Theory of Wind Driven Sea

V.E. Zakharov
Department of Mathematics, University of Arizona, Tucson, USA

Introduction to the theory

The wind-driven sea is one of the most common natural phenomena that we can observe by our own eyes without special devices. Developing of its analytical theory is possible due to presence of a natural small parameter: the ratio of atmospheric and water densities. The density of air depends on temperature and the level of moisture. It is reasonable to put

$$\epsilon = \frac{\rho_{atm}}{\rho_w} \sim 1.2 \cdot 10^{-3}$$

Let $\eta(\vec{r}, t)$, $\vec{r} = (x, y)$ be elevation. In the linear approximation

$$\eta(r, t) = Re\, \Psi\, e^{-i\,\omega_k\, t + i\,\vec{k}\,\vec{r}}$$

Here Ψ is an arbitrary complex amplitude, $\omega_k = \sqrt{g\,k + \sigma\,k^3}$ is the dispersion law, g is the gravity acceleration, and σ is the surface tension. In the presence of wind

$$\omega_k \to \omega_k + \frac{i}{2}\gamma_k, \qquad \gamma_k \sim \epsilon$$

The smallness of ϵ leads to the smallness of the parameter of nonlinearity, which is an average steepness μ. It could be defined by many ways, the most "scientific" definition is:

$$\mu^2 = \langle |\nabla \eta|^2 \rangle$$

The oceanographers prefer another definition of μ, thereafter we denote it as μ_p:

$$\mu_p^2 = \langle \eta^2 \rangle \, k_p^2 \; = \; \langle \eta^2 \rangle \, \frac{\omega_p^4}{g^2}$$

In the stationary sea the autocorrelation function of elevation

$$\hat{F}(\tau) = \hat{F}(-\tau) = \langle \eta(t) \, \eta(t+\tau) \rangle$$

does not depend on t.

The cosine Fourier transform of the function of elevation

$$F(\omega) = \frac{1}{\pi} \int_0^\infty \hat{F}(\tau)\, \cos \omega\, \tau \, d\tau$$

is traditionally called the energy spectrum of the surface. The mean squared elevation is

$$\langle \eta^2 \rangle = \hat{F}(0) = \int_0^\infty F(\omega)\, d\omega$$

Spectrum $F(\omega)$ has dimension $L^2 T$. Why is it called the energy spectrum? Let us introduce function

$$E(\omega) = \rho_w\, g\, F(\omega),$$

that has dimension M/T, the same as the spectral distribution of energy density.

In many experiments the subject of measurements is the spatial spectrum

$$I_k = 2\pi \langle |\eta_k|^2 \rangle k$$

As far as μ is small, one can use expansion in powers of μ as a basic analytic technique for study of nonlinear wave interaction. Performing this expansion we realize that we have to deal with resonant interactions of certain amount of waves that form "a resonant group". For gravity waves on two-dimensional plane the most important groups are quadruplets of waves with wave vectors $\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4$, satisfying resonant conditions

$$\begin{aligned} \vec{k}_1 + \vec{k}_2 &= \vec{k}_3 + \vec{k}_4, \\ \omega_{k_1} + \omega_{k_2} &= \omega_{k_3} + \omega_{k_4} \end{aligned}$$

All these three processes can be described by a single kinetic equation, first derived by Hasselmann in 1962 and written for the spectrum of "wave action", $N(\vec{k}, \vec{r}, t)$:

$$\frac{\partial N}{\partial t} + \frac{\partial \omega}{\partial \vec{k}} \nabla N = S_{nl} + S_{in} + S_{diss}$$

In this equation $\omega = \sqrt{gk}$ is the dispersion law for gravity waves, S_{in} is the input from wind, S_{diss} is the dissipation due to white-capping, and S_{nl} is the collision term that describes four-wave resonant interaction.

The exact definition of N_k will be done below.

$$\tilde{I}_k = \frac{\omega_k}{2} \left(N_k + N_{-k} \right) 2\pi |k|$$

$\tilde{I}_k$ is the "refined" spatial spectrum.

The main message of this talk is the following:

In the most important cases, in the real sea the four-wave nonlinear interaction is the dominating process!

It is astonishing what a large amount of information could be extracted from a careful study of the pure conservative kinetic equation

$$\frac{\partial N}{\partial t} + \frac{\partial \omega}{\partial k} \nabla N = S_{nl}$$

and even from the ultimately simple equation

$$S_{nl} = 0$$

We show that S_{nl} is actually a kind of the nonlinear elliptic equation in the k-plane. Thus the nonstationary kinetic equation can be treated as a nonlinear diffusion equation.

The stationary homogenous kinetic equation has a rich family of Kolmogorov-type solutions. Most important of them are the Kolmogorov-Zakharov spectra corresponding to the direct cascade of energy and inverse cascade of wave function. They are widely observed in experiments.

The nonstationary conservative kinetic equation has a rich family of self-similar solutions describing spacial (fetch-limited case) and temporal (duration-limited case) evolution of the wind-driven sea. They are also supported by observational data.

The Hasselmann equation describes the most well-justified "weak turbulent" theory of the wind-driven sea. This theory is good for the not very short waves. It works in the range $k_p < k < k_f$. Here k_p is the wave number of the spectral peak, $k_f \simeq 10\, k_p$. For $k > k_f$, the spectrum is defined by competition of nonlinear wave interaction and white-capping events. This part of the spectrum is called the "Phillips sea".

The weak turbulent theory can be extended for description of the Phillips sea, if the dissipation term S_{diss} is chosen by a proper way.

Kinetic Hasselmann equation

We study the weakly nonlinear waves on the surface of an ideal fluid with infinite depth in an infinite basin. The vertical coordinate is

$$-\infty < z < \eta(r,t), \qquad r = (x,y),$$

the fluid is incompressible,

$$div\, V = 0,$$

and velocity V is a potential field

$$V = \nabla\,\Phi,$$

where potential Φ satisfies the Laplace equation under boundary conditions

$$\Delta\,\Phi = 0, \qquad \Phi|_{z=\eta} = \Psi(r,t), \qquad \Phi_z|_{z=-\infty} = 0$$

The total energy of the fluid, $H = T + U$, has the following terms:

$$T = \frac{1}{2}\int d\vec{r} \int_{-\infty}^{\eta} (\nabla\,\Phi)^2\, dz = \frac{1}{2}\int \Psi\,\Phi_n\, dS, \qquad U = \frac{1}{2}\, g \int \eta^2\, d\vec{r}$$

The Dirichlet-Neumann boundary problem is uniquely resolved, thus the flow is defined by fixation of η and Ψ. They are canonical, thus evolution equations take the form:

$$\frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \Psi}, \qquad \frac{\partial \Psi}{\partial t} = -\frac{\delta H}{\delta \eta}$$

After non-symmetric Fourier transform, these equations read

$$\Psi(r) = \int \Psi(k)\, e^{ikr}\, dk, \qquad \Psi(k) = \frac{1}{(2\pi)^2}\int \Psi(r)\, e^{-ikr}\, dr,$$

$$\frac{\partial \eta}{\partial t} = \frac{\delta \tilde{H}}{\delta \Psi_k^*}, \qquad \frac{\partial \Psi}{\partial t} = -\frac{\delta \tilde{H}}{\delta \eta_k^*}, \qquad \tilde{H} = \frac{1}{4\pi^2}\, H = H_0 + H_1 + H_2 + \cdots$$

In was shown that the Hamiltonian $\tilde{H}$ can be expanded in Taylor series in powers of η:

$$H_0 = \frac{1}{2}\int \left\{k|\Psi_k|^2 + g\,|\eta_k|^2\right\} dk$$

$$H_1 = \frac{1}{2}\int L^{(1)}(k_1,k_2)\Psi_{k_1}\Psi_{k_2}\eta_{k_3}\,\delta(\vec{k_1}+\vec{k_2}+\vec{k_3})\,dk_1\,dk_2\,dk_3$$

$$H_2 = \frac{1}{2}\int L^{(2)}(k_1,k_2,k_3,k_4)\Psi_{k_1}\Psi_{k_2}\eta_{k_3}\eta_{k_4}\,\delta(k_1+k_2+k_3+k_4)\,dk_1 dk_2 \eta_{k_3}\eta_{k_4}$$

Here

$$L^{(1)}(k_1,k_2) = -(k_1,k_2) - |k_1|\,|k_2|$$

$$L^{(2)}(k_1,k_2,k_3,k_4) = \frac{1}{4}|k_1||k_2|\,\{-2|k_1| - 2|k_2| + |k_1+k_3| + |k_1+k_4| + |k_2+k_3| + |k_2+k_4|\}$$

Now we can introduce normal variables a_k:

$$\eta_k = \frac{1}{\sqrt{2}}\left(\frac{A_k}{g}\right)^{1/4}(a_k + a^*_{-k}), \qquad \Psi_k = \frac{i}{\sqrt{2}}\left(\frac{g}{A_k}\right)^{1/4}(a_k - a^*_{-k})$$

They obey the following Hamiltonian equations:

$$\frac{\partial a_k}{\partial t} + i\frac{\delta H}{\delta a^*_k} = 0$$

$$H_0 = \int \omega_k |a_k|^2 dk$$

$$H_1 = \frac{1}{2}\int V^{(1,2)}_{kk_ak_2}(a_k a^*_{k_1} a^*_{k_2} + a^*_k a_{k_1} a_{k_2})\delta(k - k_1 - k_2)dkdk_1dk_2 +$$
$$+\frac{1}{6}\int V^{(0,3)}_{kk_ak_2}(a_k a_{k_1} a_{k_2} + a^*_k a^*_{k_1} a^*_{k_2})\delta(k + k_1 + k_2)dkdk_1dk_2$$

Here

$$V^{(1,2)}_{kk_1k_2} = \frac{g^{1/4}}{2\sqrt{2}} \left\{ \left(\frac{A_k}{A_{k_1}A_{k_2}} \right)^{1/4} L^{(1)}(k_1,k_2) - \left(\frac{A_{k_1}}{A_kA_{k_2}} \right)^{1/4} \right.$$

$$\left. \times L^{(1)}(-k,k_1) - \left(\frac{A_{k_2}}{A_kA_{k_1}} \right)^{1/4} L^{(1)}(-k,k_2) \right\}$$

$$V^{(0,3)}_{kk_1k_2} = \frac{g^{1/4}}{2\sqrt{2}} \left\{ \left(\frac{A_k}{A_{k_1}A_{k_2}} \right)^{1/4} L^{(1)}(k_1,k_2) + \left(\frac{A_{k_1}}{A_kA_{k_2}} \right)^{1/4} \right.$$

$$\left. \times L^{(1)}(k,k_1) + \left(\frac{A_{k_2}}{A_kA_{k_1}} \right)^{1/4} L^{(1)}(k,k_2) \right\}$$

To separate resonant and slave harmonics we must perform the canonical transformation to new variables, excluding cubic terms in the Hamiltonian. Variables a_k are presented by infinite series in new variables b_k:

$$a_k = b_k + a_k^{(1)} + a_k^{(2)} + a_k^{(3)}$$

Now we present a_k in the form

$$a_k = \frac{1}{\sqrt{2}}(q_k + ip_k), \quad q_{-k} = q_k^*, \quad p_{-k} = p_k^*$$

Functions q_k, p_k obey equations

$$\frac{\partial q_k}{\partial t} = \frac{\delta H}{\delta p_k^*}, \quad \frac{\partial p_k}{\partial t} = -\frac{\delta H}{\delta q_k^*}$$

where H is the same Hamiltonian expressed through q_k, p_k.

