On topologies on X as points within $2^{\mathcal{P}(X)}$: lattice theory meets topology

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Outline

1. General, as background to
2. Sublattices of $2^{\mathcal{P}(X)}$ wrt completeness and compactness
3. Gauging the complexity of $Top(X)$ within $2^{\mathcal{P}(X)}$.

Powersets as lattices

Given a set X, we can form its powerset $\mathcal{P}(X)$ which naturally identifies with 2^X.

It is a complete and completely distributive Boolean lattice.

One step further: $\mathcal{P}(\mathcal{P}(X))$ identifies with $2^{\mathcal{P}(X)}$ and the same remarks apply.

"Subobjects" are sublattices.

Powersets as topological spaces

The product space $2^{\mathcal{P}(X)}$ is compact, Hausdorff and totally disconnected.

Note in particular that for any discrete space X, βX is embedded in $2^{\mathcal{P}(X)}$.

Some (sub)basics: For each $A \subseteq X$, let

$$A^+ := \{\mathcal{F} \in 2^{\mathcal{P}(X)} : A \in \mathcal{F}\} = \pi_A^{-1}(\{1\})$$

and

$$A^- := \{\mathcal{F} \in 2^{\mathcal{P}(X)} : A \notin \mathcal{F}\} = \pi_A^{-1}(\{0\}).$$

Note that $A^+ = 2^{\mathcal{P}(X)} \setminus A^-$ so that each subbasic open set is also closed.

The interval topology on a poset

Given θ and ϕ in $2^{\mathcal{P}(X)}$, $\theta^{\uparrow} = \{t : \theta \leqslant t\}$ and $\phi^{\downarrow} = \{t : t \leqslant \phi\}$ are closed subsets of $2^{\mathcal{P}(X)}$. More generally, these (closed) sets provide a subbasis for the closed sets of the *interval topology on* $2^{\mathcal{P}(X)}$.

Thus $2^{\mathcal{P}(X)}$'s product topology contains the interval topology.

Theorem (Frink, 1942)

If L is a lattice with the interval topology, then L is compact if and only if L is complete.

Completion of sublattices

Let P be a sublattice of $2^{\mathcal{P}(X)}$ and denote by $\hat{P}$ its lattice-theoretic completion. Thus $\hat{P} = \bigcap\{L \subseteq 2^{\mathcal{P}(X)} : P \subseteq L,\ L$ a sublattice of $2^{\mathcal{P}(X)}$ and $L = \hat{L}\}$.

Lemma (Morris, 2004)

$x \in \hat{P}$ if and only if $x = \bigwedge \bigvee S$ for all $S \subseteq P$ so that $x \leqslant \bigvee S$.

Lemma

Let P be a sublattice of $2^{\mathcal{P}(X)}$, let $S \subseteq P$ and let $x = \bigvee S$. If $x \notin P$, then x is a limit point of P.

Proof.

Let $\bigcap A_i^+ \cap \bigcap B_j^-$ be a basic open neighbourhood of x. Then for each of the finitely many i, there is $s_i \in S$ such that $A_i \in s_i$; furthermore $B_j \notin s_i$ for each j and each i. Thus $\bigvee_i s_i \in \bigcap A_i^+ \cap \bigcap B_j^- \cap P$ and clearly $\bigvee_i s_i \neq x$.

$\square$

The order topology on a lattice

Recall that the *interval topology* on a poset P is the one generated by $\{x^{\uparrow} : x \in P\} \cup \{x^{\downarrow} : x \in P\} \cup \{P, \emptyset\}$ as a subbase for the closed sets; we denote it by $P_{<}$.

The *order topology* P_O on a lattice P is defined in terms of Moore-Smith convergence. A filter $\mathfrak{F}$ of subsets from P is said to Moore-Smith-converge to a point $l \in P$ whenever

$$\bigwedge_{F \in \mathfrak{F}} \bigvee F = l = \bigvee_{F \in \mathfrak{F}} \bigwedge F.$$

We then take $F \subseteq P$ to be *closed* if and only if any convergent filter that contains F converges to a point in F.

