Joint Metrizability of Spaces on Families of Subspaces

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Definition. Let X be a topological space, and $\mathcal{F}$ be a family of subspaces of X. We say that X is *jointly metrizable on* $\mathcal{F}$, or that X is $\mathcal{F}$-*metrizable*, if there is a metric d on the set X such that d *metrizes all subspaces of* X *which are in* $\mathcal{F}$, that is, the restriction of d to A generates the subspace topology on A, for each $A \in \mathcal{F}$.

Example: Take an arbitrary non-metrizable space X, and let d be the standard discrete metric on X, that is, $d(x, y) = 1$, for any distinct points x, y of X. Then d doesn't metrize the space X, since X is not metrizable, but d obviously metrizes every discrete subspace of X. Thus, *every topological space X is jointly metrizable on all discrete subspaces of X*.

Proposition: Suppose that $\gamma = \{X_\alpha : \alpha \in A\}$ is a disjoint family of metrizable subspaces of a space X. Then X is jointly metrizable on γ.

Proof. For each $\alpha \in A$, fix a metric d_α on the subspace X_α which generates the topology of this subspace. We may assume that $d_\alpha(x, y) \leq 1$, for any distinct $x, y \in X_\alpha$. Now we define a metric d on X as follows: $d(x, y) = d_\alpha(x, y)$ if $x, y \in X_\alpha$ for some $\alpha \in A$; otherwise, we put $d(x, y) = 1$ if x and y are distinct. Clearly, d is a metric on X which metrizes every subspace X_α in γ.

Compactly Metrizable Spaces

Definition. We say that a topological space X is *compactly metrizable*, or that X is *jointly metrizable on compacta*, if X is jointly metrizable on the family of all compact subspaces of X.

The following result is obvious.

Proposition. Every subspace of a compactly metrizable space is compactly metrizable.

Proposition. Any compactly metrizable space X is a T_1-space.

Proof. Since every compact subspace of X is metrizable, every compact subspace of X is closed in X. But every finite space is compact. Hence, X is a T_1-space.

Remark. Not every compactly metrizable space is Hausdorff.

Example: Take an uncountable set X with the cocountable topology in which the only closed sets are countable sets and the set X. The space X is not Hausdorff, since any two non-empty open subsets of X have a non-empty intersection. On the other hand, every countable subspace of X is closed in X and discrete. Thus, all compact subspaces of X are discrete. Therefore, X is compactly metrizable.

Theorem. Every submetrizable space X is compactly metrizable.

Proof. We can fix a one-to-one continuous mapping f of X onto a metrizable space Y. Let d be a metric on the space Y generating the topology of Y. For any $x_1, x_2 \in X$, Put $\rho(x_1, x_2) = d(f(x_1), f(x_2))$. Clearly, ρ is a metric on X. Take any compact subspace C of the space X. The restriction of f to C is a homeomorphism of C onto the subspace $f(C)$ of the space Y, since C is compact and Y is Hausdorff. Clearly, the restriction of f to C is an isometry of C, metrized by the restriction of ρ, onto $f(C)$, metrized by the restriction of d. Since d metrizes the space Y, it follows that the restriction of ρ to C metrizes the subspace C of X. Hence, X is compactly metrizable.

Example. Since Sorgenfrey line is submetrizable, it follows that using the above theorem that Sorgenfrey line is compactly metrizable, while it is not metrizable.

Corollary. Every countable Tychonoff space is compactly metrizable.

Corollary. Every Tychonoff space with a countable network is compactly metrizable.

Theorem. The product of any countable family of compactly metrizable spaces is compactly metrizable.

For k-spaces, we have the following characterization of compactly metrizable spaces.

Theorem. A k-space X is compactly metrizable if and only if it is submetrizable.

Proof. It remains to prove the necessity. Let τ be the topology of the space X, and d be a metric on X which metrizes all compact subspaces of X. We denote by τ_d the topology on X generated by d. Take any $V \in \tau_d$, and any compact subspace C of the space X. Clearly, $V \cap C$ belongs to the topology generated by d on C. But this topology is precisely the topology of the subspace C of (X, τ). Thus, $V \cap C$ is open in C, for every compact subspace C of (X, τ). Since (X, τ) is a k-space, it follows that $V \in \tau$. Hence, $\tau_d \subset \tau$, i.e. the space X is submetrizable.

