Summer School on Geometry

14-25 August 2023

Last updated: 12 March

Course organisers. Dr. Álvaro del Pino Gómez and Wilfred de Graaf MSc.
Contact. `w.f.degraaf@uu.nl`
For more information and to apply: www.utrechtsummerschool.nl/courses/science/geometry.

Location. HFG611, 6th floor of the Hans Freudenthal building, Budapestlaan 6, Utrecht.
It is easy to get by bicycle to the summer school location. On the website of Utrecht Summer School you can find information about renting bicycles and using the Utrecht public transport system:
https://utrechtsummerschool.nl/practical-information/getting-around.

The Mathematical Institute will provide lunch on Monday 14 August and Friday 25 August and dinner on Monday 14 August and Tuesday 22 August. At all other times you are expected to bring your own lunch.

Saturday/Sunday 12 and 13 August	**Key pick-up.** You will find the exact key pick-up location in the pre-departure information, which becomes available after you have paid the course fee.

Monday 14 August	**Dr. Gijs Heuts and Dr. Lennart Meier** **Fixed point theorems and their applications** A fixed point of a self-map is a point that is mapped to itself. There are several theorems assuring the existence of a fixed point, the most famous being the Banach fixed point theorem and the Brouwer fixed point theorem. The latter says that every continuous self-map of an n-dimensional disk has a fixed point. We will prove this in the course and provide several applications, in particular to algebra and game theory – in particular, we will show the fundamental theorem of algebra and the existence of Nash equilibria from game theory. Our treatment of all of this will be elementary. **Expected background.** Prerequisites are minimal. Knowing what continuous maps between subsets of $\mathbb{R}^n$ are, should suffice. **09.00** Welcome, registration and lecture **11.00** Exercise class **12.30** Lunch in the Mathematics Library, 7th floor HFG **13.30** Lecture **15.30** Exercise class **17.00** We will walk along the Kromme Rijn river to a nearby restaurant

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<td>Tuesday 15 August</td>
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Dr. Fabian Ziltener: Some highlights of symplectic geometry
Symplectic geometry originated from classical mechanics, where the canonical symplectic form on phase space appears in Hamilton's equation. It is related to dynamical systems, algebraic geometry, and string theory. I will start my lectures by explaining basic notions, such as symplectic forms and Hamilton's equation. After this I will discuss the following highlights of symplectic geometry:

A famous conjecture by V. Arnol'd, which states a lower bound on the number of periodic orbits of a Hamiltonian system,
M. Gromov's celebrated nonsqueezing theorem, which states that a ball in $\mathbb{R}^{2n}$ of radius bigger than 1 does not symplectically embed into any symplectic cylinder of radius 1.

I will motivate and explain the statements. (Proofs would fill an entire lecture course.)
Expected background. Analysis in several variables and linear algebra. I expect that students know the following concepts and theorems:

total derivative of a function of several variables,
chain rule for functions of several variables,
vector space (not just $\mathbb{R}^n$),
linear map between vector spaces.

09.00 Lecture
11.00 Exercise class
12.30 Lunch break
13.30 Lecture
15.30-17.00 Exercise class
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Wednesday 16 August	**Dr. Niall Taggart and Dr. Guy Boyde: A topologist's time machine** Topology is the study of topological spaces up to continuous deformation: topologists declare two "shapes" (or spaces) to be "the same" if one can be stretched or squeezed into the "shape" of the other. To classify spaces up to this notion of "sameness" topologists have developed a wide range of algebraic tools to distinguish spaces based on their "shape". The starting point was the work of Euler on distinguishing "shapes" based on a simple formula involving the number of vertices, edges and faces. This mini-course will act as a mathematical time machine, taking us through the advances of algebraic topology starting from Euler and moving toward more modern approaches using more sophisticated algebra (groups, rings, etc). **Expected background.** Point-set topology and basic linear algebra (probably nothing beyond knowing what a vector space/matrices are). **09.00** Lecture **11.00** Exercise class **12.30** Lunch break **13.30** Lecture **15.30-17.00** Exercise class

Thursday 17 August	**Dr. Álvaro del Pino Gómez** **In how many ways can we prove the Whitney-Graustein theorem?** A map of the circle into the plane is said to be immersed if its velocity is nowhere zero. This means that it traces a closed curve in the plane that has no creases, but may have self-intersections. In a 1937 article, Whitney attributes to Graustein the following result: two such immersions can be deformed to one another if and only if they "rotate the same". This is the first classification result for immersions of manifolds, a topic that eventually became one of the cornerstones of Differential Topology. The goal of this minicourse is to introduce you to a bunch of different techniques in the study of manifolds. Each technique will show up naturally as we pursue different strategies to prove the Whitney-Graustein theorem. **Expected background.** Analysis in multiple variables. Some background on topology (e.g. the fundamental group) is helpful but not strictly necessary. **09.00** Lecture **11.00** Exercise class **12.30** Lunch break **13.30** Lecture **15.30-17.00** Exercise class

