Some properties of remainders of metrizable spaces

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A "space" in this text is a Tychonoff topological space. We discuss properties of remainders of metrizable spaces. Many recent results are presented.

A *remainder* Y of a space X is the subspace $Y = bX \setminus X$ of a Hausdorff compactification bX of X.

One of the major tasks in the theory of compactifications is to find out how the properties of a space X are related to the properties of some or of all remainders of X. We are especially interested in the *invariant* properties of the remainders, that is, in the properties that do not depend on the choice of a compactification.

The most interesting is the case when the space and the remainder are both densely located with respect to each other, that is, when each of them is dense in the compactification considered. Clearly, this happens when X is nowhere locally compact. Usually, we make this assumption below. Then Y is a remainder of X if and only if X is a remainder of Y.

A systematic study of remainders of metrizable spaces have been initiated in [5] and [4]. An important early paper on compactifications and remainders is [10]. Techniques developed in [1], [4], [11], [6], [9], [2], [3] turned out to be quite relevant to the study of remainders.

Notice that rationals and irrationals present a classical example of two metrizable spaces each of which is a remainder of the other one. These spaces are exceptionally nice. They are not only metrizable, but separable as well. They are also homogeneous; in fact, each of them is homeomorphic to a topological group.

Curiously, if a nowhere locally compact metrizable space is large, then it cannot have a metrizable remainder. This is clear from the following theorem which has been proved in [4]:

Theorem 0.1. *if a nowhere locally compact metrizable space X has a remainder Y with a G_δ-diagonal, then both X and Y are separable and metrizable spaces.*

The following question naturally arises: *how to characterize the remainders of metrizable spaces?*

This question turns out to be quite nontrivial. Indeed, let $\mathcal{Q}$ be the space of rationals and $\mathcal{I}$ be the space of irrationals. Consider also the Stone-Čech remainder $\mathcal{S}$ of $\mathcal{Q}$. The spaces $\mathcal{I}$ and $\mathcal{S}$ are very different in many respects. In particular, $\mathcal{I}$ is metrizable, while every metrizable subspace of $\mathcal{S}$ is finite. However, $\mathcal{I}$ and $\mathcal{S}$ have the same remainder $\mathcal{Q}$, which is a very standard, simple space. This suggests that it is impossible to characterize metrizability of a space by a natural property $\mathcal{P}$ of its remainders, since otherwise the space $\mathcal{Q}$, being a quite standard remainder of the metrizable space $\mathcal{I}$, would have $\mathcal{P}$, which would imply that the space $\mathcal{S}$ is metrizable, a contradiction.

Even though the above argument is informal, it tells us that there is very little hope to characterize metrizability of spaces by a natural property of its remainders. Thus, we have to concentrate on two less ambitious tasks: two find necessary properties of remainders of metrizable spaces, and to determine sufficient conditions for metrizability of a space in terms of remainders.

In the first direction, one such condition can be extracted from the next theorem of M. Henriksen and J. Isbell on remainders [10]:

Theorem 0.2. *A space X is of countable type if and only if any (some) remainder of X is Lindelöf [10].*

A space X is of *countable type* if every compact subspace P of X is contained in a compact subspace $F \subset X$ which has a countable base of open neighbourhoods in X. All metrizable spaces and all locally compact spaces, as well as all Čech-complete spaces and, more generally, all p-spaces are of countable type [1]. Every space with a point-countable base is also of countable type.

It follows from Theorem 0.2 that every remainder of a metrizable space is Lindelöf.

Since the class of metrizable spaces is contained in the class of spaces of countable type, we come to the following problem: *find a natural, and as* *narrow as possible, subclass $\mathcal{C}$ of the class of Lindelöf spaces such that every remainder of any metrizable space is in $\mathcal{C}$.*

Several nice subclasses of the class of Lindelöf spaces are known and play an important role in general topology. One of them is the class of Lindelöf p-spaces, another is the class of Lindelöf Σ-spaces. A *Lindelöf p-space* is a preimage of a separable metrizable space under a perfect mapping. A space Y is a *Lindelöf Σ-space* if Y is an image of a Lindelöf p-space under a continuous mapping. The class of Lindelöf Σ-spaces has been introduced by K. Nagami in [11]. All separable metrizable spaces are Lindelöf p-spaces, and all Lindelöf p-spaces are Lindelöf Σ-spaces.

Clearly, every separable metrizable space has a separable metrizable remainder. Here is a parallel result from [2]:

Theorem 0.3. *If X is a Lindelöf p-space, then every remainder of X is a Lindelöf p-space.*

Corollary 0.4. *[2] A nowhere locally compact space is a Lindelöf p-space if and only if every (some) remainder of X is a Lindelöf p-space.*

It follows from the last result that a nowhere locally compact space with a G_δ-diagonal is separable and metrizable if and only if every (some) remainder of X is a Lindelöf p-space. Hence, it is not true that every remainder of a non-separable metrizable space is a Lindelöf p-space. At this moment it is natural to ask whether every remainder of any metrizable space is, at least, a Lindelöf Σ-space? This question has been answered negatively in [4] where some broad sufficient conditions for a metrizable space to have a Lindelöf Σ-remainder were given.

