On the necessity of large cardinals for some constraints on the cardinality of Lindelöf indestructible spaces with small pseudocharacter

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Indestructible compact spaces

Definition (Tall 1995)

A Lindelöf space is *indestructible* if it remains Lindelöf after forcing with any countably closed partial order.

Proposition

A compact space is indestructible if and only if it remains compact after forcing with any countably closed partial order.

Proof.

Since X is Lindelöf in the extension, every open cover $\mathcal{U}$ of X has a countable subcover $\mathcal{U}_0 \subseteq \mathcal{U}$. But the forcing is countably closed, so $\mathcal{U}_0$ is in the ground model; therefore, compactness of X in the ground model implies that $\mathcal{U}_0$ (and hence $\mathcal{U}$) has a finite subcover. □

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Some recent results of Tall and Usuba

Theorem (Tall-Usuba 2012)

If it is consistent with ZFC that there is an inaccessible cardinal, then it is consistent with ZFC that every Lindelöf T_3 indestructible space of weight $\leq \aleph_1$ has size $\leq \aleph_1$.

Theorem (Tall-Usuba 2012)

If it is consistent with ZFC that there is an inaccessible cardinal, then it is consistent with ZFC that the $\aleph_1$-Borel Conjecture holds.

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If it is consistent with ZFC that there is a weakly compact cardinal, then it is consistent with ZFC that there is no Lindelöf T_1 space of pseudocharacter $\leq \aleph_1$ and size $\aleph_2$.

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Lines obtained from trees

Let T be a rooted Hausdorff tree such that all the levels and branches of T have size $\leq \aleph_1$.

We may assume that T is an initial part of $({}^{<\omega_2}(\omega_1+1), \subseteq)$ such that, for each $t \in T$, the set $\{\xi \leq \omega_1 : t^\frown(\xi) \in T\}$ is a successor ordinal.

Let $\prec$ be the ordering on the set

$$L_T = \{\bigcup B : B \text{ is a branch of } T\}$$

naturally induced by the lexicographical ordering of the branches of T, and regard L_T as a linearly ordered topological space.

Lemma (Todorčević 1984)

L_T is compact and $w(L_T) \leq |T|$.

Lemma

$\psi(L_T) \leq \aleph_1$.

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Indestructible compact lines

Proposition

The following conditions are equivalent:

(*a*) L_T is destructible;

(*b*) there is a countably closed forcing that adds a new Dedekind cut in $(L_T, \prec)$;

(*c*) there is a countably closed forcing that adds a new branch of uncountable cofinality in T;

(*d*) T contains a copy of the tree $({}^{<\omega_1}2, \subseteq)$.

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Theorem (Solovay)

If ω_2 is not inaccessible in $\mathbf{L}$, then there is a Kurepa tree.

Theorem

If T is a Kurepa tree, then L_T is a compact Hausdorff indestructible space of weight $\leq \aleph_1$ and size $\geq \aleph_2$.

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If it is consistent with ZFC that there is an inaccessible cardinal, then it is consistent with ZFC that every Lindelöf T_3 indestructible space of weight $\leq \aleph_1$ has size $\leq \aleph_1$.

Corollary

The existence of an inaccessible cardinal and the statement "every Lindelöf T_3 indestructible space of weight $\leq \aleph_1$ has size $\leq \aleph_1$" are equiconsistent.

Inaccessible cardinals and the size of indestructible spaces

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In face of the previous results, this follows from the fact that Tychonoff spaces of weight $\leq \aleph_1$ are embeddable in $[0,1]^{\omega_1}$. □

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Theorem (Jensen 1972, König 2003)

If ω_2 is not weakly compact in $\mathbf{L}$, then there is a sequence $\mathcal{F} = (f_\alpha)_{\alpha \in \mathrm{lim}(\omega_2)}$ of functions $f_\alpha : \alpha \to \alpha$ that is *coherent* — *i.e.* satisfies $f_\alpha =^* f_\beta \restriction \alpha$ for every $\alpha, \beta \in \mathrm{lim}(\omega_2)$ with $\alpha < \beta$ — and such that $(T(\mathcal{F}), \subseteq)$ is an ω_2-Aronszajn tree, where

$$T(\mathcal{F}) = \bigcup_{\xi \in \omega_2} \; \bigcup_{\alpha \in \mathrm{lim}(\omega_2) \setminus \xi} \{f \in {}^\xi\xi : f =^* f_\alpha \restriction \xi\}.$$

Lemma

If $\mathcal{F} = (f_\alpha)_{\alpha \in \lim(\omega_2)}$ is coherent, then no branch of $(T(\mathcal{F}), \subseteq)$ has cofinality ω_1.

