as for functions $f : \mathbb{R} \to \mathbb{R}$ (but note that $f'(x)$ has to be rational for all $x \in \mathbb{Q}$). In this rational world we find that basic theorems such as the Intermediate Value Theorem, Rolle's Theorem, and the Mean Value Theorem fail. How badly do they fail? What can be rescued? What can one say about solutions to differential equations such as $f' = f$, or $f'' = f$?

4.4 Analysis of holomorphic functions with special values

We denote by $\mathbb{Q}[i]$ the set of complex numbers whose real and imaginary parts are rational. Investigate holomorphic functions $f : \mathbb{C} \to \mathbb{C}$ with the property that $z \in \mathbb{Q}[i] \Rightarrow f(z) \in \mathbb{Q}[i]$.

4.5 Integration

Compare and contrast Lebesgue integration and Riemann-Stieltjes integration. [Caution: the material for such a project should cite but not repeat that given in the second-year Integration lectures.]

4.6 Measure Theory

The development of abstract measure theory yields rich rewards with a variety of generalisations and applications, for example, fractals, stochastic integration. References include: Robert Strichartz *The Way of Analysis* (Jones and Bartlett Publishers, 2000); H. L. Royden *Real Analysis* (MacMillan, 1963); Kenneth Falconer *Fractal Geometry* (Wiley 1990). [Caution: the material for such a project may cite but should not repeat that given in the third-year lecture course 'Martingales Through Measure Theory'.]

4.7 Univalent functions

A univalent map is a one-to-one conformal map $z = f(\zeta)$ from the unit disc to a domain Ω in the complex plane. It is usual to normalize f by assuming that $f(0) = 0$ and $f'(0) = 1$. Among many interesting results in the field are the Koebe $\frac{1}{4}$-Theorem (the distance from 0 to the boundary of Ω is never less than $\frac{1}{4}$, with equality for $f(\zeta) = \zeta/(1-\zeta)^2$) and the celebrated Bieberbach conjecture (proved in 1985 by De Branges) that if f is univalent and has the Taylor series $f(\zeta) = \zeta + \sum_2^\infty a_n \zeta^n$, then $|a_n| \leq n$, with equality for the same map. Proofs of the conjecture for small values of n are not hard.

4.8 Distributions

Investigate the theory of distributions and their applications. Probably the most immediate starting point is the delta function (*cf* point masses, sources, or charges) and its relation to the derivative of a function with a jump discontinuity, but a proper theoretical development is not hard to set out. Further topics include the relation with Fourier transforms.

4.9 Special functions

There are many possible project topics here, for example: Bessel functions and their applications, hypergeometric functions (and the connection with complex differential equations and conformal maps), the Riemann zeta function, the Gamma function.

4.10 Riemann surfaces

Explore the idea of the Riemann surface associated with a multi-valued conformal map.

4.11 The Schwarzian derivative

The Schwarzian derivative has many interesting properties. Find out about them.

4.12 Chaos in nonlinear ordinary differential equations

Explore the connection between the various kinds of homoclinic bifurcations and the onset of chaos in ordinary differential equations.

Investigate the occurrence of stochasticity in Hamiltonian systems, as an integrable system is perturbed more and more strongly.

The period doubling sequence for unimodal (one-humped) maps is well known. What happens for other maps, e.g. cubics?

5 Numerical Analysis

There are many topics that could be addressed as a follow-up to the Part A Numerical Analysis course or in parallel with the third-year courses in this area. Here is a small sample of possible projects; the lecturer may be able to suggest some others.

5.1 Stiff ordinary differential equations

Consider an initial value problem for the differential equation $\epsilon y' = f(x,y)$, or a system of differential equations of the form $\epsilon \mathbf{y}' = \mathbf{f}(x,\mathbf{y})$, where $0 < \epsilon \ll 1$ is a small parameter. An interesting and practically relevant question concerns the construction and analysis of numerical methods for the accurate solution of such problems. How do standard one-step and linear multi-step methods behave when ϵ is very small? How would you improve the performance of these methods by adapting the computational mesh? Is it possible to design special methods which provide accurate approximations for such *stiff* initial value problems?

5.2 Numerical approximation of singular integrals

You are probably familiar with simple numerical integration rules such as the trapezium rule or Simpson's rule; but how would you evaluate numerically the integral $\int_0^1 \frac{e^{x^2}}{x^{1/2}(1-x)^{1/3}}\mathrm{d}x$ or $\int_0^\infty \frac{\sin x^5}{x^2(1+x)^{55}}$? There are special techniques for the numerical approximation of such integrals. What are they and what can be said about their accuracy?

