Characteristic IBVP's and Magneto-Hydrodynamics

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18, 19 March, 2024, Oxford UK

Plan

Euler equations

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0 \,, \\ \partial_t(\rho v) + \operatorname{div}(\rho \, v \otimes v) + \nabla p = 0 \,, \\ \partial_t(\rho E) + \operatorname{div}((\rho E + p) v) = 0 \,, \end{cases}$$

ρ density, $v \in \mathbb{R}^3$ velocity field, $p = p(\rho, S)$ pressure, S entropy,
$E = e + \frac{1}{2}|v|^2$ total energy, $e = e(\rho, S)$ internal energy.

Euler equations

Symmetrizable in terms of (p, v, S) under the *hyperbolicity* conditions

$$\boxed{\rho > 0\,, \quad \rho_p > 0\,,}$$

$$\begin{cases} \frac{\rho_p}{\rho}(\partial_t p + v\cdot\nabla p) + \nabla\cdot v = 0, \\ \rho\{\partial_t v + (v\cdot\nabla)v\} + \nabla p = 0, \\ \partial_t S + v\cdot\nabla S = 0, \end{cases}$$

that is

$$\begin{pmatrix} (\rho_p/\rho)(\partial_t + v\cdot\nabla) & \nabla\cdot & 0 \\ \nabla & \rho(\partial_t + v\cdot\nabla)I_3 & \underline{0} \\ 0 & \underline{0}^T & \partial_t + v\cdot\nabla \end{pmatrix} \begin{pmatrix} p \\ v \\ S \end{pmatrix} = 0.$$

If $d = 3$ the size of the matrices is $N = 5$.

Euler equations

Boundary matrix:

$$A_\nu = \begin{pmatrix} (\rho_p/\rho) v\cdot\nu & \nu^T & 0 \\ \nu & \rho v\cdot\nu I_3 & \underline{0} \\ 0 & \underline{0}^T & v\cdot\nu \end{pmatrix}.$$

If $v\cdot\nu = 0$, then

$$rank\, A_\nu = 2 \qquad \text{(characteristic bdry)},$$

$$p = 1 \quad \text{(one b.c.)},$$

$$ker\, A_\nu = \{U' = (p', v', S') \,:\, p' = 0,\, v'\cdot\nu = 0\} \\ \subset ker\, M = \{U' \,:\, v'\cdot\nu = 0\} \quad \text{(reflexivity)},$$

Projection onto $(ker\, A_\nu)^\perp$:

$$P = \begin{pmatrix} 1 & \underline{0}^T & 0 \\ \underline{0} & \nu\otimes\nu & \underline{0} \\ 0 & \underline{0}^T & 0 \end{pmatrix}.$$

P has the regularity of ν.

Ideal Compressible Magneto-hydrodynamics

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho v) = 0\,, \\ \partial_t(\rho v) + \mathrm{div}(\rho\, v \otimes v - H \otimes H) + \nabla q = 0\,, \\ \partial_t H - \nabla \times (v \times H) = 0\,, \\ \partial_t(\rho E + \frac{1}{2}|H|^2) + \mathrm{div}((\rho E + p)v + H \times (v \times H)) = 0\,, \end{cases} \qquad (1)$$

ρ density, $v \in \mathbb{R}^3$ plasma velocity, $H \in \mathbb{R}^3$ magnetic field, $p = p(\rho, S)$ pressure, $q = p + \frac{1}{2}|H|^2$ total pressure, S entropy, $E = e + \frac{1}{2}|v|^2$ total energy, $e = e(\rho, S)$ internal energy.

Under a state equation $\rho = \rho(p, S)$ and the 1st principle of thermodynamics, (1) becomes a closed system for the unknown $\mathbf{U} = (p, v, H, S)$.