Proof.

Suppose, to the contrary, that $B \subseteq T(\mathcal{F})$ is a branch of cofinality ω_1. Let $g = \bigcup B$ and $\gamma = \mathrm{dom}(g) \in \lim(\omega_2)$, and fix a sequence of limit ordinals $(\gamma_\eta)_{\eta \in \omega_1}$ with $\gamma_\eta \nearrow \gamma$. Since $g \restriction \gamma_\eta \in T(\mathcal{F})$ for each $\eta \in \omega_1$ and $\mathcal{F}$ is coherent, we have $g \restriction \gamma_\eta =^* f_{\gamma_\eta} =^* f_\gamma \restriction \gamma_\eta$. It follows that

$$\omega_1 = \bigcup_{k \in \omega} \{\eta \in \omega_1 : |\{\xi \in \gamma_\eta : g(\xi) \neq f_\gamma(\xi)\}| = k\},$$

and so there are $k_0 \in \omega$ and $\eta_0 \in \omega_1$ with $|\{\xi \in \gamma_\eta : g(\xi) \neq f_\gamma(\xi)\}| = k_0$ for every $\eta \in \omega_1 \setminus \eta_0$. But then $|\{\xi \in \gamma : g(\xi) \neq f_\gamma(\xi)\}| = k_0$, which implies $g \in T(\mathcal{F})$, thus contradicting the choice of B. $\square$

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Lemma

Assume CH. If $\mathcal{F} = (f_\alpha)_{\alpha \in \lim(\omega_2)}$ is coherent, then $(T(\mathcal{F}), \subseteq)$ does not have a subtree isomorphic to $({}^{<\omega_1}2, \subseteq)$.

Proof.

Suppose there is $\{g_s : s \in {}^{<\omega_1}2\} \subseteq T(\mathcal{F})$ such that $g_s \subseteq g_t \leftrightarrow s \subseteq t$ for all $s, t \in {}^{<\omega_1}2$. By CH, we have that $\delta = \sup\{\mathrm{dom}(g_s) : s \in {}^{<\omega_1}2\} \in \omega_2$. For each $h \in {}^{\omega_1}2$, let $g_h = \bigcup\{g_{h\restriction\alpha} : \alpha \in \omega_1\}$, and then define $\tilde{g}_h = g_h \cup (f_\delta \restriction (\delta \setminus \mathrm{dom}(g_h)))$; note that $g_h \in T(\mathcal{F})$ by the previous lemma, and thus $\tilde{g}_h \in T(\mathcal{F})$. But the δ-th level of $T(\mathcal{F})$ is the set $\{f \in {}^\delta\delta : f =^* f_\delta\}$, which has cardinality $\aleph_1$; this leads to a contradiction, since this set must contain $\{\tilde{g}_h : h \in {}^{\omega_1}2\}$ and $h \mapsto \tilde{g}_h$ is one-to-one. □

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Theorem

If ω_2 is not weakly compact in $\mathbf{L}$, then there is a Lindelöf T_3 indestructible space of pseudocharacter $\leq \aleph_1$ and size $\aleph_2$.

Proof.

If CH fails, then any subspace $X \subseteq \mathbb{R}$ with $|X| = \aleph_2$ will satisfy the required conditions, since every hereditarily Lindelöf space is indestructible.

