Handbook for the Undergraduate Mathematics Courses
Supplement to the Handbook
Honour School of Mathematics & Philosophy
Syllabus and Synopses for Part A 2007–8
for examination in 2008

Contents

1 Foreword

Notice of misprints or errors of any kind, and suggestions for improvements in this booklet should be addressed to the Academic Assistant in the Mathematical Institute.

[See the current edition of the *Examination Regulations* for the full regulations governing these examinations.]

In Part A each candidate shall be required to offer the 3 written papers in Mathematics from the schedule of papers for Part A (given below). Each paper will be $2\frac{1}{4}$ hours, and the three papers taken together will be counted as equivalent to two 3 hour papers.

At the end of the Part A examination, a candidate will be awarded a 'University Standardised Mark' (USM) for each paper in Part A Mathematics. The three USMs will be carried forward into the classification awarded at the end of the third year. In the calculation of any averages used for the final classification, the three USMs will be scaled by 2/3 so as to give the equivalent of two papers.

The Schedule of Papers

Paper AC1(P) Algebra and Analysis

This paper will contain 6 short questions set on the CORE material in Algebra and Analysis for Part A of the FHS of Mathematics. Candidates are expected to answer all 6 questions. Each question is out of 10 marks.

Paper AC2(P) Algebra and Analysis

This paper will contain 6 longer questions set on the CORE material in Algebra and Analysis for Part A of the FHS of Mathematics. Each question is out of 25 marks. Candidates may submit up to 4 answers, with the best 3 counting.

Paper AO3(P) Options

This paper will contain questions on the OPTIONAL subjects drawn from Part A of the FHS of Mathematics and listed below. These will also be longer questions. Candidates may submit up to 4 answers, and the best 3 will count. There will be 1 question for each 8 lectures. Each question is out of 25 marks.

Mark Schemes

Mark schemes for questions out of 10 will aim to ensure that the following qualitative criteria hold:

- 9-10 marks: a completely or almost completely correct answer, showing good understanding of the concepts and skill in carrying through arguments and calculations; minor slips or omissions only.

- 5-8 marks: a good though not complete answer, showing understanding of the concepts and competence in handling the arguments and calculations.

Mark schemes for questions out of 25 will aim to ensure that the following qualitative criteria hold:

- 20-25 marks: a completely or almost completely correct answer, showing excellent understanding of the concepts and skill in carrying through the arguments and/or calculations; minor slips or omissions only.

- 13-19 marks: a good though not complete answer, showing understanding of the concepts and competence in handling the arguments and/or calculations. In this range, an answer might consist of an excellent answer to a substantial part of the question, or a good answer to the whole question which nevertheless shows some flaws in calculation or in understanding or in both.

OPTIONAL SUBJECTS

Groups in Action

Introduction to Fields

Number Theory

Integration

Topology

Multivariable Calculus

Candidates may also, with the support of their College tutors, apply to the Joint Committee for Mathematics and Philosophy for approval of other Optional Subjects as listed for Part A of the Honour School of Mathematics. Such an application should be made through the candidate's college and sent to the Chairman, Joint Committee for Mathematics & Philosophy, c/o Academic Administrator, Mathematical Institute to arrive by 5 pm on Monday of Week 0 of Hilary Term.

Syllabus and Synopses

The **syllabus** details in this booklet are those referred to in the *Examination Regulations* and have been approved by the Mathematics Teaching Committee for examination in Trinity Term 2008.

The **synopses** in this booklet give some additional detail, and show how the material is split between the different lecture courses. They also include details of recommended reading.

October 2007

2 CORE MATERIAL

2.1 Syllabus

2.1.1 Algebra

Vector spaces over an arbitrary field, subspaces, direct sums; quotient spaces; projection maps and their characterisation as idempotent operators.

Dual spaces of finite-dimensional spaces; annihilators; the natural isomorphism between a space and its second dual; dual transformations and their matrix representation with respect to dual bases.

Some theory of a single linear transformation on a finite-dimensional space: characteristic polynomial, minimal polynomial, Primary Decomposition Theorem, the Cayley–Hamilton Theorem; diagonalisability; triangular form.