(1) is supplemented by the divergence constraint on the initial data

$$\mathrm{div}\, H = 0\,. \qquad (2)$$

Symmetrizable in terms of (p, v, H, S) under the *hyperbolicity* conditions

$$\boxed{\rho > 0\,, \quad \rho_p > 0\,,}$$

$$\begin{cases} \rho_p(\partial_t + v\cdot\nabla)p + \rho\,\mathrm{div}\,v = 0, \\ \rho\{\partial_t v + (v\cdot\nabla)v\} + \nabla p + H\times(\nabla\times H) = 0, \\ \partial_t H + (v\cdot\nabla)H - (H\cdot\nabla)v + H\,\mathrm{div}\,v = 0, \\ \partial_t S + v\cdot\nabla S = 0, \end{cases}$$

that is

$$\begin{pmatrix} \rho_p/\rho & \underline{0}^T & \underline{0}^T & 0 \\ \underline{0} & \rho I_3 & 0_3 & \underline{0} \\ \underline{0} & 0_3 & I_3 & \underline{0} \\ 0 & \underline{0}^T & \underline{0}^T & 1 \end{pmatrix} \partial_t \begin{pmatrix} p \\ v \\ H \\ S \end{pmatrix} +$$

$$\begin{pmatrix} (\rho_p/\rho)v\cdot\nabla & \nabla\cdot & \underline{0}^T & 0 \\ \nabla & \rho v\cdot\nabla I_3 & \nabla(\cdot)\cdot H - H\cdot\nabla I_3 & \underline{0} \\ \underline{0} & H\nabla\cdot - H\cdot\nabla I_3 & v\cdot\nabla I_3 & \underline{0} \\ 0 & \underline{0}^T & \underline{0}^T & v\cdot\nabla \end{pmatrix} \begin{pmatrix} p \\ v \\ H \\ S \end{pmatrix} = 0\,.$$

A different symmetrization with the total pressure $q = p + |H|^2/2$ instead of p:

$$\begin{cases} \frac{\rho_p}{\rho}\Big((\partial_t + v\cdot\nabla)q - H\cdot(\partial_t + (v\cdot\nabla))H\Big) + \nabla\cdot v = 0, \\ \rho(\partial_t + (v\cdot\nabla))v + \nabla q - (H\cdot\nabla)H = 0, \\ (\partial_t + (v\cdot\nabla))H - (H\cdot\nabla)v - \\ \qquad\qquad -\frac{\rho_p}{\rho}H\Big((\partial_t + v\cdot\nabla)q - H\cdot(\partial_t + (v\cdot\nabla))H\Big) = 0, \\ \partial_t S + v\cdot\nabla S = 0, \end{cases}$$

that we rewrite as

$$\begin{pmatrix} \rho_p/\rho & \underline{0}^T & -(\rho_p/\rho)H^T & 0 \\ \underline{0} & \rho I_3 & 0_3 & \underline{0} \\ -(\rho_p/\rho)H & 0_3 & a_0 & \underline{0} \\ 0 & \underline{0}^T & \underline{0}^T & 1 \end{pmatrix} \partial_t \begin{pmatrix} q \\ v \\ H \\ S \end{pmatrix} + \begin{pmatrix} (\rho_p/\rho)v\cdot\nabla & \nabla\cdot & -(\rho_p/\rho)H^T v\cdot\nabla & 0 \\ \nabla & \rho v\cdot\nabla I_3 & -H\cdot\nabla I_3 & \underline{0} \\ -(\rho_p/\rho)Hv\cdot\nabla & -H\cdot\nabla I_3 & a_0 v\cdot\nabla & \underline{0} \\ 0 & \underline{0}^T & \underline{0}^T & v\cdot\nabla \end{pmatrix} \begin{pmatrix} q \\ v \\ H \\ S \end{pmatrix} = 0 \tag{3}$$

where $a_0 = I_3 + (\rho_p/\rho)H\otimes H$.