Theorem 0.5. *Suppose that X is a metrizable space, and that Y is an arbitrary remainder of X. Suppose further that $|Y| \leq 2^\omega$. Then Y is a Lindelöf Σ-space.*

Thus, it is indeed natural to look for a topological property $\mathcal{P}$ such that every remainder of a metrizable space has $\mathcal{P}$, every Lindelöf Σ-space also has $\mathcal{P}$, and the class of all spaces with the property $\mathcal{P}$ is considerably more narrow than the class of Lindelöf spaces.

The next concept, introduced in [4], plays a crucial role in the proof of Theorem 0.1.

A space X is *charming*, if there is a Lindelöf Σ-space Y such that Y is a subspace of X and, for each open neighbourhood U of Y in X, the subspace $X \setminus U$ is a Lindelöf Σ-space. It has been established in [4] that *every remainder of a metrizable space is a charming space.*

All separable metrizable spaces and all Lindelöf p-spaces are charming spaces. This is so, since every Lindelöf Σ-space is a charming space. It is easily seen that the converse is not true. However, every charming space is Lindelöf.

The class of charming spaces has some nice stability properties. An arbitrary image of a charming space under a continuous mapping is a charming space. Any preimage of a charming space under a perfect mapping is a charming space. A subspace of a charming space needn't be a charming space, but any closed subspace of a charming space is a charming space. However, the product of two charming spaces needn't be a charning space.

Recall that the *pseudocharacter* of a space X is countable if every $x \in X$ is a G_δ-point in X.

A family $\mathcal{S}$ of sets is said to be a *T_0-separator* for a pair (A, B) of sets if for every $x \in A$ and every $y \in B$ there exists $P \in \mathcal{S}$ such that $x \in P$ and $y \notin P$.

Theorem 0.6. *The cardinality of every charming space X of countable pseudocharacter does not exceed 2^ω.*

Proof. Case 1. Let us assume first that X is a Lindelöf Σ-space. Fix a compactification B of X and a countable T_0-separator S for the pair $(X, B \setminus X)$ in B such that all elements of S are compact subsets of B.

Put $\eta = \{\cap\lambda : \lambda \subset S, \cap\lambda \subset X\}$. Then, clearly, $|\eta| \leq 2^\omega$, and $\cup\eta = X$. For each $P \in \eta$ we have $|P| \leq 2^\omega$, since P is a first-countable compactum. Therefore, $|X| \leq 2^\omega$. Thus, if X is a Lindelöf Σ-space, then the conclusion holds.

Let us consider now the general case. Take a Lindelöf Σ-kernel Y of X. Then, by Case 1, $|Y| \leq 2^\omega$. Since the pseudocharacter of X at every point of Y is countable, it follows that there is a family γ of open subsets of X T_0-separating Y from $X \setminus Y$ such that $|\gamma| \leq 2^\omega$. Consider the family $E = \{\cup\lambda : \lambda \subset \gamma, |\lambda| \leq \omega, Y \subset \cup\lambda\}$. Clearly, the cardinality of E is not greater than 2^ω, and $\cup\{X \setminus U : U \in E\} = X \setminus Y$ we recall that every charming space is Lindelöf).

For each $U \in E$, $X \setminus U$ is a Lindelöf Σ-space of countable pseudocharacter; therefore, by Case 1, $|X \setminus U| \leq 2^\omega$. It follows that $|X \setminus Y| \leq 2^\omega$, and finally, that $|X| \leq 2^\omega$. $\square$

Theorem 0.1 has been improved as follows:

Theorem 0.7. *If X is a metrizable space, and Y is a remainder of X in a compactification bX such that Y has a G_δ-diagonal, then Y is separable and metrizable, and bX is an Eberlein compactum.*

A proof of this theorem depends on the following two results:

Theorem 0.8. *The closure of any countable subset in an arbitrary remainder of a metrizable space is a Lindelöf p-space.*

Theorem 0.9. *Suppose that X is a metrizable space, and that Y is an arbitrary remainder of X. Suppose further that C is a subset of Y such that $|C| \leq 2^\omega$. Then the closure of C in Y is a Lindelöf Σ-space.*

In particular, according to Theorem 0.8, *if a remainder of a metrizable space is separable, then this remainder is a Lindelöf p-space.*

If the cardinality of a remainder of a metrizable space doesn't exceed 2^ω, then this remainder is a Lindelöf Σ-space.

Some of the theorems on remainders of metrizable spaces can be extended to paracompact p-spaces or to spaces with a σ-disjoint base.