Now

$$H_0 = \frac{1}{2}\int \omega_k(|q_k|^2 + |p_k|^2)dk$$

$$H_1 = \frac{1}{2}\int L_{kk_1k_2}\, q_k\, p_{k_1}\, p_{k_2}\, \delta(k + k_1 + k_2)dkdk_1dk_2$$

$$L_{kk_1k_2} = \frac{g^{1/4}\, A_k^{1/4}}{A_{k_1}^{1/4}\, A_{k_2}^{1/2}}\, L^{(1)}_{k_1k_2}$$

We will perform the canonical transformation to new variables R_k, ξ_k using the following generation function:

$$\begin{aligned} S \quad = \quad & \int R_k\, q_k\, dk + \frac{1}{2}\int A_{kk_1k_2}\, q_k\, q_{k_1}\, R_{k_2}\, \delta(k + k_1 + k_2)\, dkdk_1dk_2 + \\ & + \frac{1}{3}\int B_{kk_1k_2}\, R_k\, R_{k_1} R_{k_2}\, \delta(k + k_1 + k_2)dkdk_1dk_2 \end{aligned}$$

The "old momentum" p_k and the "new coordinates" ξ_k are expressed as follows

$$p_k = \frac{\delta S}{\delta q_{-k}} = R_k + \int A_{-k,k_1,k_2}\, q_{k_1}\, R_{k_2}\, \delta(k - k_1 - k_2) dk_1 dk_2$$

$$\xi_k = \frac{\delta S}{\delta R_{-k}} = q_k + \frac{1}{2}\int A_{k_1,k_2,-k}\, q_{k_1}\, q_{k_2}\, \delta(k - k_1 - k_2^*)\, dk_1 dk_2 + \\ + \int B_{-k,k_1,k_2}\, R_{k_1}\, R_{k_2}\, \delta(k - k_1 - k - 2)\, dk_1 dk_2$$

We find after some calculations the following nice and elegant expressions for A, B:

$$A_{kk_1k_2} = -\frac{1}{4}\left(\frac{L_0 + L_1 + L_2}{\omega_0 + \omega_1 + \omega_2} + \frac{L_0 + L_1 - L_2}{\omega_0 + \omega_1 - \omega_2}\right) + \frac{1}{4}\left(\frac{L_0 - L_1 - L_2}{\omega_0 - \omega_1 - \omega_2} + \frac{L_1 - L_0 - L_2}{\omega_1 - \omega_0 - \omega_2}\right)$$

$$B_{kk_1k_2} = -\frac{1}{4}\left(\frac{L_0 + L_1 + L_2}{\omega_0 + \omega_1 + \omega_2} + \frac{L_0 - L_1 - L_2}{\omega_0 - \omega_1 - \omega_2}\right) - \frac{1}{4}\left(\frac{L_1 - L_0 - L_2}{\omega_1 - \omega_0 - \omega_2} + \frac{L_2 - L_0 - L_1}{\omega_2 - \omega_0 - \omega_1}\right)$$

Now we can introduce new normal complex variables b_k,

$$\xi_k = \frac{1}{\sqrt{2}} \left(\frac{k}{g}\right)^{1/4} (b_k + b^*_{-k}), \qquad R_k = \frac{i}{\sqrt{2}} \left(\frac{g}{k}\right)^{1/4} (b_k - b^*_{-k})$$

In the new variables the Hamiltonian equation takes form

$$\frac{\partial b_k}{\partial t} + i \frac{\delta \tilde{H}}{\delta b^*_k} = 0$$

$$\tilde{H} = \int \omega_k \, b_k \, b^*_k \, dk + \frac{1}{4} \int T_{k_1 k_2 k_3 k_4} \, b^*_{k_1} \, b^*_{k_2} \, b_{k_3} \, b_{k_4} \delta(k_1 + k_2 - k_3 - k_4) \, dk_1 \, dk_2 \, dk_3 \, dk_4$$

and the coupling coefficient $T_{k_1 k_2, k_4 k_3}$ satisfies the symmetry conditions:

$$T_{k_1 k_2, k_3 k_4} = T_{k_2 k_1, \, k_3 k_4} = T_{k_1 k_2, \, k_4 k_3} = T_{k_2 k_4, k_1 k_2}$$

The explicit expression for T is complicated:

$$
\begin{aligned}
T_{12,34} &= \frac{1}{2}\left(\tilde{T}_{12,34} + \tilde{T}_{21,34}\right), \\
\tilde{T}_{12,34} &= -\frac{1}{2}\frac{1}{(k_1k_2k_3k_4)^{1/4}}\Bigg\{ -12k_1k_2k_3k_4 - \\
&\quad -2(\omega_1+\omega_2)^2\Big[\omega_3\omega_4\Big((\vec{k}_1\cdot\vec{k}_2) - k_1k_2\Big) + \omega_1\omega_2\Big((\vec{k}_3\cdot\vec{k}_4) - k_3k_4\Big)\Big]\frac{1}{g^2} \\
&\quad -2(\omega_1-\omega_3)^2\Big[\omega_2\omega_4\Big((\vec{k}_1\cdot\vec{k}_3) + k_1k_3\Big) + \omega_1\omega_3\Big((\vec{k}_2\cdot\vec{k}_4) + k_2k_4\Big)\Big]\frac{1}{g^2} \\
&\quad -2(\omega_1-\omega_4)^2\Big[\omega_2\omega_3\Big((\vec{k}_1\cdot\vec{k}_4) + k_1k_4\Big) + \omega_1\omega_4\Big((\vec{k}_2\cdot\vec{k}_3) + k_2k_3\Big)\Big]\frac{1}{g^2} \\
&\quad +[(\vec{k}_1\cdot\vec{k}_2) + k_1k_2][(\vec{k}_3\cdot\vec{k}_4) + k_3k_4] + [-(\vec{k}_1\cdot\vec{k}_3) + k_1k_3][-(\vec{k}_2\cdot\vec{k}_4) + k_2k_4] \\
&\quad +[-(\vec{k}_1\cdot\vec{k}_4) + k_1k_4][-(\vec{k}_2\cdot\vec{k}_3) + k_2k_3]
\end{aligned}
$$

$$+4(\omega_1+\omega_2)^2\,\frac{[(\vec{k}_1\cdot\vec{k}_2)-k_1k_2][(\vec{k}_3\cdot\vec{k}_4)-k_3k_4]}{\omega_{1+2}^2-(\omega_1+\omega_2)^2}$$

$$+4(\omega_1-\omega_3)^2\,\frac{[(\vec{k}_1\cdot\vec{k}_3)+k_1k_3][(\vec{k}_2\cdot\vec{k}_4)+k_2k_4]}{\omega_{1-3}^2-(\omega_1-\omega_3)^2}$$

$$\left.+\,4(\omega_1-\omega_4)^2\,\frac{[(\vec{k}_1\cdot\vec{k}_4)+k_1k_4][(\vec{k}_2\cdot\vec{k}_3)+k_2k_3]}{\omega_{1-4}^2-(\omega_1-\omega_4)^2}\right\}$$

Here $\omega_i = \sqrt{g\,|k_i|}$. Then, the new Hamiltonian equation reads:

$$\frac{\partial b_k}{\partial t}+i\left(\omega_k\,b_k+\frac{1}{2}\int T_{kk_1k_2k_3}\,b^*_{k_1}b_{k_2}b_{k_3}\,\delta(k+k_1-k_2-k_3)\,dk_1dk_2dk_3\right)=0$$

The initial Hamiltonian equation has natural formal motion constants of energy and momentum,

$$\tilde{H} = const, \qquad \tilde{P} = \int \vec{k}\, b_k\, b_k^*\, dk = const$$

while the new Hamiltonian equation conserves one additional constant N:

$$\tilde{N} = \frac{1}{g} \int |b_k|^2\, dk$$

One can introduce the following correlation functions:

$$\langle \eta(r)\, \eta(r+R) \rangle = I(R)$$

$$\langle \eta_k\, \eta_{k'} \rangle = \frac{I_k}{2\pi k}\, \delta(k+k'), \quad \langle b_k\, b_{k'}^* \rangle = g\, N_k\, \delta(k-k'), \quad \langle \xi_k\, \xi_{k'} \rangle = \frac{\tilde{I}_k}{2\pi k}\, \delta(k+k')$$

$$I(R) = \int \frac{I_k}{2\pi k}\, e^{ikR}\, dk$$

The correlation functions $I(k)$ and $\tilde{I}(k)$ are close to each other. In the area of spectral maximum

$$\Delta(k) = \frac{\tilde{I}(k) - I(k)}{I(k)} \sim \mu^2$$

is small, however it grows fast at $k \to \infty$. In the first approximation

$$\tilde{I}(k) = \frac{\omega(k)}{2} (N_k + N_{-k}) 2\pi k$$

Thereafter we assume that $N = N(r, k, t)$ is also a slowly varying function on coordinate r and accept that $N = N(r, k, t)$ satisfies the Hasselmann kinetic equation.

The derivation of the resulting equation

$$\frac{dN_k}{dt} = S_{nl} = \pi g^2 \int |T_{kk_1,k_2k_3}|^2\, \delta(k + k_1 - k_2 - k_3^*)\, \delta(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}) \times$$
$$\times (N_{k_1} N_{k_2} N_{k_3} + N_k N_{k_2} N_{k_3} - N_k N_{k_1} N_{k_2} - N_k N_{k_1} N_{k_3}) dk_1 dk_2 dk_3$$

can be done by the use of standard methods of statistical physics. Here

$$\frac{dN_k}{dt} = \frac{\partial N_k}{\partial t} + \frac{\partial \omega}{\partial k} \nabla N_k$$

and $T_{kk_1k_2k_3}$ is a homogenous function of the order of 3:

$$T_{\lambda k, \lambda k_1, \lambda k_2, \lambda k_3} = \lambda^3\, T_{kk_1k_2k_3}$$

Simple calculation shows that $T_{k,k,k,k} = T = 2k^3$.