If CH holds, consider the space $L_{T(\mathcal{F})}$ obtained from a coherent family $\mathcal{F} = (f_\alpha)_{\alpha \in \lim(\omega_2)}$ given by the Jensen-König Theorem. We have that $L_{T(\mathcal{F})}$ is a compact T_2 indestructible space with $\psi(L_{T(\mathcal{F})}) \leq \aleph_1$; it remains only to show that $|L_{T(\mathcal{F})}| = \aleph_2$. On the one hand, the fact that $T(\mathcal{F})$ is ω_2-Aronszajn implies that $|L_{T(\mathcal{F})}| \geq \aleph_2$; on the other hand, since $T(\mathcal{F})$ does not have branches of uncountable cofinality, it also implies that every branch of $T(\mathcal{F})$ has countable cofinality, and thus $L_{T(\mathcal{F})} \subseteq \{\bigcup C : C \in [T(\mathcal{F})]^{\leq \aleph_0}\}$; since $|T(\mathcal{F})| = \aleph_2$, this yields $|L_{T(\mathcal{F})}| \leq {\aleph_2}^{\aleph_0} = {\aleph_1}^{\aleph_0} \cdot \aleph_2 = \aleph_2$ by CH. $\square$

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Proof.

If CH fails, then any subspace $X \subseteq \mathbb{R}$ with $|X| = \aleph_2$ will satisfy the required conditions, since every hereditarily Lindelöf space is indestructible.
If CH holds, consider the space $L_{T(\mathcal{F})}$ obtained from a coherent family $\mathcal{F} = (f_\alpha)_{\alpha \in \lim(\omega_2)}$ given by the Jensen-König Theorem. We have that $L_{T(\mathcal{F})}$ is a compact T_2 indestructible space with $\psi(L_{T(\mathcal{F})}) \leq \aleph_1$; it remains only to show that $|L_{T(\mathcal{F})}| = \aleph_2$. On the one hand, the fact that $T(\mathcal{F})$ is ω_2-Aronszajn implies that $|L_{T(\mathcal{F})}| \geq \aleph_2$; on the other hand, since $T(\mathcal{F})$ does not have branches of uncountable cofinality, it also implies that every branch of $T(\mathcal{F})$ has countable cofinality, and thus $L_{T(\mathcal{F})} \subseteq \{\bigcup C : C \in [T(\mathcal{F})]^{\leq \aleph_0}\}$; since $|T(\mathcal{F})| = \aleph_2$, this yields $|L_{T(\mathcal{F})}| \leq {\aleph_2}^{\aleph_0} = {\aleph_1}^{\aleph_0} \cdot \aleph_2 = \aleph_2$ by CH. $\square$

Theorem

If ω_2 is not weakly compact in **L**, then there is a Lindelöf T_3 indestructible space of pseudocharacter $\leq \aleph_1$ and size $\aleph_2$.

Proof.

If CH fails, then any subspace $X \subseteq \mathbb{R}$ with $|X| = \aleph_2$ will satisfy the required conditions, since every hereditarily Lindelöf space is indestructible.
If CH holds, consider the space $L_{T(\mathcal{F})}$ obtained from a coherent family $\mathcal{F} = (f_\alpha)_{\alpha \in \lim(\omega_2)}$ given by the Jensen-König Theorem. We have that $L_{T(\mathcal{F})}$ is a compact T_2 indestructible space with $\psi(L_{T(\mathcal{F})}) \leq \aleph_1$; it remains only to show that $|L_{T(\mathcal{F})}| = \aleph_2$. On the one hand, the fact that $T(\mathcal{F})$ is ω_2-Aronszajn implies that $|L_{T(\mathcal{F})}| \geq \aleph_2$; on the other hand, since $T(\mathcal{F})$ does not have branches of uncountable cofinality, it also implies that every branch of $T(\mathcal{F})$ has countable cofinality, and thus $L_{T(\mathcal{F})} \subseteq \{\bigcup C : C \in [T(\mathcal{F})]^{\leq \aleph_0}\}$; since $|T(\mathcal{F})| = \aleph_2$, this yields $|L_{T(\mathcal{F})}| \leq {\aleph_2}^{\aleph_0} = {\aleph_1}^{\aleph_0} \cdot \aleph_2 = \aleph_2$ by CH. $\square$

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