If $d = 3$ the size of the matrices is $N = 8$.
Boundary matrix:

$$A_\nu = \begin{pmatrix} (\rho_p/\rho)v\cdot\nu & \nu^T & -(\rho_p/\rho)H^T v\cdot\nu & 0 \\ \nu & \rho v\cdot\nu I_3 & -H\cdot\nu I_3 & \underline{0} \\ -(\rho_p/\rho)Hv\cdot\nu & -H\cdot\nu I_3 & a_0 v\cdot\nu & \underline{0} \\ 0 & \underline{0}^T & \underline{0}^T & v\cdot\nu \end{pmatrix}.$$

- If $v\cdot\nu = 0,\ H\cdot\nu = 0$ (perfectly conducting wall b.c.), then

$$rank\ A_\nu = 2 \quad \text{(characteristic bdry)}, \qquad p = 1 \quad \text{(one b.c.)},$$

$$ker\ A_\nu = \{U' = (q', v', H', S') \,:\, q' = 0,\ v'\cdot\nu = 0\}$$
$$\subset ker\ M = \{U' \,:\, \boxed{v'\cdot\nu = 0}\} \quad \text{(reflexivity)}$$

$(H'\cdot\nu = 0$ restriction on the initial data),

Projection onto $(ker\ A_\nu)^\perp$:

$$P = \begin{pmatrix} 1 & \underline{0}^T & \underline{0}^T & 0 \\ \underline{0} & \nu\otimes\nu & 0_3 & \underline{0} \\ \underline{0} & 0_3 & 0_3 & \underline{0} \\ 0 & \underline{0}^T & \underline{0}^T & 0 \end{pmatrix}.$$

- If $\underline{H \cdot \nu = 0}$ and $\underline{v \cdot \nu \neq 0,\ v \cdot \nu \neq \frac{|H|}{\sqrt{\rho}} \pm c(\rho)}$, then

$$ker\ A_\nu = \{0\}, \quad P = Id.$$

(Non-characteristic boundary)

- If $v \cdot \nu = 0$ and $H \cdot \nu \neq 0$, then

$$\begin{aligned} &ker\, A_\nu = \{v' = 0,\ \nu\, q' - (H \cdot \nu) H' = 0\}, \\ &rank\, A_\nu = 6. \end{aligned}$$

Projection onto $(ker\, A_\nu)^\perp$:

$$P = \begin{pmatrix} \Lambda & \underline{0}^T & -\Lambda (H\cdot\nu)\nu^T & 0 \\ \underline{0} & I_3 & 0_3 & \underline{0} \\ -\Lambda(H\cdot\nu)\nu & 0_3 & I_3 - \Lambda \nu \otimes \nu & \underline{0} \\ 0 & \underline{0}^T & \underline{0}^T & 0 \end{pmatrix}.$$

where $\Lambda := [1 + (H\cdot\nu)^2]^{-1}$.

P has the (finite) regularity of $H \cdot \nu$ (for $\partial\Omega \in C^\infty$).

Well-posedness in H^m (full regularity) [Yanagisawa Hokkaido MJ 1987, Shirota].

Characteristic hyperbolic IBVP

Consider the problem

$$\begin{cases} Lu = F & \text{in } Q_T\,, \\ Mu = G & \text{on } \Sigma_T\,, \\ u_{|t=0} = u_0 & \text{in } \Omega\,, \end{cases} \tag{4}$$

where

- $\Omega \subset \mathbb{R}^n, Q_T = \Omega \times (0,T), \Sigma_T = \partial\Omega \times (0,T)$
- $L := A_0(t,x)\partial_t + \sum_{j=1}^n A_j(x,t)\partial_{x_j} + B(t,x),\ \ A_j, B \in \mathbf{M}_{N\times N}$
- $M = M(x,t) \in \mathbf{M}_{d\times N}$, $\text{rank}(M) = d$ (maximal rank)
- $u(x,t) \in \mathbb{R}^N$, $F(x,t) \in \mathbb{R}^N$, $u_0(x) \in \mathbb{R}^N$, $G(x,t) \in \mathbb{R}^d$

Characteristic boundary

The boundary $\partial\Omega$ is characteristic if the boundary matrix

$$A_\nu := \sum_{j=1}^{n} A_j \nu_j$$

is singular at $\partial\Omega$ (not invertible). ($\nu = \nu(x)$ outward normal vector to $\partial\Omega$).