Klaus Hasselmann (25 October 1931 –)

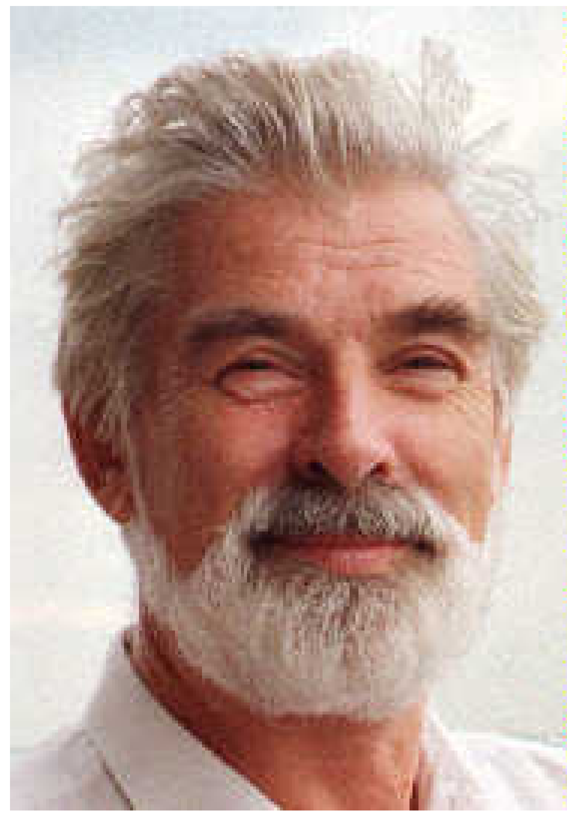


Figure 1:

Stationary solutions: the isotropic case

How to solve the stationery kinetic equation

$$S_{nl} \equiv 0\,?$$

The thermodynamically equilibrium solutions of this equation are

$$N_k = \frac{T}{\omega_k + \mu}$$

Here temperature T and μ are constants. In fact this spectrum in not the real solution of the equation. Since this moment we discuss the case of deep water only and consider $\omega = \sqrt{gk}$. Also we denote that $k = |\vec{k}|$. In two particular cases, $\mu = 0$ and $T = c\mu$, $\mu \to \infty$, these solution take the form

$$N = \frac{T}{\omega_k} = \frac{T}{\sqrt{g}}\,k^{-1/2}, \qquad N = \frac{T}{\mu}$$

The both solutions are isotropic powerlike functions

$$N_k = k^{-x}$$

with particular values $x = 1/2,\, 0$. Let us study the general powerlike solution of the stationary kinetic equation. The divergent terms in S_{nl} cancel if x is located in some "window of opportunity" $x_1 < x < x_2$. As a result,

$$S_{nl} = g^{3/2}\, k^{-3x+19/2}\, F(x)$$

Here $F(x)$ is a dimensionless function, defined inside interval $x_1 < x < x_2$. Outside the "window of opportunity", at $x < x_1$ and $x > x_2$, $F(x) = \infty$.

Let the quadruplet of waves be formed by wave vectors satisfying resonant conditions

$$\vec{k}_1 + \vec{k}_2 = \vec{k}_3 + \vec{k}_4, \qquad \omega_{k_1} + \omega_{k_2} = \omega_{k_3} + \omega_{k_4}$$

Suppose that $|k_1| \ll |k|$. One of vectors $\vec{k}_2, \vec{k}_3$ must be small. If $|k_3| \ll |k_2|$, then

$$\vec{k}_2 = \vec{k} + \vec{k}_1 - \vec{k}_3, \qquad \omega(k_2) = \sqrt{gk}\left(1 + \frac{1}{2}\frac{(k, \vec{k}_1 - \vec{k}_3)}{k^2} + \cdots\right)$$

In the first approximation by small parameter $|k_1|/|k|$ we can put

$$\omega(k_2) = \omega(k), \qquad \omega(k_1) = \omega(k_3), \qquad |k_3| \simeq |k_1|$$

In other words, vectors $\vec{k}_1, \vec{k}_3$ are small and have approximately the same length k_1.

If vector k is directed along axis x, the coupling coefficient $T_{kk_1k_2k_3}$ depends on four parameters $k, k_1, \theta_1, \theta_3$. Here θ_1, θ_3 are angles between $\vec{k}_1, \vec{k}_3$ and $\vec{k}$. Remembering that $k_1 \ll k$, we calculate the coupling coefficient in this asymptotic domain.

$$\begin{aligned} T_{kk_1k_2k_3} &\simeq \frac{1}{2} k\, k_1^2\, T_{\theta_1,\theta_3}, \\ T_{\theta_1,\theta_2} &= 2(\cos\theta_1 + \cos\theta_3) - \sin(\theta_1 - \theta_3)(\sin\theta_1 - \sin\theta_3) \end{aligned}$$

On the diagonal $k_3 = k_1$, $\theta_3 = \theta_1$ we get a very simple expression:

$$T_{kk_1} \simeq 2k_1^2 k\, \cos\theta_1$$

Suppose, the spectrum is separated to the low-frequency component $N_0(k)$ and the high-frequency component $N_1(k)$. We assume that $N_1 \ll N_0$ and take into account the interaction between N_0 and N_1 only. N_1 satisfies the linear diffusion equation

$$\frac{\partial}{\partial t} N_1 = \frac{\partial}{\partial k_i} D_{ij} \, k^2 \frac{\partial}{\partial k_j} N_1,$$

where D_{ij} is the tensor of diffusion coefficients,

$$D_{ij} = 2\pi g^{3/2} \int_0^\infty dq \, q^{17/2} \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_3 |T(\theta_1, \theta_3)|^2 \, p_i p_j N(\theta, q) N(\theta_3, q)$$

$$p_1 = \cos\theta_1 - \cos\theta_3, \quad p_2 = \sin\theta_1 - \sin\theta_3$$

The isotropic spectrum does not depend on angle θ. We get the further simplification:

$$D_{ij} = D\,\delta_{ij}, \qquad D = \frac{5}{8}\,\pi^3\,g^{3/2}\int_0^\infty q^{17/2}\,N^2(q)dq$$

The diffusion coefficient D diverges at $k \to 0$, if $x > 19/4$. Thus $x_2 = 19/4$.

In the isotopic case the diffusion equation reads

$$\frac{\partial N_1}{\partial t} = \frac{D}{k}\,\frac{\partial}{\partial k}\,k^3\,\frac{\partial}{\partial k}\,N_1$$

If $k \to 19/4$, we get the following estimate:

$$F(x) = \frac{19}{4}\cdot\frac{11}{4}\cdot\frac{5\pi^3}{16}\,\frac{1}{19/4-x} \simeq \frac{126.4}{19/4-x}$$

To find x_1, the lower end of the window, we should study the influence of short waves to the long ones.

In the isotropic case we get:

$$\begin{aligned}\frac{\partial N_k}{\partial t} &= q\, k^7\, N_k\, \frac{\partial N}{\partial k}, \\ q &= \frac{25}{16}\pi^3\, g^{3/2}\, E = \frac{25}{8}\pi^3\, g^{3/2} \int_0^\infty k^{3/2}\, N_k\, dk\end{aligned}$$

Here E is the total energy. Thus, $x_1 = 5/2$ and we get for the function $F(x)$ the following estimate:

$$F = \frac{5}{2}\,\frac{25}{8}\,\pi^3\,\frac{1}{5/2 - x} = \frac{241.86}{5/2 - x}$$

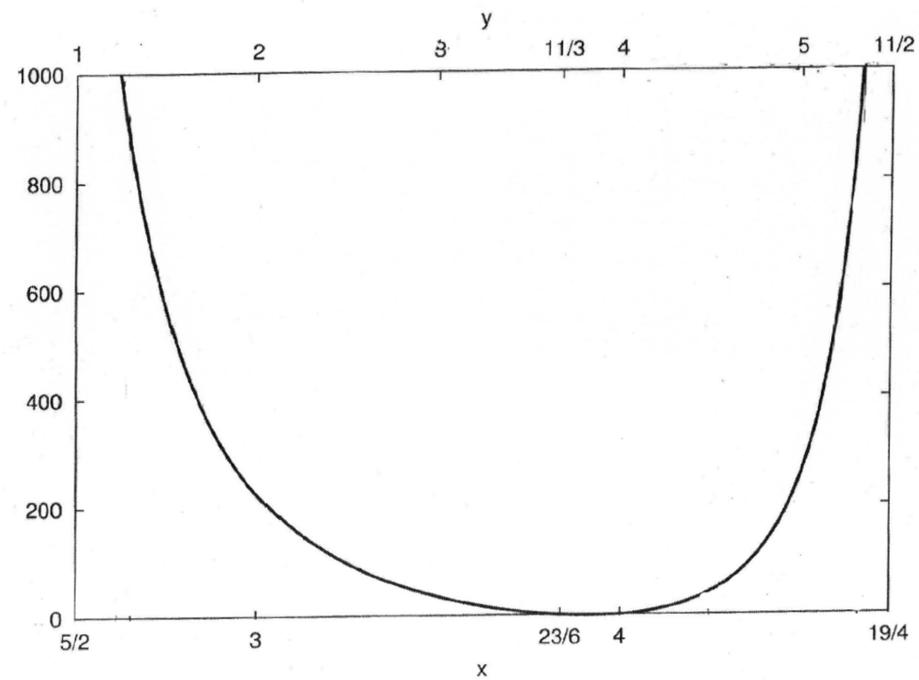


Figure 2: Plot of function $F(x)$

In figure 2 is presented a plot of the function $F(x)$ for the isotropic case, which we calculated numerically. In the interval $x_1 < x < x_2$, the function $F(x)$ has exactly two zeros at $x = y_1 = 4, \quad x = y_2 = \frac{23}{6}$

To prove this result, let us present the conservation laws of energy and wave action in the differential form:

$$\frac{\partial I_k}{\partial t} = 2\pi k\omega_k \frac{\partial N_k}{\partial t} = -\frac{\partial P}{\partial k}, \qquad P = 2\pi \int_0^k k\omega_k \, S_{nl} \, dk$$

$$2\pi k \frac{\partial N_k}{\partial t} = \frac{\partial Q}{\partial k}, \qquad Q = 2\pi \int_0^k k \, S_{nl} \, dk$$

Here, P is the flux of energy directed to high wave numbers, while Q is the flux of wave action directed to small wave numbers. The equations

$$P = P_0 = const, \quad Q = Q_0 = const$$

apparently are solutions of stationary equation $S_{nl} = 0$. We will look for the solution in the powerlike form $N = \lambda \, k^{-x}$.

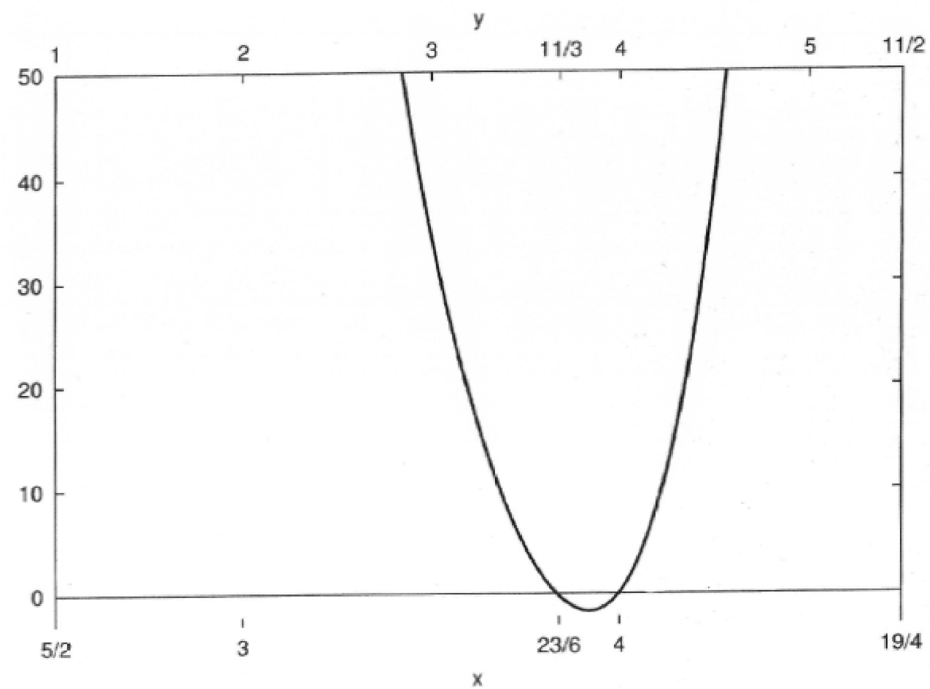


Figure 3: Plot of function $F(x)$: A zoom in the vertical direction

In Figure 3 is presented a zoom of the function $F(x)$ in the vertical coordinate. The numerics gives $F'(4) = 45.2$ and $F'(23/6) = -40.4$.