For a lattice P, $P_{<} \subseteq P_O$. [Frink, 1942]

Three topologies and a sublattice

Theorem (Frink, 1942)

If P is a lattice with the interval topology, then P is compact if and only if P is complete.

For a lattice P, $P_< \subseteq P_O$.

Lemma

Let P be a sublattice of $2^{\mathcal{P}(X)}$. Then $P_< \subseteq P \subseteq P_O$ and all three topologies coincide when P is a complete sublattice of $2^{\mathcal{P}(X)}$. Moreover, all three topologies on P are compact if and only if P is complete.

Furthermore

Theorem

Given a sublattice P of $2^{\mathcal{P}(X)}$, $\overline{P} = \hat{P}$;
that is, the topological closure of P in $2^{\mathcal{P}(X)}$ coincides with its lattice-theoretic completion.

Previously ...

Denote by $LatB(X)$ the set of all sublattices of $\mathcal{P}(X)$ that contain $\emptyset$ and X.

1. $LatB(X)$ is a (Hausdorff) compactification of $Top(X)$.
2. $Top(X)$ is co-dense in $LatB(X)$.
3. $Top(X)$ is not locally compact in $2^{\mathcal{P}(X)}$.

Is $Top(X)$ a G_δ set?

Lemma

Given $A_i \subseteq X$ for all $i \in \omega$, $\bigcap_{i\in\omega} A_i^+$ contains a bounded sublattice of $\mathcal{P}(X)$ that is not join complete; that is, $\left(\bigcap_{i\in\omega} A_i^+\right) \cap (LatB(X) \setminus Top(X)) \neq \emptyset$.

Proof.

Let $<\{A_i : i \in \omega\}>_L$ denote the sublattice of $\mathcal{P}(X)$ generated by $\{A_i : i \in \omega\}$, adding in X or $\emptyset$ if not already generated, and suppose that it is join complete (otherwise, we are done). Notice that its countable cardinality demands that only finitely many of the A_is can be singletons. Since X is infinite, we may choose a countably infinite collection of singletons $S = \{\{p\} : p \in X\}$ from $\mathcal{P}(X)$ and generate a lattice $K = <\{A_i : i \in \omega\} \cup S >_L$. Then K cannot be join complete for there are uncountably many subsets of $\cup S$ (i.e. joins of S) and only $\aleph_0$ many elements in K.

$\square$

$Top(X)$ is not a G_δ set.

Proof.

Suppose that $Top(X) = \bigcap_{k\in\omega} \mathcal{O}_k$, where

$$\mathcal{O}_k = \bigcup_{\alpha\in\beta_k} \left((\bigcap_{i_\alpha \leqslant n_\alpha} A^+_{i_\alpha}) \cap (\bigcap_{j_\alpha \leqslant m_\alpha} B^-_{j_\alpha}) \right).$$

Now, the discrete topology $\mathcal{D}$ on X must be in this intersection of open sets. Thus for each $k \in \omega$, it must belong to at least one basic open set of the form $(\bigcap_{i_\alpha \leqslant n_\alpha} A^+_{i_\alpha}) \cap (\bigcap_{j_\alpha \leqslant m_\alpha} B^-_{j_\alpha})$ and since $\mathcal{D}$ contains all sets, then no subbasic open set can be of the form B^-. That is, $\mathcal{D} \in \bigcap_{k\in\omega} A^+_k$ after some renumeration of the As. Applying the previous Lemma to $\bigcap_{k\in\omega} A^+_k$, we can find a sublattice of $\mathcal{P}(X)$ that belongs to $\bigcap_{k\in\omega} A^+_k$ and that is not join complete - a contradiction.