Real and complex inner product spaces. Orthogonal complements, orthonormal sets; the Gram–Schmidt process. Bessel's inequality; the Cauchy–Schwarz inequality.

The adjoint of a linear transformation on a finite-dimensional inner product space to itself. Eigenvalues and diagonalisability of self-adjoint linear transformations.

Commutative rings with unity, integral domains, fields; units, irreducible elements, primes.

Ideals and quotient rings; isomorphism theorems. The Chinese Remainder Theorem [classical case of $\mathbb{Z}$ only].

Maximal ideals and their quotient rings.

Euclidean rings and their properties: polynomial rings as examples, theorem that their ideals are principal; theorem that their irreducible elements are prime; uniqueness of factorisation (proof non-examinable).

Gauss' Lemma.

2.1.2 Analysis

The topology of Euclidean space and its subsets, particularly $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$: open sets, closed sets, subspace topology; continuous functions and their characterisation in terms of pre-images of open or closed sets; connected sets, path-connected sets; compact sets, Heine-Borel Theorem.

The algebra and geometry of the complex plane. Complex differentiation. Holomorphic functions. Cauchy-Riemann equations (including z, $\bar{z}$ version). Real and imaginary parts of a holomorphic function are harmonic.

Path integration. Fundamental Theorem of Calculus in the path integral/holomorphic function setting. Power series and differentiation of power series. Exponential function, logarithm function, fractional powers - examples of multifunctions.

Cauchy's Theorem (proof excluded). Cauchy's Integral formulae. Taylor expansion. Liouville's Theorem. Identity Theorem. Morera's Theorem. Laurent's expansion. Classification of singularities. Calculation of principal parts and residues. Residue theorem. Evaluation of integrals by the method of residues (straight forward examples only but to include use of Jordan's lemma and simple poles on contour of integration).

Conformal mapping: Möbius functions, exponential functions, fractional powers; mapping regions (not Christoffel transformations or Jowkowski's transformation).

2.2 Synopses of Lectures

2.2.1 Algebra — Dr Neumann — 24 lectures MT

Rings and Arithmetic [8 lectures, Michaelmas Term]

Aims and Objectives

This half-course introduces the student to some classic ring theory which is basic for other parts of abstract algebra, for linear algebra and for those parts of number theory that lead ultimately to applications in cryptography. The first-year algebra course contains a treatment of the Euclidean Algorithm in its classical forms for integers and for polynomial rings over a field. Here the idea is developed *in abstracto*. The Gaussian integers, which have applications to many questions of elementary number theory, give an important and interesting (and entertaining) illustration of the theory.

Learning Outcomes

By the end of this course students will have further developed an awareness of topics in abstract and linear algebra. They will know key elements of classical ring theory, including Euclidean rings and their properties. They will have a deeper appreciation of the theory of vector spaces and Linear transformations defined from a vector space to itself, including diagonalisation, minimal polynomials and triangular form.

Synopsis

1. **Rings**

   MT 8 Lectures

   Commutative rings with unity, integral domains, ideals, fields, examples including polynomial rings over a field and subrings of $\mathbb{R}$ and $\mathbb{C}$.

   [2 lectures]

   Ideals and quotient rings; isomorphism theorems. Examples : integers modulo a natural number; the Chinese Remainder Theorem; the quotient ring by a maximal ideal is a field.

   [2 lectures]

   Euclidean rings and their properties : units, associates, irreducible elements, primes. The Euclidean Algorithm for a Euclidean ring; $\mathbb{Z}$ and $F[x]$ as prototypes; their ideals are principal; their irreducible elements are prime; factorisation is unique (proof not examinable).

[3 Lectures]

Examples for applications: Gauss's Lemma and factorisation in $\mathbb{Q}[x]$.

[1 lecture]

2. **Further Linear Algebra**
MT 16 Lectures

Aims and Objectives

The core of linear algebra comprises the theory of linear equations in many variables, the theory of matrices and determinants, and the theory of vector spaces and linear transformations. All these topics were introduced in the Moderations course. Here they are developed further to provide the tools for applications in geometry, modern mechanics and theoretical physics, probability and statistics, functional analysis and, of course, algebra and number theory. Our aim is to provide a thorough treatment of some classical theory that describes the behaviour of linear transformations on a finite-dimensional vector space to itself, both in the purely algebraic setting and in the situation where the vector space carries a metric derived from an inner product.