The described spectra exhaust all powerlike isotropic solutions of the stationary kinetic equation $S_{nl} = 0$. This fact can be proved by the use of different methods. Thermodynamical solutions $N = const$ and $N = c/k^{1/2}$ are not the solutions of this equation, because their exponents $x = 0$ and $x = 1/2$ are far below the lower end of the "window of possibility" $x_1 = 5/2$.

The general isotropic solution describes the situation when both the energy source at small wave numbers and the wave action source exist simultaneously and have the following form:

$$N_k^{(3)} = c_p \left(\frac{P}{g^2}\right)^{1/3} \frac{1}{k^4} L\left(\frac{g^{1/2}\, Q\, k^{1/2}}{P}\right)$$

Here L is an unknown function of one variable,

$$L \to 1 \quad \text{at} \quad k \to 0, \qquad L(\xi) \to \frac{c_q}{c_p}\,\xi^{1/3} \quad \text{at} \quad k \to \infty$$

These spectra are realized if we have a source of energy at large scales and a source of wave action at small scales.

In a real sea the external force (input from wind) is distributed smoothly along all scales. Thus the KZ-spectra are observed as asymptotics in the energy-containing area. The flux-action spectrum $I_k \simeq k^{-11/3}$ is realized in area of small wave numbers, while the KZ-spectrum is the high-frequency asymptotics. The typical picture is the following:

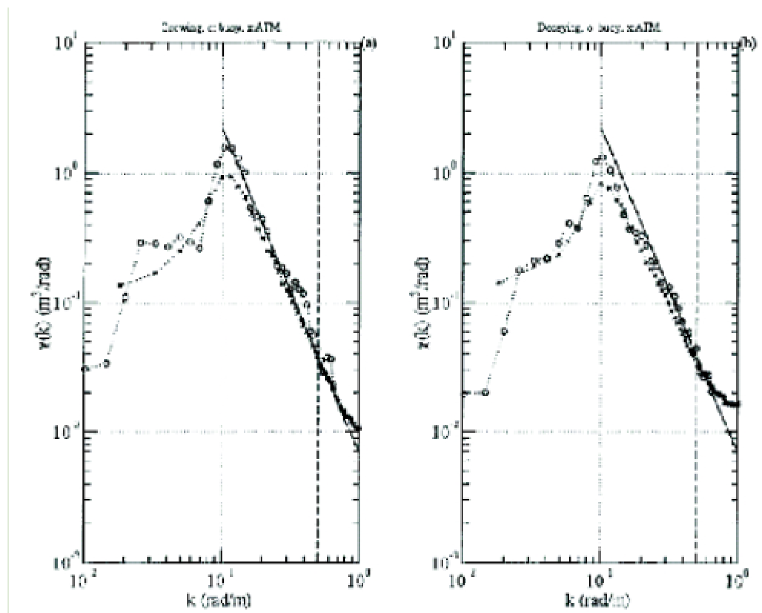


Figure 4: Experimental spatial spectra

The same scenario is routinely observed in experiments on numerical simulation of the kinetic equation

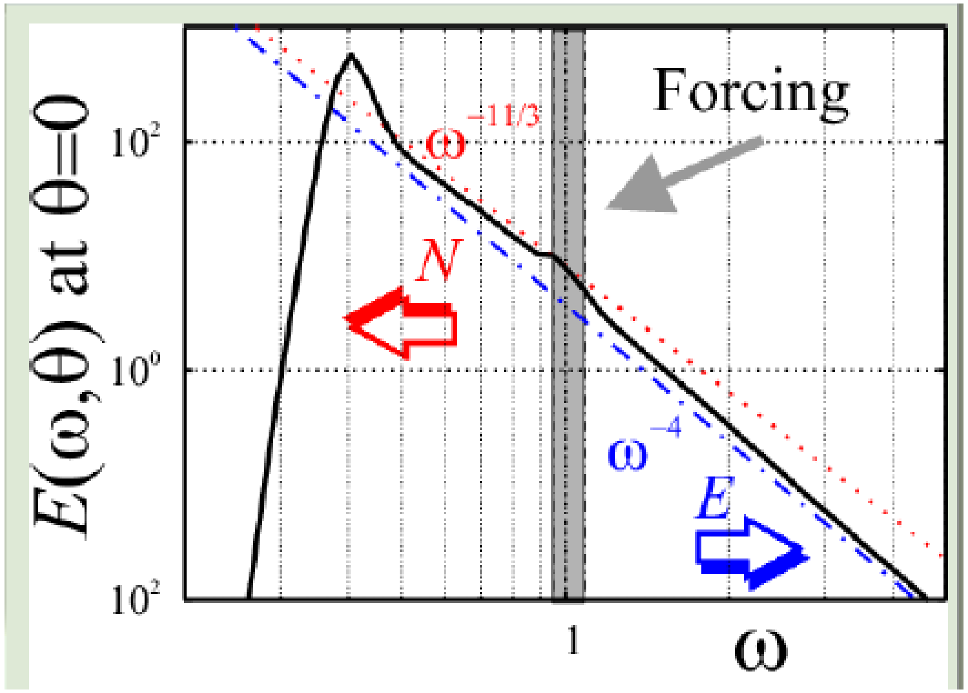


Figure 5: Numerical simulation

Stationary solutions: the anisotropic case

We introduce polar coordinates on the k-plane and put $k^2 = \omega/g$. Thereafter, we will use the notation

$$N(\omega, \phi)\, d\omega d\phi = N(\vec{k})\, d\vec{k},$$
$$N(\omega, \phi) = \frac{2\omega^3}{g^2}\, N(\vec{k})$$

In the spatially homogenous case, $N(\omega, \phi)$ satisfies the equation

$$\frac{\delta N(\omega, \phi)}{\partial t} = S_{nl}(\omega, \phi)$$

In new variables:

$$
\begin{aligned}
S_{nl}(\omega,\phi) = 2\pi g^2 \int & |T_{\omega,\omega_1,\omega_2,\omega_3}|^2\, \delta(\omega+\omega_1-\omega_2-\omega_3)\times \\
& \times\delta(\omega^2\cos\phi+\omega_1^2\cos\phi_1-\omega_2^2\cos\phi_2-\omega_3^2\cos\phi_3)\times \\
& \times\delta(\omega^2\sin\phi+\omega_1^2\sin\phi_1-\omega_2^2\sin\phi_2-\omega_3^2\sin\phi_2)\times \\
& \times\left\{\omega^3\, N(\omega_1,\phi_1)\, N(\omega_2,\phi_2)\, N(\omega_3,\phi_3)+\omega_1^3\, N(\omega,\phi)\, N(\omega_2,\phi_2)\, N(\omega_3,\phi_3)-\right. \\
& \left.-\omega_2^2\, N(\omega,\phi)\, N(\omega_1,\phi_1)\, N(\omega_3,\phi_3)-\omega_3^2\, N(\omega,\phi)\, N(\omega_1,\phi_1)\, N(\omega_2,\phi_2)\right\} \\
& \qquad d\omega_1\, d\omega_2\, d\omega_3\, d\phi_1\, d\phi_2\, d\phi_3
\end{aligned}
$$

Exactly this form of S_{nl} is used for numerical simulation of the Hasselmann equation.

Suppose that $N(\omega, \phi) = \omega^{-z}$ is the isotropic spectrum. Then,

$$S_{nl} = \frac{\omega^{-3z+13}}{4g^4} F\left(\frac{z+3}{2}\right) = \frac{G(z)}{g^4} \omega^{-3z+13},$$

where $F(x)$ was defined before. Now the "window of opportunity" is $2 < z < 13/2$. Zeros of $G(z)$ are posed at $z_1 = 5$ and $z_2 = 14/3$, and near these zeros $G(z)$ can be presented as parabola,

$$G(z) \simeq \, 16.05(z-5)(z-14/3)$$

To make the motion constants more conspicuous, we introduce the elliptic differential operator

$$L\, f(\omega, \phi) = \left(\frac{\partial^2}{\partial \omega^2} + \frac{2}{\omega^2}\frac{\partial^2}{\partial \phi^2}\right) f(\omega, \phi), \qquad 0 < \omega < \infty, \;\; 0 < \phi < 2\pi$$

Then, the equation

$$L\,G = \delta(\omega - \omega')\,\delta(\phi - \phi')$$

with boundary conditions

$$G|_{\omega\to 0} = 0, \qquad G_{\omega\to\infty} < \infty, \qquad G(2\pi) = G(0)$$

can be resolved as

$$G(\omega, \omega', \phi - \phi') = \frac{1}{4\pi}\sqrt{\omega\omega'} \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')} \times$$

$$\times \left[\left(\frac{\omega}{\omega'}\right)^{\Delta_n} \Theta(\omega' - \omega) + \left(\frac{\omega'}{\omega}\right)^{\Delta_n} \Theta(\omega - \omega') \right]$$

where $\Delta_n = 1/2\sqrt{1 + 8n^2}$

Now we present S_{nl} in the form:

$$A(\omega,\phi) = \int_0^\infty d\omega' \int_0^{2\pi} d\phi' \, G(\omega,\omega',\phi-\phi') \, S_{nl}(\omega',\phi')$$

Note that $A(\omega,\phi)$ is a regular integral operator and suppose that $N(\omega,\phi) = \omega^{-z}$. Then

$$A[\omega^{-z}] = \frac{\omega^{-3z+15}}{g^4} H(z),$$
$$H(z) = \frac{G(z)}{9(z-5)(z-14/3)}$$

The function $H(z)$ is positive and has no zeros. If $G(z)$ is presented by parabola, then:

$$H(min) = H_0 = 16.05/9 = 1.83$$

The fact that $H(z)$ is just a constant, leads to a bold idea. If we assume that

$$A = \frac{H_0}{g^4}\,\omega^{15}\,N^3,$$

the nonlinear term S_{nl} turns to the elliptic operator:

$$S_{nl} = \frac{H_0}{g^4}\left(\frac{\partial^2}{\partial\omega^2} + \frac{2}{\omega^2}\frac{\partial^2}{\partial\phi^2}\right)\omega^{15}\,N^3$$

This is the so-called "diffusion approximation". Being very simple, it grasps the basic features of the wind-driven sea theory.

The stationary kinetic equation is now simplified to the linear equation

$$\left(\frac{\partial^2}{\partial\omega^2} + \frac{2}{\omega^2}\frac{\partial^2}{\partial\phi^2}\right) A = 0$$

It has the anisotropic KZ solution

$$A = \frac{1}{2\pi}\left\{P + \omega\, Q + \frac{R_x}{\omega}\,\cos\phi\right\},$$

where P and R_x are fluxes of energy and momentum at $\omega \to \infty$ and Q is the flux of wave action directed to small wave numbers. In the general case, this is the nonlinear integral equation but in the diffusion approximation the KZ solution can be found in the explicit form:

$$N(\omega,\phi) = \frac{1}{(2\pi\, H_0)^{1/3}}\,\frac{g^{4/3}}{\omega^5}\,\left(P + \omega\, Q + \frac{R_x}{\omega}\,\cos\phi\right)^{1/3}$$

In this case

$$c_p = c_q = \frac{1}{2(2\pi\, H_0)^{1/3}} = 0.223, \qquad H_0 = 1.83$$

This is exactly the arithmetic mean between the values of Kolmogorov constants.