The boundary $\partial\Omega$ is characteristic of constant multiplicity if

$$\dim A_\nu = \text{const} \qquad \text{at} \quad \partial\Omega.$$

Full regularity (existence in usual Sobolev spaces $H^m(\Omega)$) can't be expected, in general, because of the possible loss of normal regularity at $\partial\Omega$.

[Tsuji, Proc. Japan Acad. 1972]

MHD [Ohno & Shirota, ARMA 1998], ill-posedness for the MHD eqn.s (with perfectly conducting wall b.c.) in $H^m(\Omega)$ for $m \geq 2$

Generally speaking, one normal derivative (w.r.t. $\partial\Omega$) is controlled by two tangential derivatives. Natural function space is the

weighted anisotropic Sobolev space

$$H^m_*(\Omega) := \{u \in L^2(\Omega) : \; Z^\alpha \partial^k_{x_1} u \in L^2(\Omega), \; |\alpha| + 2k \leq m\},$$

where

$$Z_1 = \sigma(x_1)\partial_{x_1} \quad \text{and} \quad Z_j = \partial_{x_j} \quad \text{for } j = 2, \dots, n,$$

if $\Omega = \{x_1 > 0\}$, where $\sigma \in C^\infty(\overline{R}_+)$ such that $\sigma(x_1) = x_1$ in a neighborhood of the origin and $\sigma(x_1) = 1$ for x_1 large.

[Chen Shuxing, Chinese Ann. Math. 1982],
[Yanagisawa & Matsumura, CMP 1991].

Results

Most of results have been proved for

Symmetric hyperbolic systems

maximal non-negative boundary conditions on the characteristic boundary of constant multiplicity:

- Linear L^2 theory [Rauch, Trans. AMS 1985],
- Existence theory in $H^m_*(\Omega)$ [Guès, CPDE '90], [Ohno-Shizuta-Yanagisawa, JM Kyoto U '95], [Secchi, DIE '95, ARMA '96, Arch. Math. 2000], [Shizuta, Proc. Japan Acad. MS 2000], [Casella-S.-Trebeschi, IJPAM 2005, DIE 2006],
- Application to MHD: [Secchi, Arch. Math. 1995, NoDEA 2002]
  Current-vortex sheets: [Chen-Wang, ARMA 2008], [Trakhinin, ARMA 2009], [Morando-S.-Trebeschi-Yuan ARMA 2023]

  Plasma–vacuum: [S.-Trakhinin, Nonlinearity 2014], [Morando-S.-Trakhinin-Trebeschi-Yuan 2023]

  FBP [Lindblad-Zhang, ARMA 2023]

Consider (3) with perfectly conducting wall b.c.s

$$v \cdot \nu = 0\,, \qquad H \cdot \nu = 0\,. \tag{5}$$

The corresponding i.b.v.p. has the form (4) ($F = 0, G = 0$) for $u = (q, v, H)^T$.
Assume

$$\Omega = \mathbb{R}^3_+ = \{x \in \mathbb{R}^3 \,:\, x_1 > 0\}.$$

(so that $\partial\Omega = \{x_1 = 0\}$, $\nu = (-1, 0, 0)$).