Synopsis

Fields; $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ and $\mathbb{F}_p$ as examples. Vector spaces over an arbitrary field, subspaces, direct sums; projection maps and their characterisation as idempotent operators.

[2 Lectures]

Dual spaces of finite-dimensional spaces; annihilators; the natural isomorphism between a finite-dimensional space and its second dual; dual transformations and their matrix representation with respect to dual bases.

[2 Lectures]

Some theory of a single linear transformation on a finite-dimensional space: characteristic polynomial, minimal polynomial, Primary Decomposition Theorem, the Cayley–Hamilton Theorem (economically); diagonalisability; triangular form.

[4 Lectures]

Real and complex inner product spaces: examples, including function spaces [but excluding completeness and L^2]. Orthogonal complements, orthonormal sets; the Gram–Schmidt process. Bessel's inequality; the Cauchy–Schwarz inequality.

[4 Lectures]

Some theory of a single linear transformation on a finite-dimensional inner product space: the adjoint; eigenvalues and diagonalisability of a self-adjoint linear transformation.

[4 Lectures]

(16 lectures)

Reading

Peter J Cameron, *Introduction to Algebra*, OUP 1998, ISBN 0-19-850194-3. Chapter 2.

Richard Kaye and Robert Wilson, *Linear Algebra*, OUP 1998, ISBN 0-19-850237-0. Chapters 2–13. [Chapters 6, 7 are not entirely relevant to our syllabus, but are interesting.]

Alternative and further reading:

Joseph J Rotman, *A First Course in Abstract Algebra*, (Second edition), Prentice Hall, 2000, ISBN 0-13-011584-3. Chapters 1, 3.

I N Herstein, *Topics in Algebra* (Second edition) Wiley 1975, ISBN 0-471-02371-X. Chapter 3. [Harder than some, but an excellent classic. Widely available in Oxford libraries; still in print.]

P M Cohn, *Classic Algebra*, Wiley 2000, ISBN 0-471-87732-8. Various sections. [This is the third edition of his book previously called *Algebra I*.]

David Sharpe, *Rings and Factorization*, CUP 1987, ISBN 0-521-33718-6. [An excellent little book, now sadly out of print; available in some libraries, though.]

Paul R Halmos, *Finite-dimensional Vector Spaces*, (Reprint 1993 of the 1956 second edition), Springer Verlag ISBN 3-540-90093-4. §§1–15, 18, 32–51, 54–56, 59–67, 73, 74, 79. [Now over 50 years old, this idiosyncratic book is somewhat dated but it is a great classic, and well worth reading.]

Seymour Lipschutz and Marc Lipson, *Schaum's Outline of Linear Algebra* (3rd edition, McGraw Hill 2000), ISBN 0-07-136200-2. [Many worked examples.]

C W Curtis, *Linear Algebra—an Introductory Approach*, Springer (4th edition), reprinted 1994

D T Finkbeiner, *Elements of Linear Algebra*, Freeman, 1972 [Out of print, but available in many libraries]

There are very many other such books on abstract and linear algebra in Oxford libraries.

2.2.2 Analysis — Dr Melcher — 24 lectures MT

Aims and Objectives

The theory of functions of a complex variable is a rewarding branch of mathematics to study at the undergraduate level with a good balance between general theory and examples. It occupies a central position in mathematics with links to analysis, algebra, number theory, potential theory, geometry, topology, and generates a number of powerful techniques (for example, evaluation of integrals) with applications in many aspects of both pure and applied mathematics, and other disciplines, particularly the physical sciences.

In these lectures we begin by introducing students to the language of topology before using it in the exposition of the theory of (holomorphic) functions of a complex variable. The central aim of the lectures is to present Cauchy's theorem and its consequences, particularly series expansions of holomorphic functions, the calculus of residues and its applications.

The course concludes with an account of the conformal properties of holomorphic functions and applications to mapping regions.