Countably Metrizable Spaces

Definition: We say that a topological space X is *countably metrizable* if X is jointly metrizable on the family of all countable subspaces of X.

The following result is obvious.

Proposition. Every subspace of a countably metrizable space is countably metrizable.

Question. Is every submetrizable space countably metrizable?

Example. Take any non metrizable countable Tychonoff space X. Clearly, any such space is submetrizable. On the other hand, it is not countably metrizable.

Moreover, the above example illustrates that joint metrizability on compact subspaces does not imply joint metrizability on countable subspaces.

Question. Is every countably metrizable space compactly metrizable?

Note that Sorgenfrey line is first countable. Thus, every countable subspace of Sorgenfrey line is metrizable.

Question. Is Sorgenfrey line countably metrizable?

Proposition. Suppose that X is a space of countable tightness, and that d is a metric on the set X which metrizes all countable subspaces of the space X. Then d generates the topology of the space X.

Proof. Let $x \in X$ and $A \subset X$. We shall show that $x \in \overline{A}$ if and only if $d(x, A) = 0$.

First, assume that $d(x, A) = 0$. For every positive integer n, fix $a_n \in A$ such that $d(x, a_n) < 1/n$. This provides us with a countable subset B of A such that $d(x, B) = 0$. Put $C = B \cup \{x\}$. The set C is also countable. By the assumption, the restriction of d to C metrizes the subspace C of the space X. Therefore, it follows from $d(x, B) = 0$ that $x \in \overline{B}$. Hence, $x \in \overline{A}$, since $B \subset A$.

Conversely, assume that $x \in \overline{A}$. Since the tightness of X is countable, it follows that $x \in \overline{B}$, for some countable subset B of A. The set $C = B \cup \{x\}$ is countable. Therefore, by the assumption, the restriction of d to C metrizes the subspace C of the space X. Hence, $d(x, B) = 0$, which implies that $d(x, A) = 0$. Thus, d generates the topology of the space X.

Theorem. A topological space X is metrizable if and only if X is countably metrizable and the tightness of X is countable.

Theorem. Every countably metrizable space X is jointly metrizable on all countably compact subspaces of X.

Corollary. Every countably metrizable space is compactly metrizable.

Corollary. Every countably metrizable countably compact space is metrizable.

The above corollary says more. It says that compactness and countable compactness are equivalent in the class of countably metrizable spaces.

Theorem. If a k-space X is countably metrizable, then X is metrizable.

Theorem. The product of any countable family of countably metrizable spaces is countably metrizable.

Jointly Partially Metrizable Spaces

Definition. A topological space X is *jointly partially metrizable*, or a JPM*-space*, if there is a metric d on X which metrizes all metrizable subspaces of X.

The following result is obvious.

Proposition. Every subspace of a JPM-space is a JPM-space.

Proposition. If all metrizable subspaces of a space X are discrete, then X is a JPM-space.

If X is a space with no non-trivial convergent sequences, then all metrizable subspaces of X are discrete, and hence, X is a JPM-space. Therefore, every extremally disconnected Hausdorff space is a JPM-space. In particular, we have:

Example. The Stone-Čech compactification $\beta(\omega)$ of the discrete space ω of natural numbers is a JPM-space. This space is compact and not metrizable. Hence, it is not compactly metrizable.

Proposition. Suppose that X is a space, and that d is a metric on X which metrizes all countable metrizable subspaces of the space X. Then d metrizes every metrizable subspace Y of X.

Corollary. Every countably metrizable space is a JPM-space.

Corollary. Suppose that all countable subspaces of a space X are metrizable. Then X is countably metrizable if and only if X is a JPM-space.

Theorem. Every regular first countable JPM space X is metrizable.

Proof. It is clear that every countable subspace of X is metrizable. Since X is a JPM-space, it follows that X is countably metrizable. But the tightness of X is countable. Hence, it follows that X is metrizable.

Example. The Sorgenfrey line is not a JPM-space by the above theorem since it is first countable, regular, and not metrizable.

This example also demonstrates that a compactly metrizable space need not be a JPM-space.

Theorem. Every Hausdorff sequential JPM-space is metrizable.

Theorem. The product of any countable family of Hausdorff JPM-spaces is a Hausdorff JPM-space.

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