We write the operator L in (3) in the form

$$L := A_0\partial_t + \sum_{j=1}^{3}(A_j + C_j)\partial_{x_j}\,,$$

where

$$A_j = A_j(u) \qquad j = 0, 1, 2, 3,$$

C_j are constant symmetric matrices,

$$A_\nu := \sum_{j=1}^{3} A_j\nu_j = -A_1 \equiv 0 \quad \text{on } \Sigma_T.$$

The boundary matrix is $-A_1 - C_1$, where

$$C_1 = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & & & & \dots \\ 0 & & \dots & & 0 \end{pmatrix}$$

(rank $C_1 = 2$). Accordingly, decompose

$$A_i = \begin{pmatrix} A_i^{I\,I} & A_i^{I\,II} \\ A_i^{II\,I} & A_i^{II\,II} \end{pmatrix}, \qquad i = 0, 1, 2, 3,$$

$$(A_i^{I\,I} \in \mathbf{M}_{2\times 2}, \quad A_i^{I\,II} \in \mathbf{M}_{2\times 5}, \quad A_i^{II\,I} \in \mathbf{M}_{5\times 2}, \quad A_i^{II\,II} \in \mathbf{M}_{5\times 5}),$$

$$u = (u^I, u^{II})^T,$$

$$u^I = (q, v_1)^T \qquad \text{(non-characteristic part)}$$
$$u^{II} = (v_2, v_3, H_1, H_2, H_3)^T \qquad \text{(characteristic part).}$$

Then

$$\partial_{x_1} u^I = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \left(A_0 \partial_t u + A_1 \partial_{x_1} u + \sum_{j=2}^3 (A_j + C_j) \partial_{x_j} u \right)^I \qquad (6)$$

where $A_1 \partial_{x_1} u$ behaves like $Z_1 u = \sigma(x_1)\partial_{x_1}$, because $A_1 = 0$ on Σ_T.

Therefore $\partial_{x_1} u^I$ is equal to tangential derivatives only.

It suggests the introduction of a second weighted anisotropic Sobolev space with some more regularity:

$$H^m_{**}(\Omega) := \{u \in L^2(\Omega) : \; Z^\alpha \partial^k_{x_1} u \in L^2(\Omega), \; |\alpha| + 2k \leq m+1, |\alpha| \leq m\}.$$

In fact:

$$H^m_{**}(\Omega) = \{u \in H^m_*(\Omega) : \partial_\nu u \in H^{m-1}_*(\Omega)\}.$$

(Here $\partial_\nu = \partial_1$)

To prove the apriori estimates of the solution to (3), (5) we use

Lemma (1)

Let $U=(Q,V,W)$ *(with* $Q\in\mathbb{R}$ *and* $V,W\in\mathbb{R}^2$*) be a sufficiently smooth solution of*

$$A_0\partial_t U+\sum_{j=1}^n(A_j+C_j)\partial_j U=F$$

be such that

$$\text{either } Q=0 \quad \text{or} \quad V_1=0 \quad \text{on} \quad \Sigma_T.$$

Then[a]

$$a_0||U(t)||^2_{L^2(\Omega)}\le a_0^{-1}||U(0)||^2_{L^2(\Omega)}+\int_0^t\left(2\int_\Omega F\cdot U\,dx+|Div\vec{A}|_{L^\infty}||U||^2_{L^2(\Omega)}\right)ds$$

for all $t\in[0,T]$*, where* $Div\vec{A}=\partial_t A_0+\sum_j\partial_j A_j$*.*

[a] we assume $a_0\le A_0\le 1/a_0$ for a given constant $0<a_0<1$.

- We apply Lemma 1 for the estimate of the tangential derivatives $U = \boxed{Z^\alpha u}$ ($|\alpha| \le m$), because $V_1 = Z^\alpha v_1 = 0$ at $\partial\Omega$ (by tangential differentiation of $v_1 = 0$ along the boundary). (Also true for time derivatives)

Lemma

Assume $m \ge 5$. Let u be a sufficiently smooth solution to (3), (5) *such that*

$$|||u|||_{5,**} \le K \quad \text{in } Q_T. \tag{7}$$

Let $\boxed{D^\alpha_\star = \partial_t^{\alpha_0}\partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}}$ *for* $\alpha = (\alpha_0, \dots, \alpha_3)$ *with* $|\alpha| \le m$. *Then* $D^\alpha_\star u$ *satisfies*

$$a_0||D^\alpha_\star u(t)||^2 \le a_0^{-1}||D^\alpha_\star u(0)||^2 +$$
$$+C\int_0^t \{\Phi(K)||D^\alpha_\star u||^2 + |||A|||_{m,**}|||u|||^2_{m,**} + |||u|||_{m,**}|||\partial_1 q|||_{m-1,*}\}ds$$

for all $t \in [0,T]$.