Learning Outcomes

Students will have been introduced to point-set topology and will know the central importance of complex variables in analysis. They will have grasped a deeper understanding of differentiation and integration in this setting and will know the tools and results of complex analysis including Cauchy's Theorem, Cauchy's integral formula, Lioville's theorem, Laurent's expansion and the theory of residues.

Synopsis

(1-4) Topology of Euclidean space and its subsets, particularly $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$. Open sets, closed sets, subspace topology; continuous functions and their characterisation in terms of preimages of open or closed sets; connected sets, path-connected sets; compact sets, Heine-Borel Theorem (covered in Chapter 3 of Apostol).

(5-7) Review of algebra and geometry of the complex plane. Complex differentiation. Holomorphic functions. Cauchy-Riemann equations. Real and imaginary parts of a holomorphic function are harmonic.

(8-11) Path integration. Power series and differentiation of power series. Exponential function and logarithm function. Fractional powers - examples of multifunctions.

(12-13) Cauchy's Theorem. (Sketch of proof only -students referred to various texts for proof) Fundamental Theorem of Calculus in the path integral/holomorphic situation.

(14-16) Cauchy's Integral formulae. Taylor expansion. Liouville's Theorem. Identity Theorem. Morera's Theorem

(17-18) Laurent's expansion. Classification of singularities. Calculation of principal parts, particularly residues.

(19-21) Residue theorem. Evaluation of integrals by the method of residues (straight forward examples only but to include use of Jordan's lemma and simple poles on contour of integration).

(22-23) Conformal mapping: Möbius functions, exponential functions, fractional powers; mapping regions (not Christoffel transformations or Jowkowski's transformation).

(24) Summary and Conclusion.

Reading

Main texts

Apostol, *Mathematical Analysis*, Addison–Wesley (1974)(Chapter 3 for the topology).

Priestley, *Introduction to Complex Analysis*, (second edition, 2003), Oxford Science Publications.

Jerold E Marsden, Michael J Hoffman, *Basic Complex Analysis*, W.H. Freeman (1996).

Further Reading

Theodore Gamelin, *Complex Analysis*, Springer (2000).

Reinhold Remmert, *Theory of Complex Functions*, Springer (1989) (Graduate Texts in Mathematics 122)

Mark J Ablowitz, Athanassios S Focas, *Complex Variables, Introduction and Applications*, Cambridge Texts in Applied Mathematics (2nd edition 2003).

3 OPTIONS

3.1 Syllabus

3.1.1 Groups in Action

Groups: subgroups, normal subgroups and quotient groups; elementary results concerning symmetric and alternating groups; important examples of groups, including the general Linear groups. Actions of groups on sets; examples, including coset spaces, groups acting on themselves by translation and conjugation, the Möbius groups, Linear groups acting on sets of subspaces. Orbits, transitivity, stabilisers, equivalence of a transitive space with a coset space, kernels of such actions, examples. Symmetry groups of geometric objects including regular polyhedra. Combinatorial applications: the formula for counting orbits and its use in enumeration problems.

3.1.2 Introduction to Fields

Fields, subfields, finite extensions; examples. Degree of an extension, the Tower Theorem. Simple algebraic extensions; splitting fields, uniqueness (proof not to be examined); examples. Characteristic of a field. Finite fields: existence; uniqueness (proof not to be examined). Subfields. The multiplicative group of a finite field. The discrete logarithm.

3.1.3 Number Theory

The ring of integers; congruences; rings of integers modulo n; the Chinese Remainder Theorem. Wilson's Theorem; Fermat's Little Theorem for prime modulus. Euler's phi-function; Euler's generalisation of Fermat's Little Theorem to arbitrary modulus. Quadatic residues modulo primes. Quadratic reciprocity. Factorisation of large integers; basic version of the RSA encryption method.

3.1.4 Integration

Measure spaces. Outer measure, null set, measurable set. The Cantor set. Lebesgue measure on the real line. Counting measure. Probability measures. Construction of a non-measurable set (non-examinable). Measurable function, simple function, integrable function. Reconciliation with the integral introduced in Moderations.