5.3 Newton's method for nonlinear systems

Newton's method is a standard technique for solving a nonlinear equation of the form $f(x) = 0$ where f is a continuously differentiable function. Consider the generalisation of Newton's method for solving the nonlinear system $f(x,y) = 0$, $g(x,y) = 0$. What can be said about the convergence of Newton's method? What is a good choice of starting value and how to locate it? How does the speed of convergence of Newton's method compare with that of a simple fixed-point iteration?

5.4 Finite element methods for singularly perturbed problems

Conventional finite element methods are known to provide poor approximations when applied to singularly perturbed two-point boundary value problems of the form $-\epsilon u'' + u' = 0$, $u(0) = 0$, $u(1) = 1$: while the analytical solution is a smooth function, its numerical approximation exhibits unacceptable oscillations. Why does this happen? Can you improve matters by using a suitable non-uniform mesh? How should such a mesh be designed? Would generalising the concept of Galerkin finite element method by allowing a trial space that is different from the test space cure the problem?

5.5 A posteriori error analysis of finite element methods

Conventional *a priori* error estimates for finite element approximations of boundary value problems are of limited practical use since they bound the computational error in terms of powers of the mesh size and norms of derivatives of the *unknown* analytical solution. How would you derive an *a posteriori* bound on the error by exploiting a computed solution? Now suppose that you have estimated the size of the error by means of an *a posteriori* bound; how would you adapt the computational mesh to ensure that the error in a given norm does not exceed a given tolerance?

5.6 Fast iterative methods for systems of linear equations

Gaussian elimination is a standard technique for solving a system of linear equations of the form $Ax = b$ where A is a non-singular matrix. Suppose that A is a sparse matrix (i.e. it contains a very large number of zero entries). Is it a good idea to solve such a system by Gaussian elimination, or would it perhaps be better to use an iterative method instead? Consider the performance of various iterative methods (such as the Conjugate Gradient Method and its relatives) for the solution of sparse systems. How would you accelerate the convergence of an iterative method by *preconditioning* A, i.e. by pre-multiplying the system by a non-singular matrix P such that $P \approx A^{-1}$, and solving $PAx = Pb$ instead? How would you choose P?

6 Applied mathematics

6.1 Mathematics and the environment

Part C half-unit projects.

Surging glaciers.
Observations and theories concerning glacier surges.

Waves on rivers
Formation of waves on rivers (roll waves, tidal bores).

Formation of aeolian and fluvial bedforms such as dunes.

6.2 Mathematical physiology

- Respiratory control modelling.
- Modelling of cardiac arrhythmias.
- Mathematical models of the heart.
- Mathematical models of the respiratory control system.

6.3 Mathematical models in finance

Possible topics might include a rigorous discussion of Itô's lemma and its relation to random walks; the rôle of martingales in models of markets; stochastic volatility; time series methods. The lecturer may be able to suggest others.

6.4 Fluid dynamics

There are many topics that could be done in parallel with the course for Paper B6; consult the lecturer for possibilities.

Describe some of the phenomena which occur in rotating flows, and their application in meteorology.

Linear and nonlinear stability theory of Rayleigh–Bénard convection.

Transition to turbulence in shear flows.

6.5 Mechanics of Solids

Write on the linear theory of elastic solids, including for instance some of the following topics: the stress–strain relation in an anisotropic crystalline material and its relation to the symmetries of the material; Saint-Venant torsion of a prismatic beam of isotropic material; wave motion in an isotropic elastic material, including waves in thin rods and thin plates.

6.6 Mathematical biology

The Hodgkin-Huxley equations for signal propagation in nerve membranes and their later mathematical derivatives, e.g. the Fitzhugh-Nagumo equations.

- Reaction-diffusion models and their role in developmental morphogenesis.

- The dynamics of coupled oscillators and their role in development.

- Chemotactic cellular migration: travelling waves.

- The investigation of wave speeds in systems that exhibit travelling waves.

- Investigation of the behaviour of cellular swimmers.

- Modelling chemical transport in biological tissues.

6.7 Mathematical Genealogy and the Evolution of Research Groups

BE Essay or CD Dissertation

The student will analyze mathematics genealogies using real data and various techniques from network analysis. Among other things, this is expected to include the deployment of modified versions of specialized network structures that have been used previously to study family genealogies. The genealogical data to be used includes things such as geographical location of Ph.D.s and limited information on mathematics subject classification, so it can be used to analyze the time evolution of mathematics departments/research groups and it might be possible to use it to examine the evolution of mathematics subfields.