Multiplying the explicit form by $2\pi\omega$, we get the general KZ spectrum in the diffusion approximation:

$$F(\omega) = 2.78\, \frac{g^{4/3}}{\omega^4} \left(P + \omega\, Q + \frac{R_x}{\omega}\, \cos\phi \right)^{1/3}$$

We must be sure that in the isotropic case $R_x = 0$, the expression

$$F(\omega) = 2.78\, \frac{g^{4/3}}{\omega^4}\, (P + \omega\, Q)^{1/3}$$

approximates the generic isotropic KZ spectrum with accuracy up to a few percent.

Damping due to nonlinear interaction

In the presence of wind input and dissipation the stationary Hasselmann equation, or the balance equation, reads

$$S_{nl} + S_{in} + S_{diss} = 0$$

We can assume

$$S_{in} = \gamma_{in}(k)\, N(k), \qquad S_{dis} = -\gamma_{dis}(k)\, N(k)$$

If

$$\gamma(k) = \gamma_{in}(k) - \gamma_{dis}(k)$$

the balance kinetic equation reads

$$S_{nl} + \gamma(k)\, N_k = 0$$

We can present the S_{nl} term as

$$S_{nl} = F_k - \Gamma_k \, N_k$$

The definition of Γ_k and F_k are given as follow:

$$F_k = \pi g^2 \int |T_{kk_1k_2k_3}|^2 \, \delta(k{+}k_1{-}k_2{-}k_3) \, \delta(\omega_k{+}\omega_{k_1}{-}\omega_{k_2}{-}\omega_{k_3}) \, N_{k_1} N_{k_2} N_{k_3} \, dk_1 dk_2 dk_3$$

$$\Gamma_k = \pi g^2 \int |T_{kk_1,k_2k_3}|^2 \, \delta(k + k_1 - k_2 - k_3) \, \delta(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}) \times$$
$$\times (N_{k_1} N_{k_2} + N_{k_1} N_{k_3} - N_{k_2} N_{k_3}) \, dk_1 dk_2 dk_3$$

The solution of the balance equation is the following:

$$N_k = \frac{F_k}{\Gamma_k - \gamma_k}$$

The positive solution exists if $\Gamma_k > \gamma_k$. The term Γ_k can be treated as the nonlinear damping that appears due to four-wave interaction. This damping has a very powerful effect. The main source of Γ_k is the interaction between long and short waves. To estimate it, we assume that the spectrum of long waves is narrow in angle, $N(k_1, \theta_1) = \tilde{N}(k_1)\,\delta(\theta_1)$. Long waves propagate along the axis x and $\vec{k}$ is the wave vector of the short wave propagating in direction θ. For the coupling coefficient we must put $T_{kk_1,k_2,k_3} \simeq 2k_1^2 k \cos\theta$. Then

$$\Gamma_k = 8\pi g^{3/2}\, k^2\, \cos^2\theta \int_0^\infty k_1^{13/2}\, \tilde{N}^2(k_1)\, dk_1$$

Even for the most mildly decaying KZ spectrum, $N_k \simeq k^{-23/6}$, the integrand behaves like $k_1^{-7/6}$ and the integral diverges. For steeper KZ spectra, the divergence is stronger.

Let us estimate Γ_k for the case of a "mature sea", when the spectrum can be taken in the form

$$N_k \simeq \frac{3}{2}\frac{E}{\sqrt{g}}\frac{k_p^{3/2}}{k^4}\,\theta(k-k_p)$$

Here E is the total energy. For this case we get the equation

$$\Gamma_\omega = 36\,\pi\omega\left(\frac{\omega}{\omega_p}\right)^3 \mu_p^4\cos^2\theta,$$

which includes a huge enhancing factor: $36\pi \simeq 113.04$. For the very modest value of steepness, $\mu_p \simeq 0.05$, we get

$$\Gamma_\omega \simeq 7.06\cdot 10^{-4}\omega\left(\frac{\omega}{\omega_p}\right)^3 \cos^2\theta$$

These results show that the four-wave nonlinear interaction is a very strong effect. The strong turbulence of the near-surface air boundary layer makes the development of a reliable theory of air-water interaction, including a well-justified analytical calculation of γ_k, an extremely difficult task. making field and laboratory measurements of γ_k is also difficult, and the scatter in determination of γ_k is itself of the order of γ_k. Anyway, a comparison of the above calculated Γ_k with experimental data on γ_k shows that Γ_k surpasses γ_k at least by an order of magnitude. This fact is demonstrated in figure 6.

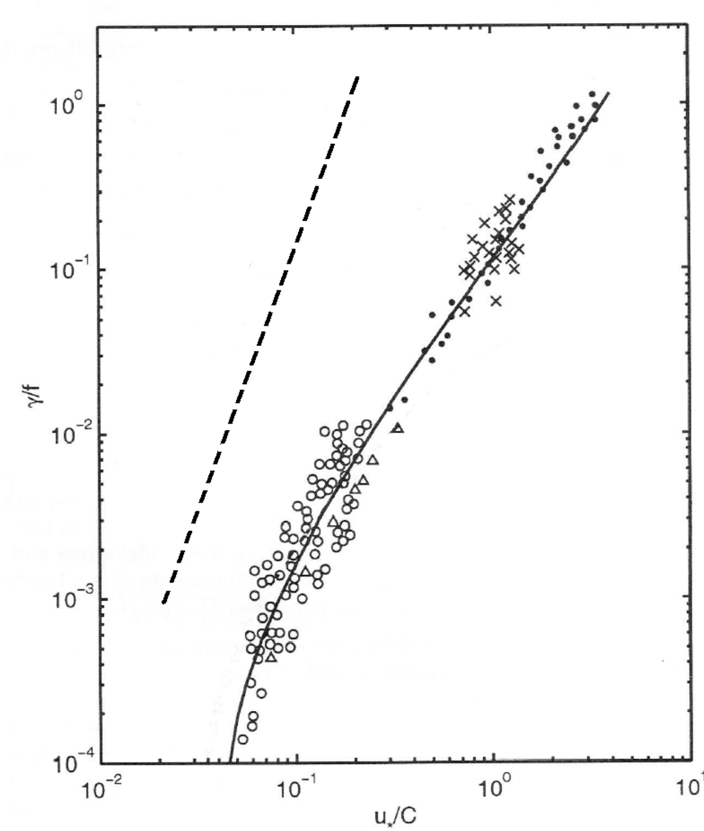


Figure 6: Comparison of experimental data for the wind-induced growth rate $2\pi\,\gamma_{in}(\omega)/\omega$ and the damping due to four-wave interactions $2\pi\,\Gamma(\omega)/\omega$, calculated for the narrow in angle spectrum at $\mu \simeq 0.05$ (dashed line)

As a result, we can make the conclusion that S_{nl} is the leading term in the balance equation and that the rear face of the spectrum is describes by solution of the stationary kinetic equation, which has a rich family of solutions. In particulary, this equation describes the angular spreading.

In Figure 7, we demonstrate that for the nonlinear interaction term $S_{nl} = F_k - \Gamma_k \, N_k$ the magnitudes of constituents F_k and $\Gamma_k \, N_k$ essentially exceed their difference. They are one order higher than the magnitude of S_{nl}!

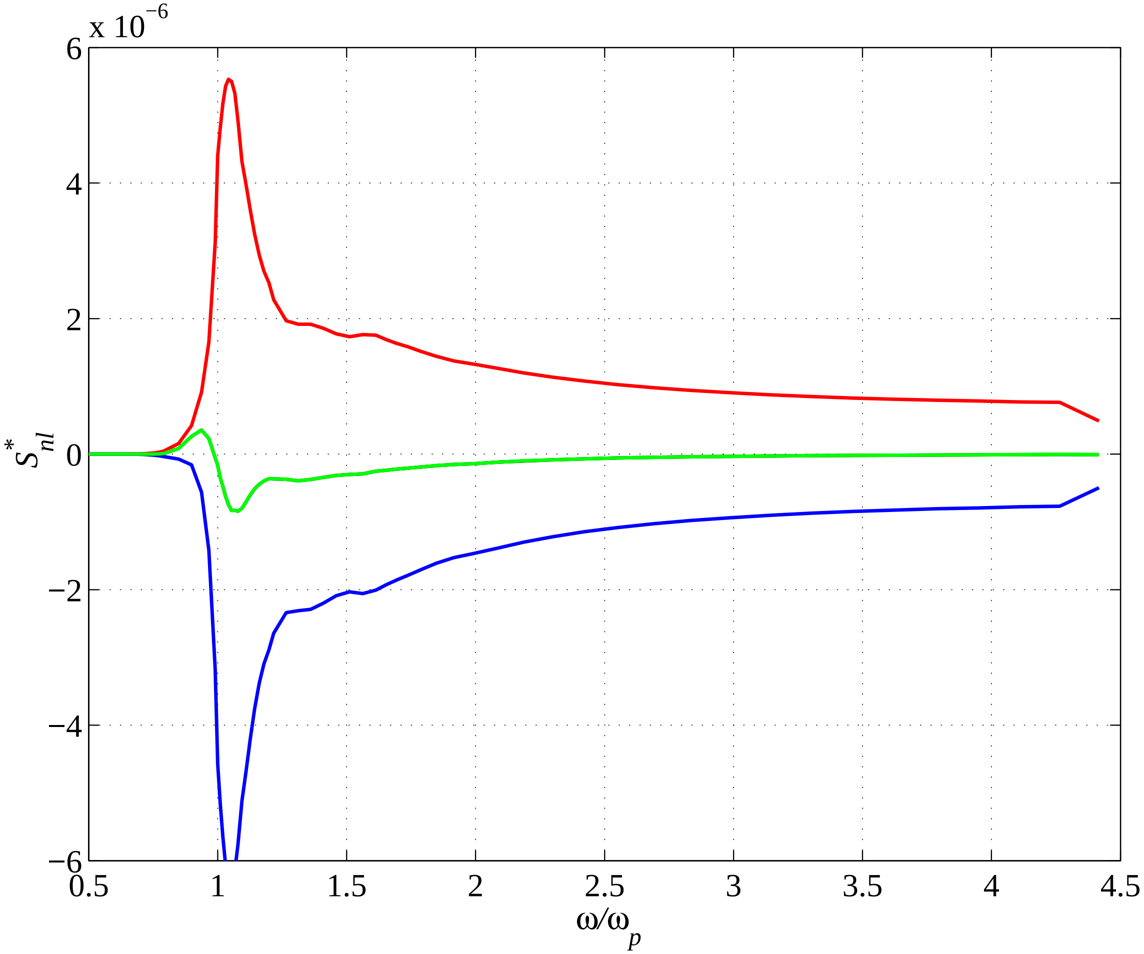
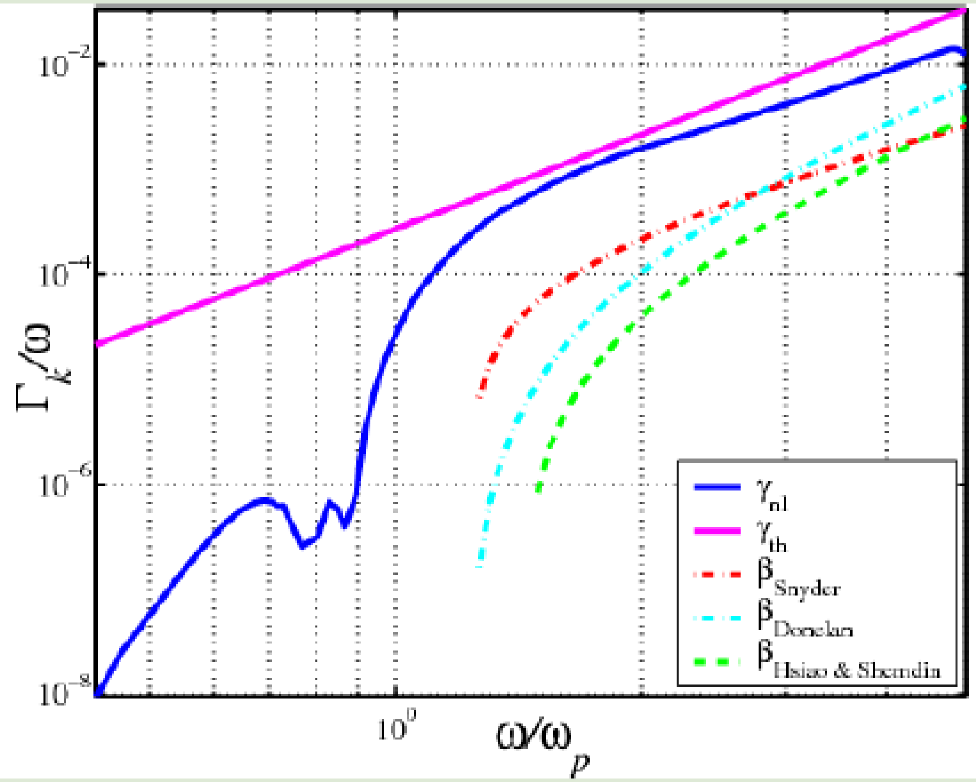