A simple comparison Theorem. Integrability of polynomial and exponential functions over suitable intervals. Fatou's Lemma (proof not examinable). Montone Convergence Theorem (proof not examinable). Dominated Convergence Theorem. Corollaries and applications of the Convergence Theorems (including term by term integration of series).

Theorems of Fubini and Tonelli (proofs not examinable). Differentiation under the integral sign.

Fundamental Theorem of Calulus. Change of variable. The 'devil's staircase'.

Brief introduction to L^p spaces. Hölder and Minkowski inequalities (proof not examinable).

3.1.5 Topology

Metric spaces. Examples to include metrics derived from a norm on a real vector space, particularly l^1, l^2, l^∞ norms on $\mathbb{R}^n$, the *sup* norm on the bounded real-valued functions on a set, and on the bounded continuous real-valued functions on a metric space. Continuous functions (ϵ, δ definition). Uniformly continuous functions. Open balls, open sets, accumulation points of a set. Completeness (but not completion). Contraction mapping theorem. Completeness of the space of bounded real-valued functions on a set, equipped with the *sup* norm, and the completeness of the space of bounded continuous real-valued functions on a metric space, equipped with the *sup* metric.

Axiomatic definition of an abstract topological space in terms of open sets. Continuous functions, homeomorphisms. Closed sets. Accumulation points of sets. Closure of a set ($\bar{A} = A$ together with its accumulation points). Continuity if $f(\bar{A}) \subseteq \overline{f(A)}$. Examples to include metric spaces (definition of topological equivalence of metric spaces), discrete and indiscrete topologies, subspace topology, cofinite topology, quotient topology. Base of a topology. Product topology on a product of two spaces and continuity of projections. Hausdorff topology.

Connected spaces: closure of a connected space is connected, union of connected sets is connected if there is a non-empty intersection, continuous image of a connected space is connected. Path-connectedness implies connectedness. Connected open subset of a normed vector space is path-connected.

Compact sets, closed subset of a compact set is compact, compact subset of a Hausdorff space is closed. Heine-Borel Theorem in $\mathbb{R}^n$. Product of two compact spaces is compact. A continuous bijection from a compact space to a Hausdorff space is a homeomorphism. Equivalence of sequential compactness and abstract compactness in metric spaces.

Further discussion of quotient spaces: simple classical geometric spaces such as the torus and Klein bottle.

3.1.6 Multivariable Calculus

Definition of a derivative of a function from $\mathbb{R}^m$ to $\mathbb{R}^n$; examples; elementary properties; partial derivatives; the chain rule; the gradient of a function from $\mathbb{R}^m$ to $\mathbb{R}$; Jacobian. Continuous partial derivatives imply differentiability. Higher order derivatives; symmetry of multiple partial derivatives.

The inverse function theorem (proof non-examinable). The implicit function theorem (statement only).

The definition of a submanifold of $\mathbb{R}^m$, its tangent space at a point. Examples, defined parametrically and implicitly, including curves and surfaces in $\mathbb{R}^3$, and $SO(n)$.

Lagrange multipliers.

3.2 Synopses of Lectures

3.2.1 Groups in Action — Dr Collins — 8 lectures HT

Aims and Objectives

The set of bijections on any set form a group under the composition of functions; in the case of finite sets, we get the symmetric groups that were met in Mods. More generally, if one imposes some structure on the set, one can study the groups of those bijections that preserve that structure. For example, in the case of vector spaces, the set of all invertible linear transformations will form a group, while for $\mathbb{R}^2$, with the further restriction of preserving Euclidean distance, the dihedral groups were met in Mods, which preserve a regular polygon.

The aims and objectives of this course are to explore these ideas further, both in an algebraic and a geometric context. Much will be "example-driven", building upon the above examples. Further examples that have been met previously are given by studying just a group; left and right translations, or conjugation by an element, also yield bijections,and the theory that will be developed behind these examples allows results to be obtained about the groups themselves.