$\square$

The Borel hierarchy

In fact, that Lemma proves something much stronger. Define recursively:

$$
\begin{aligned}
G_\delta^0 &:= \{\text{all } G_\delta \text{ sets}\} \\
G_{\delta\sigma}^0 &:= \{\text{all countable unions of } G_\delta \text{ sets}\} \\
G_\delta^\beta &:= \{\text{all countable intersections of } G_{\delta\sigma}^{\beta-1} \text{ sets}\} && (\beta \text{ a succ. ordinal}) \\
G_{\delta\sigma}^\beta &:= \{\text{all countable unions of } G_\delta^\beta \text{ sets}\} && (\beta \text{ a succ. ordinal}) \\
G_\delta^\gamma &:= \bigcup_{\beta\in\gamma} G_\delta^\beta && (\gamma \text{ limit ordinal}) \\
G_{\delta\sigma}^\gamma &:= \bigcup_{\beta\in\gamma} G_{\delta\sigma}^\beta && (\gamma \text{ limit ordinal})
\end{aligned}
$$

$Top(X) \notin G_\delta^\beta$ for $\beta \in \omega_1$

In other words, it is not possible to generate (in the sense of Borel) $Top(X)$ from open sets in $2^{\mathcal{P}(X)}$.

Corollary

$Top(X)$ is not Čech complete.

Proof.

We showed above that any countable intersection of open sets from $2^{\mathcal{P}(X)}$ containing the discrete topology on X contains an element of $LatB(X) \setminus Top(X)$. Hence, $Top(X)$ is not a G_δ set in $LatB(X)$. $\square$

Is $Top(X)$ an F_σ set?

Strategy: We will show that if $Top(X) = \bigcup_{k\in\omega} C_k$ where each C_k is a closed set, at least one such closed set must contain a (convergent) sequence of topologies whose limit is not a topology. It then follows that $Top(X)$ cannot be such a union.
We prove the above for $|X| = \aleph_0$ and note that the same is true for any X with $|X| \geqslant \aleph_0$.

Let $k : [0,1] \to \mathcal{P}(\mathbb{N})$ be an injective order morphism so that $\forall a \in [0,1]$, $\bigcup_{b<a} k(b) = k(a)$ and $k(1) \neq \mathbb{N}$.
That is, $k([0,1])$ is a dense and uncountable linear order in $\mathcal{P}(\mathbb{N})$ where $a < b \Rightarrow k(a) \subset k(b)$.

Next, for any a define $\tau_a = \mathcal{P}(k(a)) \cup \{\mathbb{N}\}$.

Notice that $\forall a \in [0,1]$, $\tau_a \in Top(\mathbb{N})$ and $\bigcup_{b<a} \tau_b \notin Top(\mathbb{N})$ (since $k(a) \notin \bigcup_{b<a} \tau_b$), and $\{\tau_a\}_{a\in[0,1]}$ is an uncountable dense linear order in $Top(\mathbb{N})$.

$Top(X)$ is not an F_σ set

If $Top(\mathbb{N}) = \bigcup_{k\in\omega} C_k$ where each C_k is closed then there must exist one set C from $\{C_k\}_{k\in\omega}$ which contains an uncountable set $D \subset \{\tau_a\}_{a\in[0,1]}$.

We immediately get that D must contain a densely ordered subset that in turn contains a strictly increasing sequence, call it S.

Now $\bigcup S \notin Top(\mathbb{N})$ yet $\bigcup S \in C$, a contradiction.

The wishful....

We wish to consider a notion of *evolution of topologies*:

Is there a path $p : [0,1] \to Top(X)$ such that, for example,

- $(X, p(0)) \cong \mathbb{R}$ and $(X, p(1)) \cong \omega_1$ (CH), or
- $(X, p(0))$ is Hausdorff and $(X, p(1))$ is compact, or
- if $(X, \sigma) \cong (X, \tau)$, then $p(0) = \sigma$ and $p(1) = \tau$
- and so on ...?