$|||u(t)|||^2_{m,*} = \sum_{k=0}^m ||\partial_t^k u(t)||^2_{m-k,*}$, $\qquad |||u(t)|||^2_{m,**} = \sum_{k=0}^m ||\partial_t^k u(t)||^2_{m-k,**}$

- The alternative possibility $Q = 0$ suggests that we can exploit the property

$$\partial_1 q = -\rho(\partial_t v_1 + v \cdot \nabla v_1) + H \cdot \nabla H_1 = 0 \quad \text{at} \quad \Sigma_T$$

  (because of $v_1 = H_1 = 0$ at Σ_T) for the estimate of one normal derivative.
- Again, we use Lemma 1 for the estimate of tangential and one normal derivatives $U = \boxed{Z^\beta \partial_1 u}$ $(|\beta| \leq m-1)$, with $Q = Z^\beta \partial_1 q = 0$ at $\partial\Omega$ (by tangential differentiation of $\partial_1 q = 0$ along the boundary).
- The better estimate allows to solve the problem for u in $H^m_{**}(\Omega)$ (instead of $H^m_*(\Omega)$).

Lemma

Assume $m \geq 6$. *Let* u *be a sufficiently smooth solution to* (3), (5) *satisfying* (7). *Let* $\beta = (\beta_0, \ldots, \beta_3)$ *a multi-index with* $|\beta| \leq m-1$. *Then* $D_\star^\beta \partial_1 u$ *satisfies*

$$a_0 ||D_\star^\beta \partial_1 u(t)||^2 \leq a_0^{-1} ||D_\star^\beta \partial_1 u(0)||^2 + \int_0^t \{\Phi(K) ||D_\star^\beta \partial_1 u||^2 + |||A|||_{m,**} |||u|||_{m,**}^2 + \boxed{|||\partial_1 v_1|||_{m-1,**}} |||\partial_1 q|||_{m-1,*}\} ds$$

for all $t \in [0, T]$.

The norm in the red box requires more regularity than $u \in H^m_{**}(\Omega)$.

- To complete the estimate of $\partial_1 u^{II}$ in $H^{m-1}_*(\Omega)$ we need to estimate $Z^\gamma \partial_1^k u^{II}$ for $|\gamma| + 2k \leq m+1, k \geq 2$. This is obtained from rows 3 to 7 of (3) that we write as

$$(A_0^{II\,II}\partial_t + \textstyle\sum_{j=1}^3 A_j^{II\,II}\partial_j)u^{II} = G \tag{8}$$

with

$$G = -\left((A_0^{II\,I}\partial_t + \sum_{j=1}^3 A_j^{II\,I}\partial_j)u^I\right),$$

that is

$$G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}, \quad G_1 = -\begin{pmatrix} \partial_2 q \\ \partial_3 q \end{pmatrix}, \quad G_2 = \tfrac{\rho_p}{\rho} H(\partial_t q + v\cdot\nabla q) + \begin{pmatrix} H\cdot\nabla v_1 \\ 0 \\ 0 \end{pmatrix},$$

where (8) is a transport-type equation because the boundary matrix $A_1^{II\,II} \equiv 0$ at $\partial\Omega$, and no boundary condition is required.