References

Newman, M. E. J. "The Structure and Function of Complex Networks." SIAM Review, 2003, 45(2), 167-256.

Newman, M. E. J., Networks: An Introduction, 2010, Oxford University Press.

Schweizer, T. & White, D. R. (Eds), Kinship, Networks, and Exchange, 1998, Cambridge University Press.

Further information from Mason Porter (`porterm@maths.ox.ac.uk`)

6.8 Nonlinear Schrödinger Equations with Lattice and Superlattice Potentials

CD Dissertation

Although nonlinear Schrödinger (NLS) equations have been extensively studied for several decades, advances in optics and Bose-Einstein condensation have brought new life to their investigation. Of particular interest has been the study of cubic NLS equations in lattice and superlattice potentials, which allow one to investigate the competition between nonlinearity and spatial inhomogeneity (and the localized wave structures that can result from it). The former are spatially periodic, and the latter are either spatially periodic or quasiperiodic. Closely related problems involve the consideration of spatially periodic and quasiperodic coefficients of the nonlinearity. In this project, the student will study such systems, with an eye towards examining solitary waves with multiple humps and 'snaking' bifurcation phenomena that relate waves with different numbers of pulses to each other.

References

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Beck, M.; Knobloch, J.; Lloyd, D. J. B.; Sandstede, B.; & Wagenknecht, T. "Snakes, Ladders, and Isolas of Localised Patterns." SIAM Journal of Mathematical Analysis, 41(3), 936-972.

Porter, M. A. & Kevrekidis, P. G. "Bose-Einstein Condensates in Superlattices." SIAM Journal on Applied Dynamical Systems, 2005, 4(4), 783-807.

Porter, M. A.; Kevrekidis, P. G.; Malomed, B. A.; & Frantzeskakis, D. J. "Modulated Amplitude Waves in Collisionally Inhomogeneous Bose-Einstein Condensates." Physica D, 2007, 229(1), 104-115.

Zhang, Y. & Wu, B. "Composition Relation between Gap Solitons and Bloch Waves in Nonlinear Periodic Systems." Physical Review Letters, 2009, 102, 093905.

Further information from Mason Porter (`porterm@maths.ox.ac.uk`)

6.9 Community Structure in Multislice Networks

CD Dissertation

One of the most popular and important areas of network science is the study of the "community structure" of a network. A network, usually modelled as a graph, consists of a set of entities and the connections between them. Communities are important because they are thought to have a strong bearing on functional units in many networks. In this project, the student will study community structure in multislice networks, which can be multiplex (consisting of multiple types of connections), time-dependent, multiscale, and multiplex. In this project, the student will apply recent theoretical advances in Ref. [2] to interesting data sets. Depending on the data set, this project could also entail building on the theoretical work of Ref. [2].

References

Fortunato, S. "Community Detection in Graphs" Physics Reports, 2010, 486, 75-174.

Mucha, P. J.; Richardson, T.; Macon, K.; Porter, M. A.; & Onnela, J.-P. "Community Structure in Time-Dependent, Multiscale, and Multiplex Networks." arXiv:0911.1824.

Newman, M. E. J. "The Structure and Function of Complex Networks." SIAM Review, 2003, 45(2), 167-256.

Newman, M. E. J., Networks: An Introduction, 2010, Oxford University Press.

Porter, M. A.; Onnela, J.-P.; & Mucha, P. J. “Communities in Networks” Notices of the American Mathematical Society, 2009, 56(9), 1082-1097, 1164-1166.

Further information from Mason Porter (`porterm@maths.ox.ac.uk`)

Dr Porter is also happy to supervise other projects (not specifically listed) in networks, complex systems, or nonlinear dynamics. Contact him to discuss details

7 Theoretical physics

7.1 Extremum principles in theoretical physics

For example: (a) Fermat’s principle; (b) Hamilton’s principle; . . .; (f) variational methods in quantum mechanics; . . .; (r) geodesics in general relativity for a (given) Schwarzschild metric; . . .; (z) path integral formulations of quantum mechanics [rather advanced].

7.2 Hamilton–Jacobi theory

Action-angle variables and the early development of quantum theory.

7.3 Concepts of quantum mechanics

For example: Schrödinger’s cat; Bell’s inequality. Another possibility is a critical summary of the Bohr–Einstein dialogues.