Synopsis

Reminders about groups, subgroups, normal subgroups and quotient groups; reminders about the symmetric and alternating groups; important examples of groups, including the general Linear groups. [1-2 lectures]

Actions of groups on sets, equivalence of actions, examples including coset spaces, groups acting on themselves by translation and conjugation, Möbius groups. [1 lecture]

Linear groups acting on sets of subspaces. [1 lecture]

Orbits, transitivity, stabilisers, equivalence of a transitive action with a coset space, kernels of such actions, examples. [2 lectures]

Symmetry groups of geometric objects including regular polyhedra. Combinatorial applications: Combinatorial applications: the formula for counting orbits and its use in enumeration problems.

[2 lectures]

Reading

Peter M Neumann, G A Stoy, E C Thompson, *Groups and Geometry*, (OUP 1994, reprinted 2002), ISBN 0-19-853451-5. Chapters 1-9, 15.

Further Reading

Geoff Smith, Olga Tabachnikova, *Topics in Groups Theory* (Springer Undergraduate Mathematics Series, 2002) ISBN 1-85233-2. Chapter 3.

M A Armstrong, *Groups and Symmetry*, (Springer 1988), ISBN 0-387-96675-7. Chapters 1-19.

Joseph J Rotman, *A First Course in Algebra*, (Second Edition, Prentice Hall, 2000). Chapter 2

3.2.2 Introduction to Fields — Dr Collins — 8 lectures HT

Aims and Objectives

Informally, finite fields are generalisations of systems of real numbers such as the rational or the real numbers— systems in which the usual rules of arithmetic (including those for division) apply. Formally, fields are commutative rings with unity in which division by non-zero elements is always possible. It is a remarkable fact that the finite fields may be completely classified. Furthermore, they have classical applications in number theory, algebra, geometry, combinatorics, and coding theory, and they have newer applications in other areas. The aim of this course is to show how their structure may be elucidated, and to present the main theorems about them that lead to their various applications.

Synopsis

Fields, subfields, finite extensions; examples. Degree of an extension, the Tower Theorem. [2 lectures]

Simple algebraic extensions; splitting fields, uniqueness (proof only sketched); examples. [2 lectures]

Characteristic of a field. Finite fields: existence; uniqueness (proof only sketched). Subfields. The multiplicative group of a finite field. The discrete logarithm. [4 lectures]

Reading

Joseph J Rotman, *A First Course in Abstract Algebra*, (Second Edition, Prentice Hall, 2000), ISBN 0-13-011584-3. Chapters 1,3.

Dominic Welsh, *Codes and Cryptography*, Oxford University Press 1988, ISBN 0-19853-287-3. Chapter 10.

Further Reading

Peter J Cameron, *Introduction to Algebra*, Oxford Univeristy Press 1998, ISBN 0-19-850194-3 Parts of 2.4, 7.3.

I N Herstein, *Topics in Algebra*, (Wiley 1975). ISBN 0-471-02371-X 5.1, 5.3, 7.1. [Harder than some, but an excellent classic. Widely available in Oxford libraries; still in print.]

P M Cohn, *Classic Algebra*, (Wiley 2000), ISBN 0-471-87732-8, parts of Chapter 6. [This is the third edition of his book on abstract algebra, in Oxford libraries.]

There are many other such books on abstract algebra in Oxford libraries

3.2.3 Number Theory — Prof. Heath-Brown — 8 lectures TT

Aims and Objectives

Number theory is one of the oldest parts of mathematics. For well over two thousand years it has attracted professional and amateur mathematicians alike. Although notoriously 'pure' it has turned out to have more and more applications as new subjects and new technologies have developed. Our aim in this course is to introduce students to some classical and important basic ideas of the subject.

Synopsis

The ring of integers; congruences; rings of integers modulo n; the Chinese Remainder Theorem. [2 lectures]

Wilson's Theorem; Fermat's Little Theorem for prime modulus; Euler's generalization of Fermat's Little Theorem to arbitrary modulus; primitive roots.[2 lectures]

Quadratic residues modulo primes. Quadratic reciprocity. [2 lectures]

Factorisation of large integers; basic version of the RSA encryption method. [2 lectures]

Reading

Alan Baker, *A Concise Introduction to the Theory of Numbers*, Cambridge University Press ISBN: 0521286549 Chapters 1,3,4

David Burton, *Elementary Number Theory*, (McGraw-Hill, 2001).