Lemma

Assume $m \geq 6$*. Let* u *be a sufficiently smooth solution to* (3), (5) *satisfying* (7) *and decomposed in the form* $u = (u^I, u^{II})$*. Then* u^{II} *satisfies*

$$a_0 ||D_\star^\gamma \partial_1^k u^{II}(t)||^2 \leq a_0^{-1} ||D_\star^\gamma \partial_1^k u^{II}(0)||^2 + C \int_0^t \{\Phi(K) ||D_\star^\gamma \partial_1^k u^{II}||^2 +$$

$$(1 + |||A|||_{m,**}) \left(|||u|||_{m,**} + \boxed{|||\partial_1 q|||_{m-1,**} + |||\partial_1 v_1|||_{m-1,**}} \right) |||u|||_{m,**} \} ds$$

for all $t \in [0, T]$*, and for any multi-index* γ *and integer* $k \geq 2$ *such that* $|\gamma| + 2k \leq m + 1$*.*

The norms in the red box require more regularity than $u \in H^m_{**}(\Omega)$.

We introduce a third function space:

$$H^m_{***}(\mathbb{R}^3_+) = \{u \in H^m_{**}(\mathbb{R}^n_+) : \partial_1 u \in H^{m-1}_{**}(\mathbb{R}^n_+)\}.$$

- At last, the estimate of $\boxed{Z^\gamma \partial_1^k u^I}$ $(|\gamma| + 2k \leq m+1, k \geq 2)$ is obtained by a direct estimate of (6):

$$\partial_{x_1} u^I = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \left(A_0 \partial_t u + A_1 \partial_{x_1} u + \sum_{j=2}^3 (A_j + C_j) \partial_{x_j} u \right)^I .$$

Lemma

Let u be a sufficiently smooth solution to (3), (5) *satisfying* (7). *Then for all $t \in [0,T]$:*
(i) the pressure q satisfies the estimate

$$|||\partial_1 q|||_{m-1,**} \leq \Phi(K) \left(|||u|||_{m,**} + |||v_1, H_1|||_{m,***} \right),$$

(ii) the first component of v satisfies

$$|||\partial_1 v_1|||_{m-1,**} \leq \Phi(K) |||u|||_{m,**},$$

(iii) the non-characteristic part (q_1, v_1) and H_1 satisfy

$$|||q_1, v_1, H_1|||_{m,***} \leq \Phi(K) |||u|||_{m,**}.$$

Some technical tools:

- Imbeddings:

$$H_*^{[(n+1)/2]+1}(\Omega) \hookrightarrow C_B^0(\overline{\Omega}),$$

$$H_{**}^m(\Omega) \hookrightarrow C_B^0(\overline{\Omega}) \quad \text{for} \quad m > n/2\,.$$

- Product of 2 functions:

$$||uv||_{H_*^m(\Omega)} \leq C||u||_{H_*^m(\Omega)}||v||_{H_*^r(\Omega)} \qquad m \geq 1,\ r = \max\{m, [\frac{n+1}{2}]+2\},$$

$$||uv||_{H_*^m(\Omega)} \leq C||u||_{H_*^m(\Omega)}||v||_{H_{**}^r(\Omega)} \qquad m \geq 1, 1, r = \max\left\{m, 2\left[\tfrac{n}{2}\right]+2\right\},$$

$$||uv||_{H_{**}^m(\Omega)} \leq C||u||_{H_{**}^m(\Omega)}||v||_{H_{**}^r(\Omega)} \qquad m \geq 1, 1, r = \max\left\{m, 2\left[\tfrac{n}{2}\right]+2\right\}.$$

- Compactness:
  $\Omega \subset \mathbb{R}^n$ open bounded set with C^∞ boundary. The imbedding $H_{**}^m(\Omega) \hookrightarrow H_*^{m-1}(\Omega)$ is compact.