7.4 Symmetries in quantum mechanics

Various groups which occur naturally in quantum mechanics, particle-field systems can be studied in a simple manner.

7.5 Quantum mechanics of scattering

There are many calculational examples which are within reach of the advanced undergraduate. Examples: the Born approximation; simple atoms and molecules (in electric and/or magnetic fields); one- or two-electron systems.

7.6 Waveguides

The undergraduate can start with the excellent exposition on waveguides in the Feynman Lectures, and go a bit further from there.

7.7 The microwave background

A systematic list of the consequences of the microwave background would make a worthwhile and approachable project.

7.8 Tests of general relativity

For example, simple calculations involved in gravitational wave experiments; gravitational lensing effect (quite a lot of interesting geometry there).

7.9 Hot big bang versus steady state in cosmology

This is a topic that can be expanded in various interesting ways, expository and historical.

7.10 Appearance of moving objects in special relativity

In spite of Lorentz contraction, a spherical object does look spherical to an observer. Why?

7.11 Study of a special metric

Even apart from the Schwarzschild metric, general relativity is full of special metrics ("exact solutions") which repay even simplistic studies. In particular, the NUT metrics have intriguing topological properties.

7.12 SO(3), SU(2), Euler angles and angular momentum

Work out the exact 2-to-1 map of SU(2) to SO(3); generalize Euler angles to higher dimensions; angular momentum and spin.

7.13 Momentum space in quantum mechanics

States as functions of momentum. Fourier transform. Plancherel theorem. Time evolution.

7.14 Translation of some well known theorem in euclidean space to Minkowski space

Many interesting problems relating to Euclidean and hyperbolic geometry can be tackled.

7.15 Crystal symmetry, non-periodic tiling and quasicrystals

Write an essay to accompany the display in the foyer of the Mathematical Institute!

8 Probability and Statistics

8.1 Mathematical models in evolution

Modern methods for understanding genetic data, and using this to learn about the processes of evolution, rely heavily on mathematical models. These models usually involve probability, to capture the various sources of randomness in genetics processes. Study these models and their uses to untangle evolutionary questions, such as how different species are related, or what we can learn about early human evolution from genetics data.

8.2 Duality and Random Walks

Duality, the relation between a set of paths and the reversed set is a classical tool in the study of random walks. One project would be to review some problems in which this technique has been successfully applied, notably in the queuing literature.

For the very able student there is the possibility of novel work on the study and classification of dual times for walks in $\mathbb{R}^n$ or in the understanding of duality relations that have arisen from recent work on stopping rules for Markov chains.

8.3 The Coupling Method

Loosely, coupling refers to the study of one or two marginal probability distributions by way of the construction of an appropriate joint distribution. The success of this method lies in its ability to convert difficult analytic problems into ones of probabilistic construction. There are many areas in applied probability in which to study how this most elegant method is used, including Poisson Approximation, Random Walks and Markov Chains.

8.4 Applied Probability

Queues are often used to model communication and manufacturing systems: a topic with manufacturing applications would be to discuss the stability and behaviour of networks of queues when the arrivals at a network almost saturate its service capacity; a topic with applications in the telecommunications context would be to discuss rare events in large systems.

8.5 Operational Research

Gather appropriate data from a filling station or a supermarket, and use it, with the help of an appropriate simulation software package, to investigate the characteristics of the queues which would form under different possible service regimes.

9 History of Mathematics

It is difficult to offer specific projects in the history of mathematics because the possibilities are so varied and the choice will depend very much on each student's personal inclination and skills. Those who have taken O1 as a third-year option will already have a good grounding in the history behind the present day mathematics curriculum and may choose to go more deeply into a particular topic, person, or debate. Others may wish to work on a more general theme. There is also plenty of untranslated source material and those with some Latin, French, or German might like to undertake a translation and commentary; there is no better way to enter into the mind of a first rank mathematician, and Euler, Lagrange, and Cauchy, for example, all offer material that is both accessible and engaging.

To give some idea of the range and style of projects that are possible, here are examples of some topics that have been the subject of some recent OE essays or OD dissertations:

Mathematics and World War II

Early modern popularisation of mathematics

Robert Recorde's presentation of Euclidean geometry

Hilbert's seventh and eighth problems

The lives and work of Emilie du Chatelet and Sophie Germain

The life and work of Edmund Halley

Arthur Cayley's contribution to group theory

A translation (from Latin) on summation of series by Euler

A translation (from Italian) of Bombelli on complex numbers

A comparison of the contributions to analysis of Cauchy and Bolzano

Fibonacci and the introduction of Hindu-Arabic numerals to Europe

A translation (from French) of Galois' work on finite fields

Anyone interested in working on a historical subject is encouraged to come and discuss ideas with Jackie Stedall (`jackie.stedall@queens.ox.ac.uk`) before the end of Trinity Term.