Dominic Welsh, *Codes and Cryptography*, Oxford University Press 1988, ISBN 0-19853-287-3. Chapter 11

3.2.4 Integration — Prof. Etheridge — 16 lectures HT

Aims & Objectives

The course will emphasize two aspects of the theory of integration:

1. The theory of integration is extremely flexible and does not just apply to functions on $\mathbb{R}$.

2. The power of the Convergence Theorems.

Although we shall be at pains to point out that Lebesgue integration, probability theory and summation of series can all be seen as part of one framework, students who have not studied probability in Mods will *not* be disadvantaged.

Students will not be expected to learn technical proofs and indeed there will be very few proofs in the course (although there will be sufficient examples of what can go wrong to justify the need for conditions in the Theorems). The treatment of measures will (necessarily at this stage) be extremely light touch. To some extent it should be seen as a methods course. We will not allow ourselves to be detained by technical details. We will emphasize that this is an *extension* of what went before, not a replacement (so their techniques of integration are still valid).

The *principal aim* of the course is to familiarise students with the statements and applications of the main convergence theorems.

Synopsis

Motivation: Why do we need a more general theory of integration?
The notion of measure.
Key examples: Lebesgue measure, probability measure, counting measure. (*No previous knowledge of probability will be assumed.*)
Integrable functions (via simple functions). Reconcilliation with Mods Analysis III.
Comparison Theorem.
Fatou's Lemma.
Montone Convergence Theorem.
Dominated Convergence Theorem.
Corollaries and applications of the Convergence Theorems (term by term integration of series etc).
Fubini–Tonelli.
Differentiation under the integral sign.
A very brief introduction to L^p spaces. Hölder and Minkowski inequalities.

Reading

There will be a complete set of typed notes, but students should also refer to texts. Suggestions for useful references include:

M. Capinski & E. Kopp, *Measure, Integrability and Probability*, 2nd Edition. Springer (2004).
E.M. Stein & R. Shakarchi, *Real Analysis: Measure Theory, Integration and Hilbert Spaces*, Princeton Lectures in Analysis III, Princeton University Press (2005).
F. Jones, *Lebesgue Integration on Euclidean Space*, 2nd Edition, Jones & Bartlett (2000).
H.L. Royden, *Real Analysis*, 3rd Edition, Macmillan (1988).

Further reading

R.G. Bartle, *The Elements of Integration*, Wiley (1966).
D.S. Kurtz & C.W.Swartz, *Theories of Integration*, Series in Real Analysis Vol.9, World Scientific (2004).

3.2.5 Topology — Dr Drutu — 16 lectures HT

Learning Outcomes

The ideas, concepts and constructions in general topology arose from extending the notions of continuity and convergence on the real line to more general spaces. The first class of general spaces to be studied in this way were metric spaces, a class of spaces which includes many of the spaces used in analysis and geometry. Metric spaces have a distance function which allows the use of geometric intuition and gives them a concrete feel. They allow us to introduce much of the vocabulary used later and to understand the formulation of continuity which motivates the axioms in the definition of an abstract topological space.

The axiomatic formulation of a topology leads to topological proofs of simplicity and clarity often improving on those given for metric spaces using the metric and sequences. There are many examples of topological spaces which do not admit metrics and it is an indication of the naturality of the axioms that the theory has found so many applications in other branches of mathematics and spheres in which mathematical language is used.

The outcome of the course is that a student should understand and appreciate the central results of general topology and metric spaces, sufficient for the main applications in geometry, number theory, analysis and mathematical physics, for example.

Synopsis

Metric spaces. Examples to include metrics derived from a norm on a real vector space, particularly l^1, l^2, l^∞ norms on $\mathbb{R}^n$, the *sup* norm on the bounded real-valued functions on a set, and on the bounded continuous real-valued functions on a metric space. Continuous functions (ϵ, δ definition). Uniformly continuous functions. Open balls, open sets, accumulation points of a set. Completeness (but not completion). Contraction mapping theorem. Completeness of the space of bounded real-valued functions on a set, equipped with the *sup* norm, and the completeness of the space of bounded continuous real-valued functions on a metric space, equipped with the *sup* metric. [3 lectures].

