Approximation theorems

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I will prove the following theorem (Theorem 1) on Monday. It is almost what is called Dirichlet's approximation theorem. Anyway, feel free to use it if/when you attempt to solve the problem of the week (see below).

Theorem 1. *Given any real θ and any positive integer N, there exists relatively prime integers h and k with $0 < k \leqslant N$ such that $\mid k\theta - h \mid < \frac{1}{N}$.*

Corollary 1. *For every real θ there exists integers h and k with $k > 0$ and $(h,k) = 1$ such that $\mid \theta - \frac{h}{k} \mid < \frac{1}{k^2}$.*

Proof. In Theorem 1 we have $1/(Nk) \leqslant 1/k^2$ because $k \leqslant N$. □

Problem of the week: Show that $\theta \in \mathbb{R}$ is irrational if and only if there exist infinitely many ordered pairs of integers (h,k) with $k > 0$ and $(h,k) = 1$ such that $\mid \theta - \frac{h}{k} \mid < \frac{1}{k^2}$.

Hint for "only if"-part: Proof by contradiction...Let $S(\theta)$ denote the set of all such ordered pairs of integers (h,k). We suppose that $S(\theta)$ is finite and thus want to derive a contradiction. Consider the well-defined

$$\alpha := \min_{(h,k)\in S(\theta)} \mid \theta - \tfrac{h}{k} \mid .$$

Since θ is irrational, α is positive. Then applying Theorem 1 with any integer $N > 1/\alpha$ will yield a contradiction to the definition of α.

Hint for "if"-part: Proof by contradiction...Suppose that $\theta = a/b$. It is obvious that if the set $S(\theta)$ is infinite then it must contain pairs (h,k) with arbitrarily large k. Pick $(h,k) \in S(\theta)$ with $k > b$. Then use

$$0 <\mid \tfrac{a}{b} - \tfrac{h}{k} \mid < \tfrac{1}{k^2}$$

to derive a contradiction.

Remark: We will go through this on Monday and probably there will be time over for me to prove a slightly stronger statement (Hurwitz's theorem) than the one given in the problem of the week.