Figure 7: Split of the nonlinear interaction term S_{nl} (central curve) into F_k (upper curve) and $\Gamma_k\, N_k$ (lower curve)

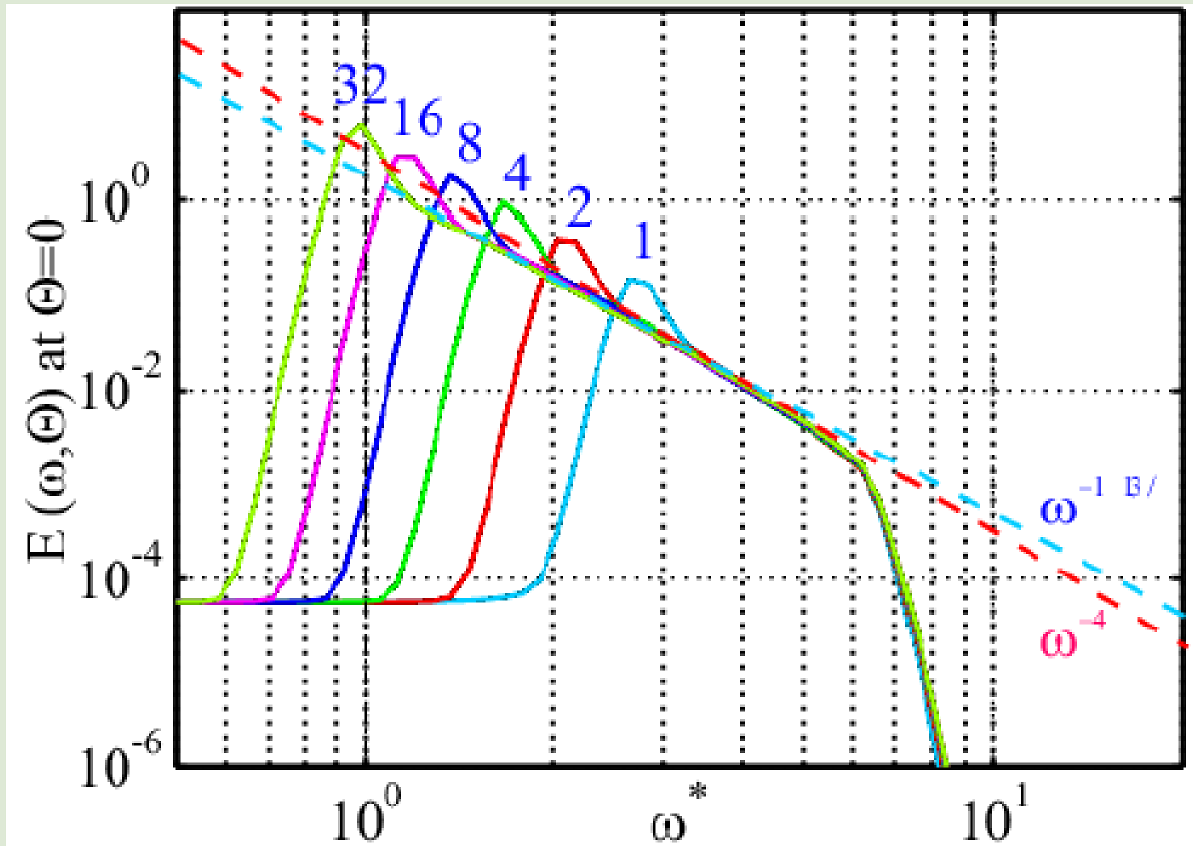
Compare nonlinear damping decrement and wind input increment



S_{nl} surpasses S_{in} and S_{diss} in order of magnitude !

Figure 8:

KE solutions with EXACT S_{nl}



Look at spectral slopes

Figure 9: Numerical solutions with the wind input

The approximation procedure splits wave balance into two parts when S_{nl} dominates

$$\begin{cases} dn_k/dt = S_{nl} \\ d\langle n_k\rangle/dt = \langle S_{in}\rangle + \langle S_{diss}\rangle \end{cases} \quad (*)$$

- We do not ignore input and dissipation, we put them into appropriate place !
- Self-similar solutions (duration-limited) can be found for (*) for power-law dependence of net wave input on time

$$N = at^{\alpha}U_{\beta}(b\mathbf{k}t^{\beta}),$$
$$\text{when } \langle n_k\rangle \sim t^{r};\ \ r = \alpha - 2\beta$$

Figure 10:

We have two-parametric family of self-similar solutions where relationships between parameters are determined by property of homogeneity of collision integral S_{nl}

$$\boxed{a = b^{19/4}; \quad \alpha = \frac{19\beta - 2}{4}}$$

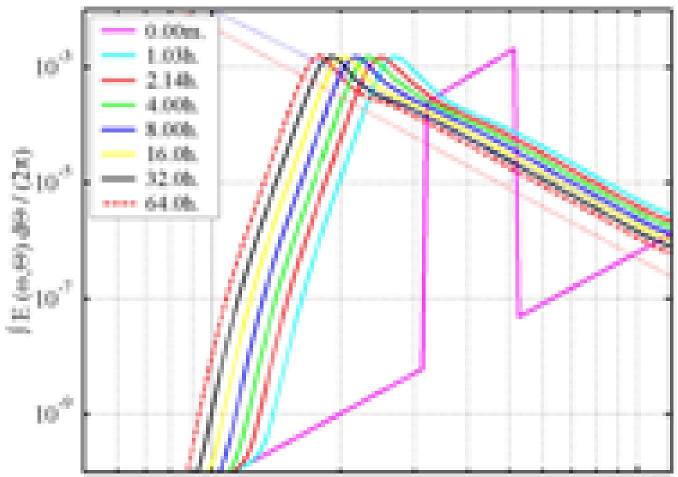
and function of self-similar variable $U_\beta(\xi)$ obeys integro-differential equation

$$\boxed{\alpha U_\beta + \beta \xi \nabla_\xi U_\beta = S_{nl}[U_\beta(\xi)] \quad (**)}$$

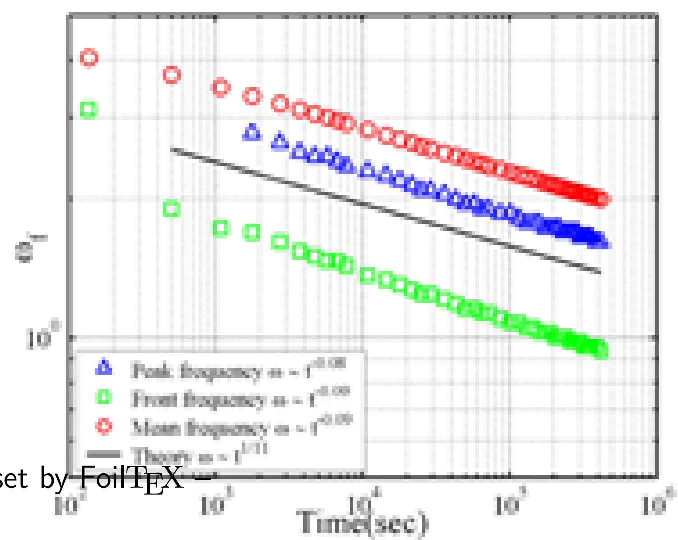
Stationary Kolmogorov-Zakharov solutions appear to be particular cases of the family of non-stationary (or spatially non-homogeneous) self-similar solutions when left-hand and right-hand sides of (**) vanish simultaneously !!!

Figure 11:

Self-similar solutions for wave swell (no input and dissipation)



$$n = t^{2/11} U_0(\omega^2 t^{1/11}, \Theta)$$



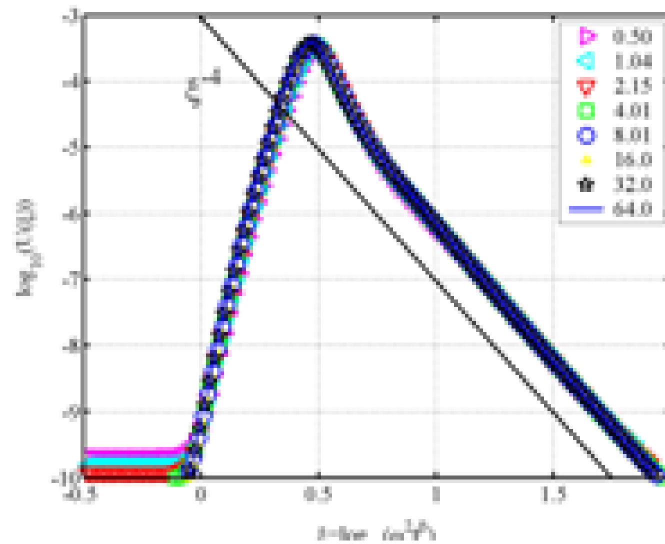
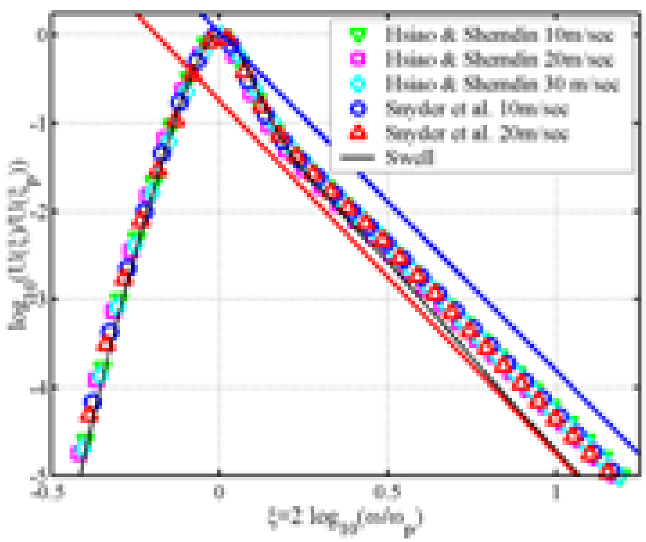