Theorem 0.10. *Suppose that X is a space with a σ-disjoint base $\mathcal{B}$ such that $|\mathcal{B}| \leq 2^\omega$, and that Y is a remainder of X in a compactification bX of X. Then Y is a Lindelöf Σ-space.*

Theorem 0.11. *Suppose that X is a space with a σ-disjoint base $\mathcal{B}$, and that Y is a remainder of X in a homogeneous compactification bX of X. Then Y is a Lindelöf σ-space.*

Corollary 0.12. *Suppose that a space X is the union of a countable family η of dense metrizable subspaces and that $|X| \leq 2^\omega$. Then every remainder of X in a compactification is a Lindelöf Σ-space.*

In the opposite direction, we have:

Theorem 0.13. *If a perfect space X has a remainder which is a Lindelöf Σ-space, then X is a p-space.*

Recall that a space X is *perfect* if every closed subset of X is a G_δ-set in X.

Theorem 0.14. *Suppose that X is a Lindelöf space with a σ-disjoint base $\mathcal{B}$, and that Y is a remainder of X in a compactification bX of X. Then Y is a Lindelöf Σ-space.*

Fix a space Z and an arbitrary family $\mathcal{S}$ of open subsets of Z, and let $\mathcal{S}_\delta$ be the family of all sets that can be represented as the intersection of some subfamily of $\mathcal{S}$. We will say that the family $\mathcal{S}$ is a *source* of a subspace X in Z if X is the union of some subfamily of $\mathcal{S}_\delta$.

We call a Tychonoff space X an *s-space* if it has a countable source in some compactification of X [6].

Every Lindelöf p-space is an s-space. However, an s-space needn't be a p-space [6].

The next statement shows how s-spaces are related to Lindelöf Σ-spaces.

Proposition 0.15. *A space X is an s-space if and only if any (some) remainder of X is a Lindelöf Σ-space.*

The next result was obtained in [6]:

Theorem 0.16. *If a perfect space X is an s-space, then X is a p-space.*

The next result immediately follows from Proposition 0.15 and Theorem 0.16.

Corollary 0.17. *If a perfect space X has a remainder which is a Lindelöf Σ-space, then X is a p-space.*

Theorem 0.23 can be used to show that certain Lindelöf spaces cannot be represented as remainders of metrizable spaces. For example, we see that no Lindelöf version L of the Michael line is a remainder of a metrizable space, since it has a G_δ-diagonal, but is not metrizable. Observe that any version of the Michael line has a metrizable remainder. We see that the two properties: to have a metrizable remainder, and to be a remainder of a metrizable space, are not equivalent. Clearly, this can occur only in the case of spaces that are locally compact at some point.

Theorem 0.18. *Under the Continuum Hypothesis CH, there exists a Lindelöf space X with a point-countable base such that no remainder of X is a Lindelöf Σ-space.*

Proof. Assuming CH, E.K. van Douwen, F.D. Tall, and W. Weiss have constructed a non-metrizable hereditarily Lindelöf space X with a point-countable base [7].

Claim 1: No remainder Y of X in a compactification is a Lindelöf Σ-space.

Assume the contrary. Then, by Proposition 0.15, X is an s-space. The space X is also perfect, since it is hereditarily Lindelöf. Therefore, it follows from Theorem 0.16 that X is a p-space. Since X is a Lindelöf p-space with a point-countable base, we conclude that X is metrizable, - a contradiction. Claim 1 is established. □

Problem 0.19. *Is there in ZFC a Lindelöf space X with a point-countable base such that no remainder of X is a Lindelöf Σ-space?*

Problem 0.20. *Is any remainder of an arbitrary space with a σ-disjoint base a charming space?*

Let n be a positive natural number. We will call a space X *n-charming*, if X is homeomorphic to a closed subspace of the product of n charming spaces. A space X is *ω-charming*, if X is homeomorphic to a closed subspace of the product of a countable family of charming spaces.

Theorem 0.21. *If a space X is the union of $\leq n$ (of countably many) of dense metrizable subspaces, then its remainder in any compactification is an n-charming space (an ω-charming space, respectively).*

Theorem 0.22. *If a remainder Y of a metrizable space X in a compactification bX is a subspace of a symmetrizable space, then Y is separable and metrizable, and bX is an Eberlein compactum.*

Every compact space can be easily represented as the remainder of a discrete (hence, metrizable) space. Therefore, the next theorem can be interpreted as a generalization of the well known theorem on metrizability of every compact space with a point-countable base [8].

Theorem 0.23. *Suppose that X is a paracompact p-space, and that Y is a remainder of X with a point-countable base. Then Y is separable and metrizable.*

The last result confirms the point of view (see [5], [4]) that remainders of metrizable spaces often behave like compacta. Probably, this is especially true for theorems on cardinal invariants.

There are many interesting open questions ivolving the concept of charming space and its generalization to other topological properties.

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