Axiomatic definition of an abstract topological space in terms of open sets. Continuous functions, homeomorphisms. Closed sets. Accumulation points of sets. Closure of a set ($\bar{A} = A$ together with its accumulation points). Continuity if $f(\bar{A}) \subseteq \overline{f(A)}$. Examples to include metric spaces (definition of topological equivalence of metric spaces), discrete and indiscrete topologies, subspace topology, cofinite topology, quotient topology. Base of a topology. Product topology on a product of two spaces and continuity of projections. Hausdorff topology. [5 lectures]

Connected spaces: closure of a connected space is connected, union of connected sets is connected if there is a non-empty intersection, continuous image of a connected space is connected. Path-connectedness implies connectedness. Connected open subset of a normed vector space is path-connected. [2 lectures]

Compact sets, closed subset of a compact set is compact, compact subset of a Hausdorff space is closed. Heine-Borel Theorem in $\mathbb{R}^n$. Product of two compact spaces is compact.

A continuous bijection from a compact space to a Hausdorff space is a homeomorphism. Equivalence of sequential compactness and abstract compactness in metric spaces. [4 lectures]

Further discussion of quotient spaces explaining some simple classical geometric spaces such as the torus and Klein bottle. [2 lectures]

Reading

W A Sutherland, *Introduction to Metric and Topological Spaces*, OUP (1975). Chapters 2-6, 8, 9.1-9.4
(New edition to appear shortly.)

Further Reading

B Mendelson, *Introduction to Topology*, Allyn and Bacon (1975). (cheap paper back edition available).

G Buskes, A Van Rooij, *Topological Spaces*, Springer (1997).

J R Munkres, *Topology, A First Course*, Prentice Hall (1974).

N Bourbaki, *General Topology*, Springer (1998).

R Engelking, *General Topology*, Heldermann Verlag Berlin (1989) (for the last word, goes far beyond the syllabus).

3.2.6 Multivariable Calculus — Prof. Niethammer — 8 lectures TT

Aims and Objectves

In this course, the notion of a derivative for a function $f: \mathbb{R}^m \to \mathbb{R}^n$ is introduced. Roughly speaking, this is an approximation of the function near each point in $\mathbb{R}^n$ by a linear transformation. This is a key concept which pervades much of mathematics, both pure and applied.

Synopsis

Definition of a derivative of a function from $\mathbb{R}^m$ to $\mathbb{R}^n$; examples; elementary properties; partial derivatives; the chain rule; the gradient of a function from $\mathbb{R}^m$ to $\mathbb{R}$; Jacobian. Continuous partial derivatives imply differentiability. Higher order derivatives; symmetry of multiple partial derivatives. [3 lectures]

The inverse function theorem (proof non-examinable), via the contraction mapping theorem. The implicit function theorem (statement only) [2 lectures]

The definition of a submanifold of $\mathbb{R}^m$. Its tangent space at a point. Examples, defined parametrically and implicitly, including curves and surfaces in $\mathbb{R}^3$, and $SO(n)$. [2 lectures]

Lagrange multipliers. [1 lecture]

Reading

Theodore Shifrin, *Multivariable Mathematics*, Wiley (2005). Chapters 3-6.

T M Apostol, *Mathematical Analysis: Modern Approach to Advanced Calculus (World Students)* Addison Wesley (1975) Chapters 12 and 13.

S Dineen, *Multivariate Calculus and Geometry*, (Springer 2001) Chapters 1-4.

J J Duistermaat and J A C Kolk, *Multidimensional Real Analysis I, Differentiation*, Cambridge University Press (2004).

B D Craven, *Functions of Several Variables* (Chapman & Hall 1981) ISBN: 0-412-23330-4 (out of print) Chapters 1-3.

Further reading

William R Wade, *An Introduction to Analysis*, 2nd Edition, 2000, Prentice Hall, Chapter 11

C Dixon, *Advanced Calculus* (Wiley 1981) (out of print)

M P Do Carmo, *Differential Geometry of Curves and Surfaces*, 1976.

Stephen G. Krantz and Harold R. Parks, *The Implicit Function Theorem: History, Theory and Applications* (Birkhaeuser 2002)