10 General

A host of tractable and interesting problems are to be found in journals such as

- *The American Mathematical Monthly*,
- *The Mathematical Gazette*,
- *The Mathematical Intelligencer*,
- *SIAM Review*.

Many of these have a past, and ramifications, the tracing of which would provide a project—including, perhaps, the solution of the problem, though this would not be essential.

11 Titles of Previous Projects

Listed below are the titles of some projects undertaken by students in recent years.

11.1 Extended Essays

Neural Models for Pattern Formation

Nowhere Differentiable Continuous Functions

A Study of the Riemann Hypothesis and the Prime Number Theorem

An Introduction to the Riemann–Stieltjes Integral and a Comparison with the Lebesgue Integral

Applied Probability: Some Aspects of Queueing Theory

Definable Subset of Finite Structures

Fast Iterative Methods for Systems of Linear Equations—the Conjugate Gradient Method

Finite Difference Methods Applied to Singularity Perturbed Differential Equations

Fractals—Theory and Application

Fundamental Principles of Astrodynamics with Applications to the Apollo II Mission to the Moon

Periodic and Non-periodic Tiling

Personal Probabilities and their Reappraisal in Light of New Information

Systems with Limit Cycle Behaviour

The Application of Symmetry in the Development of Crystallography

The Mathematics of Gambling

The Science of Music, a Mathematical Approach

The Three Body Problem and Orbit Switching

The Three Classical Problems of Ancient Greece

11.2 Dissertations

The Appearance of Relativistically Moving Spheres

Turbulent boundary Layers and Drag Reduction

2,500 Years of Ruler and Compasses: a Historical Study of Geometric Construction

Evolutionary Pattern Formation

Climatic Energy Balance Models

Toric Interpretations of Counterexamples to the Motzkin Conjecture Regarding Simplicial Polytopes

Is Cantor's Conception of Set Present in ZF Set Theory

Reaction-Diffusion Models for Dorsoventral Patterning in Prosophila

The Solomon Descent Algebra

An Investigation into the contributions of Bernhard Bolzano and Augustin-Lovis Cauchy to the development of real analysis during the early 19th century.

Hair Modelling

A Review of Some Mathematical Models for Chemotactic Pattern Formation

A Study of Methods for Forecasting Age-specific Mortality and Fertility Rates

Characterisation of Character's Theorem and its Applications

Mathematical Models of Dermal Wound Healing

Number Theory and Cryptography

The Theory of Flight

Set Theories: a Comparative Study with Particular Reference to the Axiom of Choice and the Urelemente

Efficient Deterministic Factorisation of Polynomials Over Finite Fields

A Discussion of the Use of Cavalieris Method in John Wallis "De Sectionibus Conicis"

A Study of Extreme Value Theory with Hydrological Applications

A Study of Small World Networks in One and Two Dimensions

A Theorem of Philip Hall about Soluble Groups

Algebraic Curves and their Singularities

An Account of the Application of the Generalised Riemann Hypothesis to Primality Testing

An Efficient Presentation for the Covering Group of A_8

Analogues of Perfect Numbers in Quadratic Number Fields

Analysis of Polynomial Time Primality Tests

Calculation of the Character Tables of Some Finite Simple Groups

Chain Conditions for Rings and Modules

Coalescent Theory and its Applications to Evolutionary Genetics

Diffusions on the Fractal Tree

Fluvial Dunes

Geometric Group Theory

Mathematical Modelling and the Respiratory Control System

Models of Security in Modern Cryptography and Hyper-Encryption

On Linear Representations and Characters of Finite Groups

Ruin Theory

Simple Groups of Odd Order

Singularities of Complex Curves and their Resolutions

Symbolic Integration

The Axiom of Choice and the Prime Ideal Theorem

The Ends of Finitely Generated Groups

The Sine-Gordon Equation; a Completely Integrable System

The Use of Kullbeck–Leibler Divergence as a Measure of Information Gain Between Prior and Posterior Distributions

Translation of and Commentary on John Wallis: 'Demonstatio Postulati Quinti Euclidis; D de Postulato Quinto et Definitione Quinta Lib. 6 Euclidis; Disceptatio Geometrica', *Opera Mathematica*, Vol II (1963) pp. 674–678