Friday 18 August	**Dr. Marta Pieropan: Geometry of numbers** How many points with integer coordinates lie inside a circle? How does this number grow as the radius increases? Is the area of the circle a good approximation? What if the circle is replaced by other regions? This course formalizes the notion of point with integer coordinates via the theory of lattices, and introduces the basics of Minkowski theory for lattice points in convex bodies. We will discuss the equivalence between the Euclidean norm and the sup norm in real vector spaces. We will also consider applications to some elementary number theory problems such as: In how many distinct ways can a natural number be represented as the sum of four squares? **Expected background.** Second year students who have taken basic linear algebra and analysis (limits, integration in several variables, perhaps a bit measure theory). **09.00** Lecture **11.00** Exercise class **12.30** Lunch break **13.30** Lecture **15.30-17.00** Exercise class

Saturday/Sunday 19 and 20 August	For the social programme organised by UU for all the summer school students, see: https://utrechtsummerschool.nl/.

Monday 21 August	**Dr. Valentijn Karemaker: Modular forms** The number theorist Martin Eichler (1912-1992) famously said: "There are five elementary arithmetical operations: addition, subtraction, multiplication, division, and modular forms." Assuming only familiarity with the first four, some linear algebra and a pinch of complex analysis, in this course we will introduce modular forms and modular functions. We will define them geometrically, starting from the action of the modular group on the complex upper-half plane, and study some of their remarkable algebraic and analytic properties. In addition, we will point out some of their fundamental connections to other algebraic and geometric objects, that explain their importance in number theory. **Expected background.** Basic complex analysis (one-variable) and algebra (the definitions of groups and rings). **09.00** Lecture **11.00** Exercise class **12.30** Lunch break **13.30** Lecture **15.30-17.00** Exercise class

Tuesday 22 August	**Dr. Stefano Marseglia** **Discrete logarithm problem for elliptic curves over finite fields** One of the most widely used protocols to secure and certify online communications uses elliptic curves (defined over a finite field). Such curves are special because their points form a group. The safety of the protocol relies on the assumption that the Elliptic Curve Discrete Logarithm Problem is hard to solve. In this course, we will discuss the mathematics behind the ECDLP and compare it to the classical DLP, which built on the multiplicative group of a finite field. **Expected background.** Basic algebra. The notion of group and elliptic curve will be recalled in the lectures. **09.00** Lecture **11.00** Exercise class **12.30** Lunch break **13.30** Lecture **15.30-17.00** Exercise class **18.00** Dinner

Wednesday 23 August	**Prof. Dr. Ieke Moerdijk and Dr. Tobias Lenz** **Representations of finite groups** The representation theory of finite groups studies the ways a finite group can act on a vector space, or equivalently, the ways in which a finite group can be mapped into a matrix group. The result is a beautiful mathematical theory, reducing the problem completely to understanding certain ("conjugate-invariant") functions from the group into the complex numbers, the so-called characters of the group. We will explain some first steps of this theory, study a couple of examples, and explain how representations of different groups are related. **Expected background.** Linear algebra and basic group theory. **09.30-10.15** Lecture (theory) **10.30-11.15** Lecture (theory) **11.30-12.30** Examples/exercises **12.30-13.30** Lunch break **13.30-14.30** Lecture (theory) **15.00-15.45** Lecture (more examples) **16.00-17.00** Exercises and discussion

Thursday 24 August	Dr. Wioletta Ruszel: Random geometry of equilibrium phases
	Equilibrium statistical mechanics intends to describe and explain the macroscopic behavior of systems in thermal equilibrium in terms of the microscopic interaction between their great many constituents. The equilibrium states with respect to the given interaction are described by the associated Gibbs measures. These are probability measures on the space of configurations which have prescribed conditional probabilities with respect to fixed configurations outside of finite regions. Since the early days of statistical mechanics, geometric notions have played a role in elucidating certain aspects of the theory. For example, the thermodynamic formalism, as first developed by Gibbs, already admits some geometric interpretations primarily related to convexity. The geometry considered here is a way of visualizing the structure in the typical realizations of the system's constituents using percolation theory. It serves as a toy model for the study of statistical equilibrium properties in geometric terms. **Expected background.** Solid knowledge of probability theory and analysis. Knowledge of measure theory is advisable. Pre-knowledge in physics is helpful but not necessary. **09.00** Lecture **11.00** Exercise class **12.30** Lunch break **13.30** Lecture **15.30-17.00** Exercise class

Friday 25 August	Prof. Dr. Gunther Cornelissen: Zeta functions
	Zeta functions are generating functions with very strong analytic properties, whose special values and poles encode information from the number-theoretical, combinatorial, or geometrical input. The prototype is the Riemann zeta function, that can be used to study the distribution of prime numbers and properties of prime ideals, but there are also zeta functions that encode lengths of geodesics, walks in graphs, points on algebraic curves, spectra of operators, etc. The lecture will be a very gentle introduction to the theory, and in the second part, you pick your favourite zeta function and work with it yourself. **Expected background.** End of second year students. **09.30** Lecture **11.00** Exercise class **12.30** Lunch in the Mathematics Library, 7th floor HFG **N.B.** You can store your luggage in HFG610, next to HFG611.