Jensen's Theorem and a Simple Application

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A frequent problem in complex analysis is determining the location of the zeroes of a function. More specifically one wishes to calculate the number of zeroes of an analytic function[1] in some region. One way to undertake such a task is via Jensen's Theorem.

Theorem 1 *If, for a complex variable $s = re^{i\theta}$*

- *the function $f(s)$ is analytic in the region $|s| \le R$; and*
- *$f(s)$ has no zeroes on $|s| = R$; and*
- *$f(0) = 1$, then*

$$(2\pi)^{-1} \int_0^{2\pi} \log|f(Re^{i\theta})|\, d\theta = \int_0^R \frac{n(r)}{r}\, dr, \tag{1}$$

where the function $n(r)$ is the number of zeroes of $f(s)$ inside the disc $|s| = r$

Some Comments

The first two conditions are required to use Cauchy's integral theorem later on. The condition that $f(0) = 1$ is not necessary, but is used for convenience. Indeed as the prove unfolds this fact will be made apparent.

Proof

First we recall that logarithms of complex numbers require more attention to detail than their real counterparts. Indeed for a complex variable z, one writes $\log z = \log|z| + i \arg z + 2\pi k i$, where k (an integer) is only determined when the corresponding branch cut in the complex plane is determined: in particular

$$\operatorname{Re} \log z = \log|z|. \tag{2}$$

Furthermore once a branch cut has been determined, one can write

$$\log f(Re^{i\theta}) - \log f(0) = \int_0^R \frac{df(re^{i\theta})}{f(re^{i\theta})}\, dr. \tag{3}$$

[1] The terms *regular, analytic, holomorphic* are all identically used to describe a function which is differentiable in every point of some region A. If this region is not specified, the implication is that the function is analytic everywhere, in which case it is said to be *entire.*

One combines these two facts to write the right hand side of (1) as

$$(2\pi)^{-1}\int_0^{2\pi}\log|f(Re^{i\theta})|\,d\theta = (2\pi)^{-1}\int_0^{2\pi}\mathrm{Re}\left\{\int_0^R \frac{df(re^{i\theta})}{f(re^{i\theta})}\right\}d\theta, \tag{4}$$

where the term $\log f(0)$ has been evaluated to be zero, by assumption (see that it can be carried through until the end). Now one writes $df(re^{i\theta}) = f'(re^{i\theta})e^{i\theta}\,dr$, and after a change of variable $s = re^{i\theta}$ it follows that

$$(2\pi)^{-1}\int_0^{2\pi}\mathrm{Re}\left\{\int_0^R \frac{df(re^{i\theta})}{f(re^{i\theta})}\right\}d\theta = (2\pi i)^{-1}\int_0^R \frac{1}{r}\mathrm{Re}\left\{\left(\int_{|s|=r}\frac{f'(s)}{f(s)}\,ds\right)\right\}dr \tag{5}$$

Now to use Cauchy's integral formula[2] we note that the only[3] poles are those zeroes of the function $f(s)$. One can check[4] that, if $f(s)$ has a zero of multiplicity m at some point a, then the residue of $f'(s)/f(s)$ is equal to m. Thus the right hand side of the above equation[5] is equal to

$$\int_0^R \frac{n(r)}{r}\,dr, \tag{6}$$

and all is right with the world.

Further Comments

One can also write this last equation (6) in another format. Suppose that the zeroes of the function $f(s)$ are at points $s_1, s_2, \ldots, s_n$, such that each s_i is located at a distance r_i. Then one may write

$$\begin{aligned}
\int_0^R \frac{n(r)}{r}\,dr &= \left(\int_0^{r_1} + \int_{r_1}^{r_2} + \ldots + \int_{r_n}^{R}\right)\frac{n(r)}{r}\,dr && (7)\\
&= \int_0^{r_1}\frac{0}{r}\,dr + \int_{r_1}^{r_2}\frac{1}{r}\,dr + \int_{r_2}^{r_3}\frac{2}{r}\,dr + \ldots + \int_{r_n}^{R}\frac{n}{r}\,dr && (8)\\
&= \log(r_2 - r_1) + 2\log(r_3 - r_2) + \ldots + n\log(R - r_n) && (9)\\
&= n\log R - (\log r_1 + \log r_2 + \ldots + \log r_n) && (10)\\
&= \frac{\log R^n}{|s_1 \cdot s_2 \cdots s_n|} && (11)
\end{aligned}$$

Benediction - More in the Seminar

Jensen's Theorem may be used to show the correct upper bound on the order of magnitude for the number of zeroes of the zeta-function to height T.

[2] That is the integral of an analytic function around a closed contour is $2\pi i$ times the sum of the residues. The residue of a function $f(s)$ at the point a is just the coefficient of the term $(s-a)^{-1}$ in the (Laurent series) expansion.

[3] These are the only poles since $f(s)$ is analytic.

[4] Write $f(s) = g(s)(s-a)^m$, since the function $g(s)$ must be analytic and be free of zeroes at $s = a$ (why?) then ...

[5] Notice that the 'real part' has slipped away ... how?