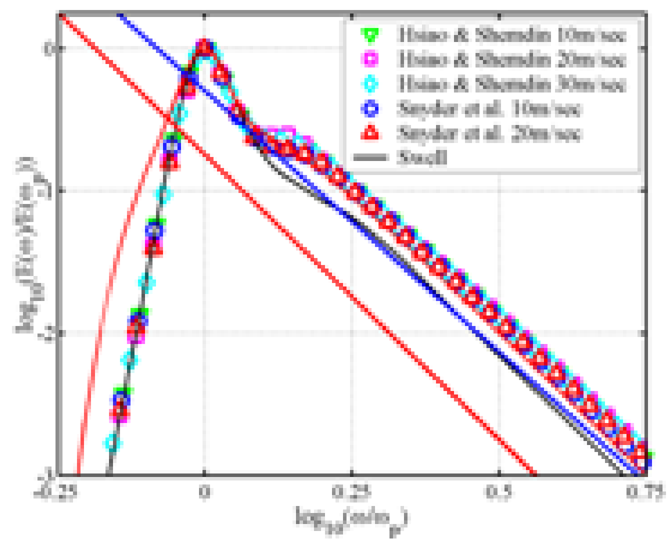
Figure 12:

Quasi-universality of wind-wave spectra

Spatial down-wind spectra



ω−spectra



Dependence of spectral shapes on indexes of self-similarity is weak

Figure 13:

Numerical solutions for duration-limited case *vs* non-dimensional frequency $\omega^*=\omega U/g$

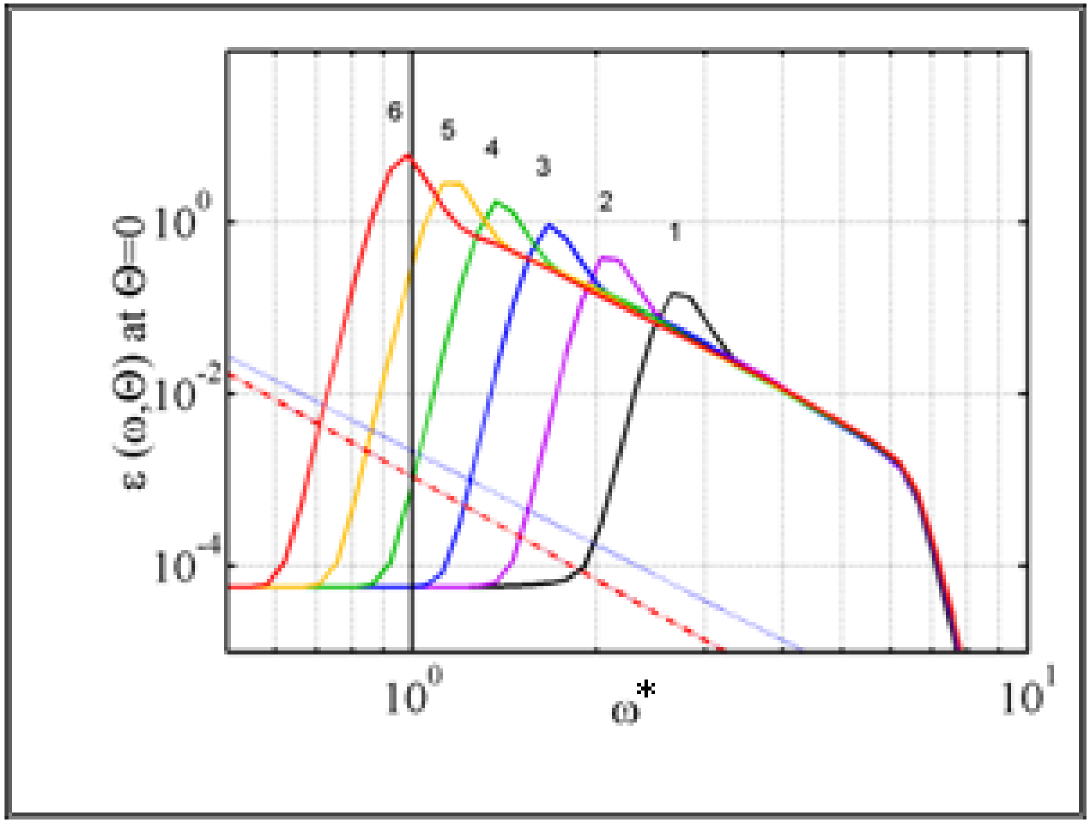


Figure 14:

In this case

$$\beta = \frac{6}{11}, \qquad \alpha = \frac{23}{11}$$

$$r = 1, \qquad < n > \simeq t$$

Time-(fetch-) independent spectra grow as power-law functions of time (fetch) but experimental wind speed scaling

$$\tilde{E} = \frac{Eg^2}{U_h^4}; \qquad \tilde{\omega} = \frac{\omega U_h}{g}$$

is not consistent with our "spectral flux approach"

1. Duration-limited growth $$\tilde{E} = \tilde{E}_0 \tau^p; \quad \tilde{\omega} = \tilde{\omega}_0 \tau^{-q}$$

2. Fetch-limited growth $$\tilde{E} = \tilde{E}_0 \chi^p; \quad \tilde{\omega} = \tilde{\omega}_0 \chi^{-q}$$

Experimental dependencies use 4 parameters. Our two-parameteric self-similar solutions dictate two relationships between these 4 parameters

For case 2

$$\alpha_{ss} = \left(\frac{2E_0^2 \omega_0^{10}}{p_\chi} \right)^{1/3}; \quad p_\chi = \frac{10 q_\chi - 1}{2}$$

α_{ss} – **self-similarity parameter**

Figure 15:

Experimental power-law fits of wind-wave growth. Thanks to Paul Hwang

Something more than an idealization?

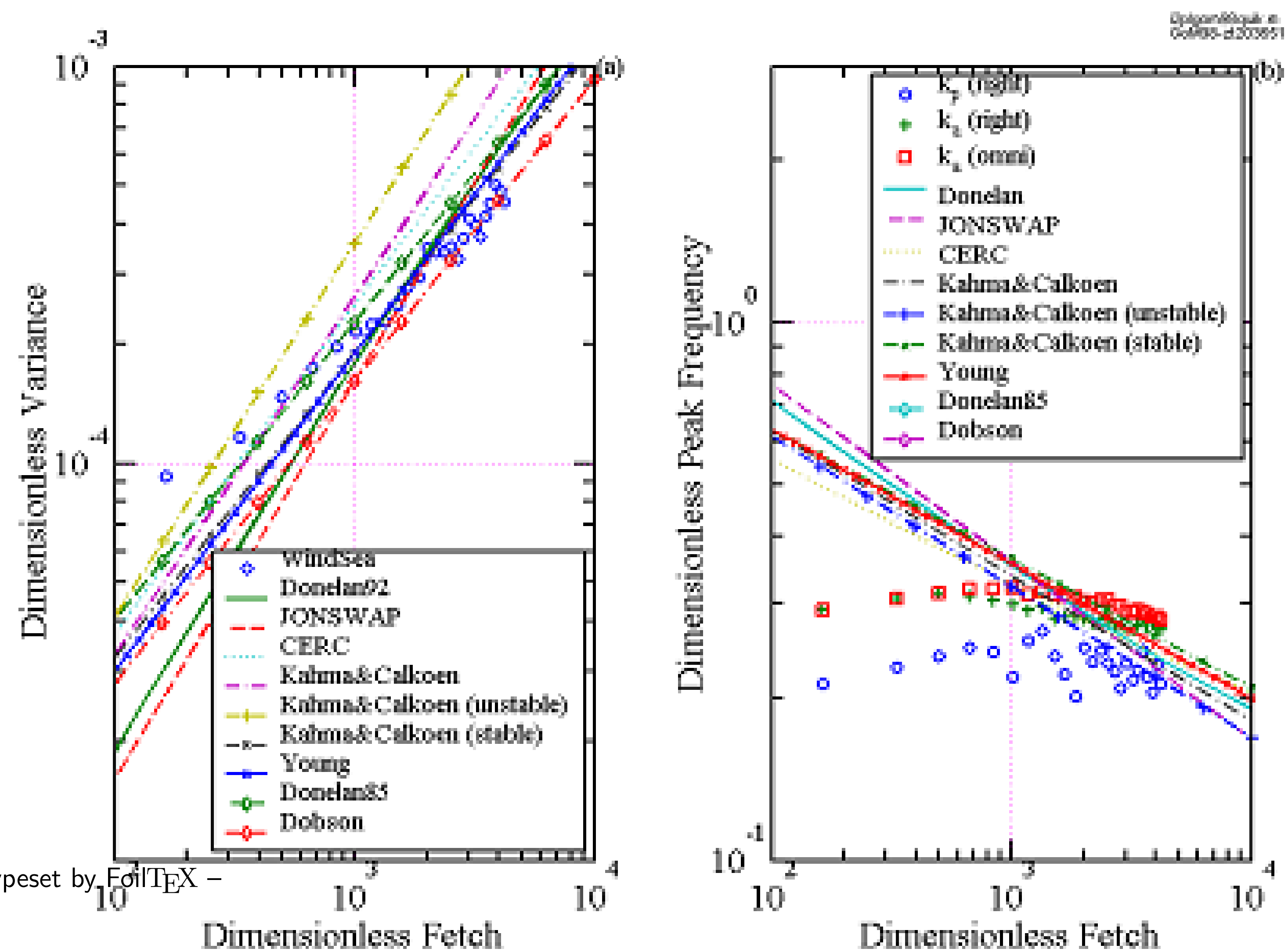
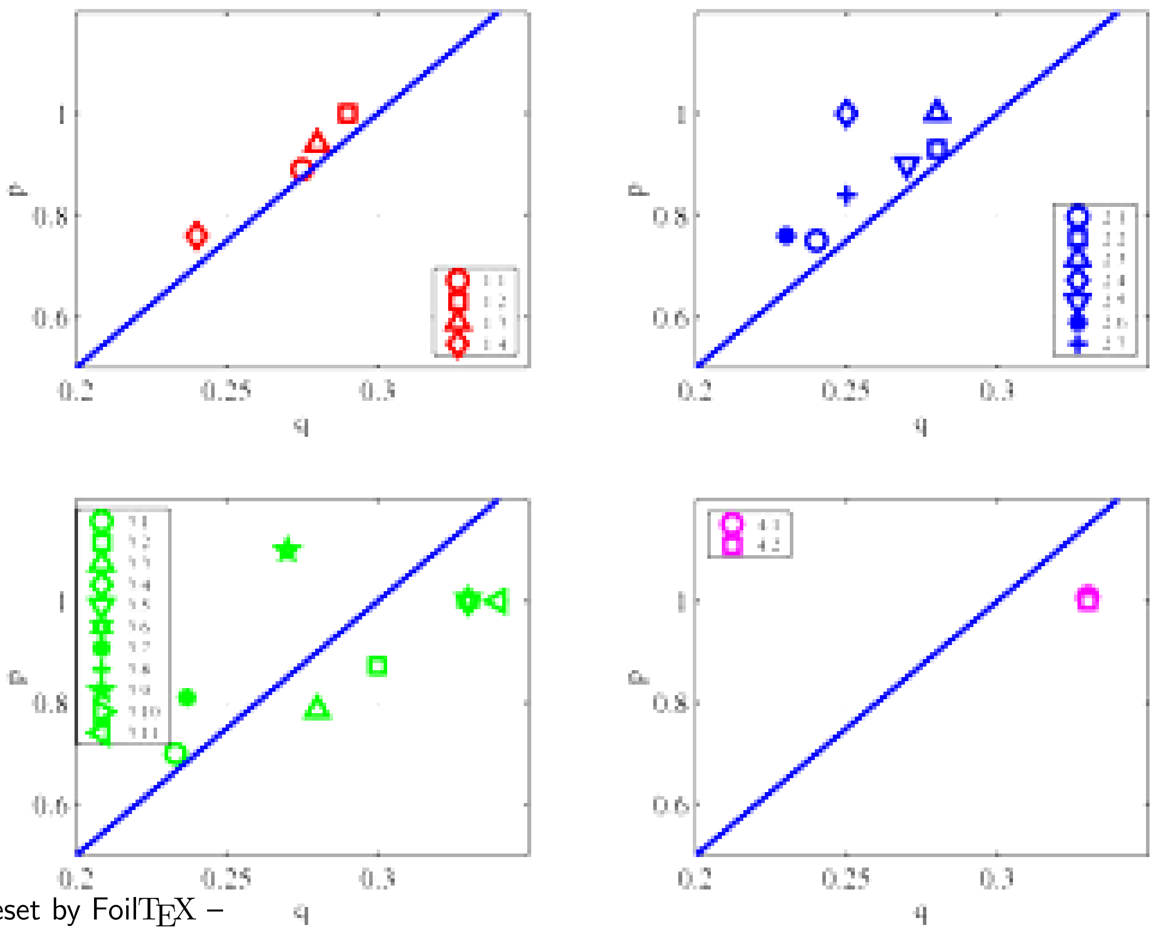