Hardy-type inequalities:

- If $A = 0$ on $\partial\Omega = \{x_1 = 0\}$, let's define $U(x_1, x') = A(x_1, x')/\sigma(x_1)$. Then

$$\|U\|_{H_*^{m-2}(\mathbb{R}^n_+)} \leq C\|A\|_{H_*^m(\mathbb{R}^n_+)},$$

$$\|U\|_{H_{**}^{m-2}(\mathbb{R}^n_+)} \leq C\|A\|_{H_{**}^m(\mathbb{R}^n_+)},$$

$$\|U\|_{H_*^{m-1}(\mathbb{R}^n_+)} \leq C\|A\|_{H_{**}^m(\mathbb{R}^n_+)},$$

$$\|U\|_{H_{**}^{m-1}(\mathbb{R}^n_+)} \leq C\|A\|_{H_{***}^m(\mathbb{R}^n_+)},$$

where

$$H^m_{***}(\mathbb{R}^3_+) = \{u \in H^m_{**}(\mathbb{R}^n_+) : \partial_1 u \in H^{m-1}_{**}(\mathbb{R}^n_+)\}.$$

Examples of application

Estimate of $A\partial_1 u$, given $u \in H^m_{**}(\mathbb{R}^n_+)$ and $A \in H^m_{**}(\mathbb{R}^n_+)$ such that $A = 0$ at $\partial\Omega$.

- $$A \cdot \partial_1 u \in H^m_{**}(\mathbb{R}^n_+) \cdot H^{m-2}_{**}(\mathbb{R}^n_+) \subset H^{m-2}_{**}(\mathbb{R}^n_+), \qquad m \geq 2\left[\frac{n}{2}\right] + 2,$$
  $$A \cdot \partial_1 u \in H^m_{**}(\mathbb{R}^n_+) \cdot H^{m-1}_{*}(\mathbb{R}^n_+) \subset H^{m-1}_{*}(\mathbb{R}^n_+), \qquad m \geq 2\left[\frac{n}{2}\right] + 2,$$
- $$\frac{A}{\sigma} \cdot \sigma\partial_1 u \in H^{m-1}_{*}(\mathbb{R}^n_+) \cdot H^{m-1}_{**}(\mathbb{R}^n_+) \subset H^{m-1}_{*}(\mathbb{R}^n_+), \qquad m-1 \geq \left[\frac{n+1}{2}\right] + 2,$$
- If $A \in H^m_{***}(\mathbb{R}^n_+)$:
  $$\frac{A}{\sigma} \cdot \sigma\partial_1 u \in H^{m-1}_{**}(\mathbb{R}^n_+) \cdot H^{m-1}_{**}(\mathbb{R}^n_+) \subset H^{m-1}_{**}(\mathbb{R}^n_+), \qquad m-1 \geq 2\left[\frac{n}{2}\right] + 2.$$

Theorem (NoDEA 2002; 2024)

Let $m \geq 6$ be an integer. Let $\rho \in C^{m+1}$ and $u_0 = (q_0, v_0, H_0) \in H^m(\Omega)$, let $p_0 = q_0 - |H_0|^2/2$ be such that $\rho(p_0) > 0, \rho_p(p_0) > 0$ in $\overline{\Omega}$, $\nabla \cdot H_0 = 0$ in Ω. The data satisfy the compatibility conditions $\partial_t^k v_{|t=0} \cdot \nu = 0$ for $k = 0, \dots, m-1$, $H_0 \cdot \nu = 0$ on $\partial\Omega$.

Then there exists $T > 0$ such that the mixed problem (3), (5), *$u_{|t=0} = u_0$, has a unique solution on $[0, T]$*

$$u = (q, v, H) \in \cap_{k=0}^{m} C^k([0,T]; H^{m-k}_{**}(\Omega))$$

with $\rho(p) > 0, \rho_p(p) > 0$, for $p = q - |H|^2/2$, and $\nabla \cdot H = 0$ in Q_T.

Moreover $\partial_t^k v(t)_{|\partial\Omega} \in H^{m-k-1/2}(\partial\Omega)$ for $k = 0, \dots, m-1$, $H(t)_{|\partial\Omega} \cdot \nu \in H^{m-1/2}(\partial\Omega)$, for each $t \in [0, T]$.