Figure 16:

Exponents p_χ (energy growth) vs q_χ (frequency downshift) for 24 fetch-limited experimental dependencies. Hard line – theoretical dependence

$$p_\chi=(10q_\chi-1)/2$$



1. "Cleanest" fetch-limited
2. Fetch-limited composite data sets
3. One-point measurements converted to fetch-limited one
4. Laboratory data included

Figure 17:

To the theory of Phillips sea

Owen Martin Phillips
1930–2010

Figure 18:

Modified kinetic equation

Phillips' idea: Clavus clavo pellitur (Like cures like)

$$\frac{dE}{dt} = -\frac{\partial P}{\partial \omega} - \Phi(P\omega^3/g^2)\frac{P}{\omega}$$

Dissipation is governed by spectral flux $P(\omega) = \int_0^\omega S_{nl}(\omega)\mathrm{d}\omega$

Dimensional consideration: spectral flux to wave steepness

$$\frac{P\omega^3}{g^2} \sim \left(\frac{E(\omega)\omega^5}{g^2}\right)^3 = \mu_w^6$$

reconcile two opposite (at the first glance) approaches

- weakly turbulent "flux" approach;
- O. Phillips dissipative approach (for the case $\Phi(\mu_w^6)$=const)

Figure 19:

Solutions for the stationary case. Phillips' spectrum ω^{-5}

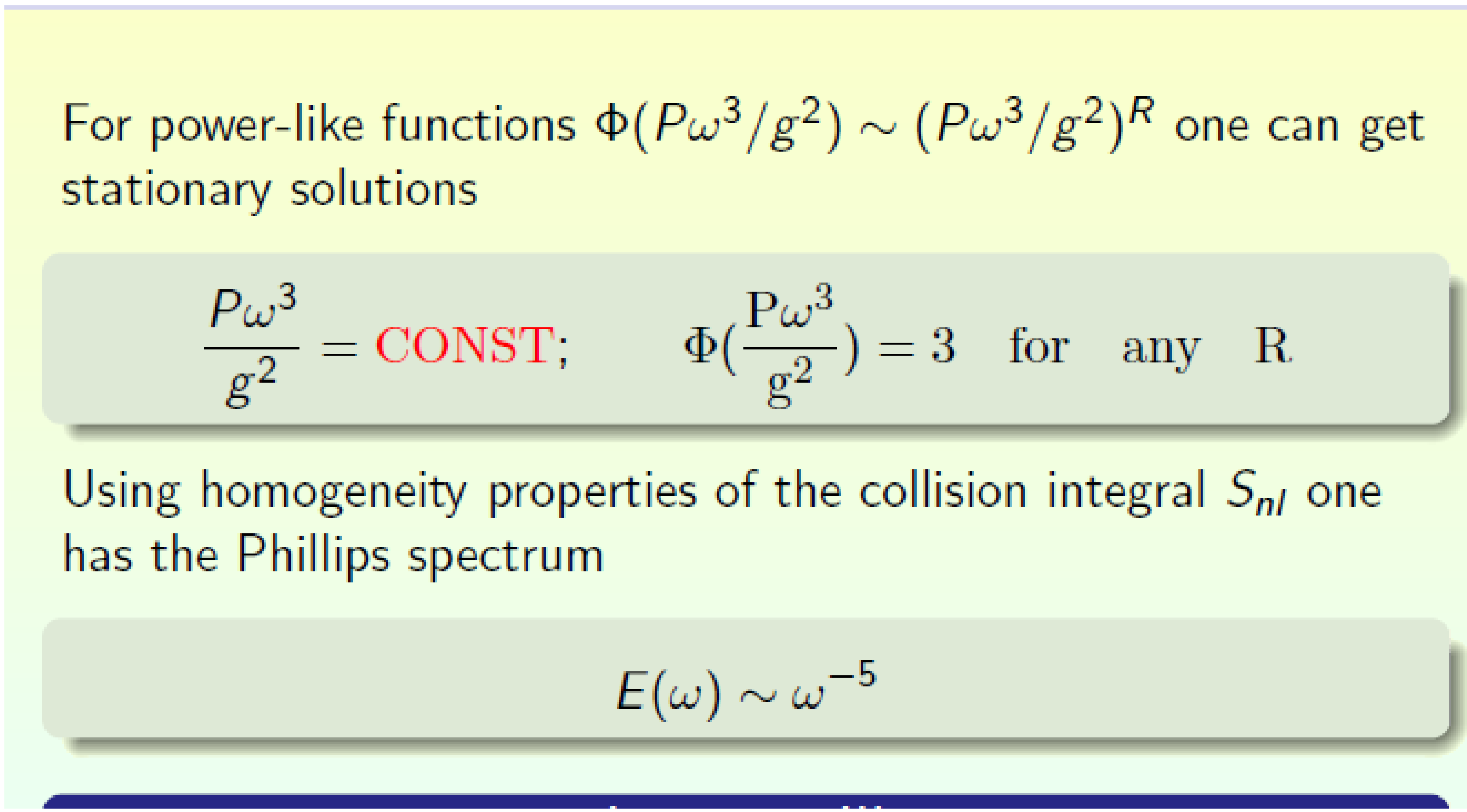


Figure 20:

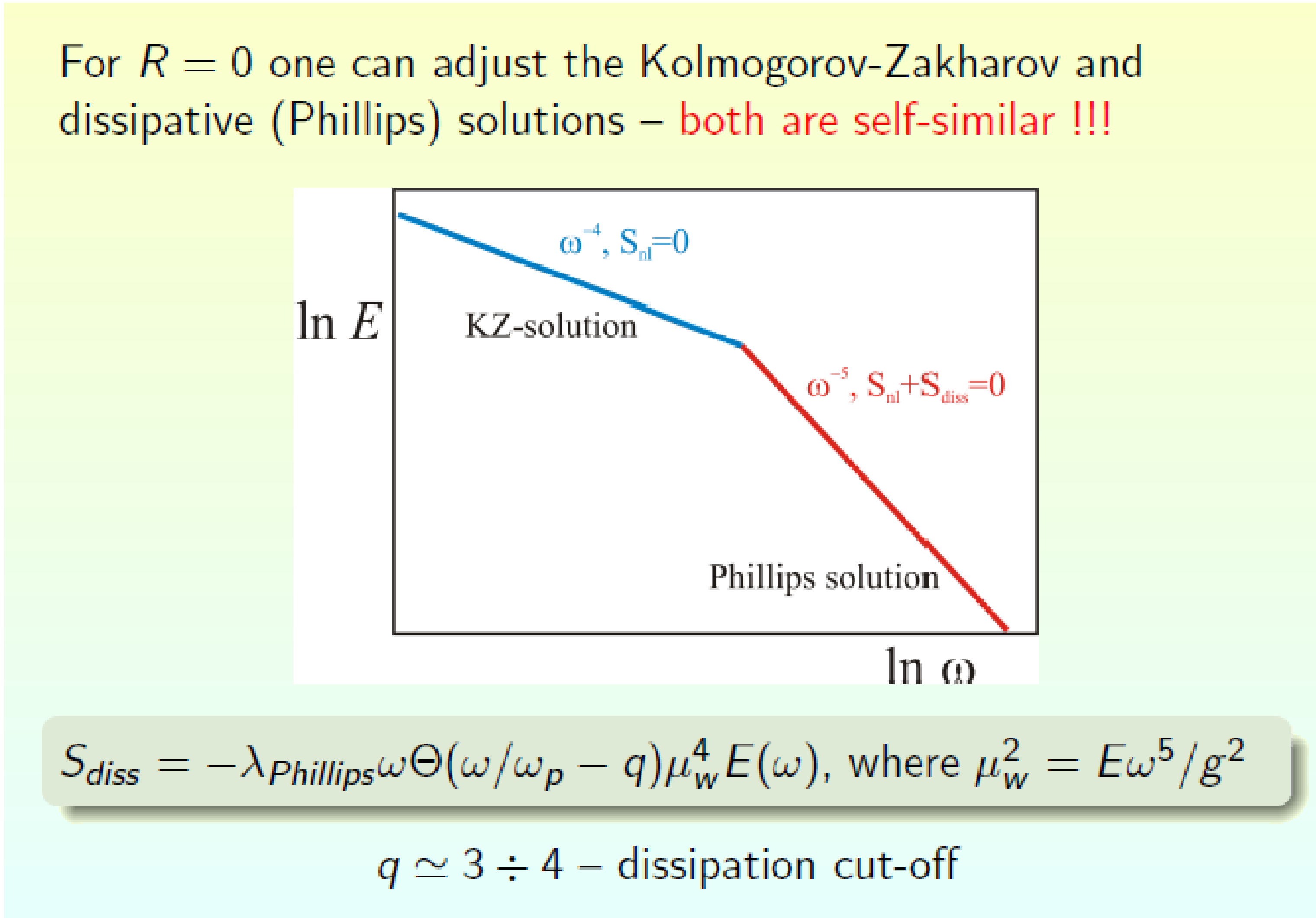


Figure 21: From the "purely-nonlinear" KZ-spectrum to "nonlinear-dissipative" Phillips' spectrum

A minor contribution to Phillips finding

Dissipation term

$$S_{diss} = -\lambda_{Phillips} \omega \Theta(\omega/\omega_p - q)\mu_w^4 E(\omega)$$

has the same homogeneity properties as the collision integral

$$S_{nl}(aE,\ b\omega) = a^3 b^{11} S_{nl}(E,\omega)$$

For power-like spectra $E \sim \omega^{-z}$

$$S_{nl} = C(z)\omega\Theta(\omega/\omega_p - q)\mu_w^4 E(\omega)$$

Spectral slope -5 fixes the dissipation rate

$$\lambda_{Phillips} = C(5) = 2.19$$

Figure 22:

The essential of minor contribution

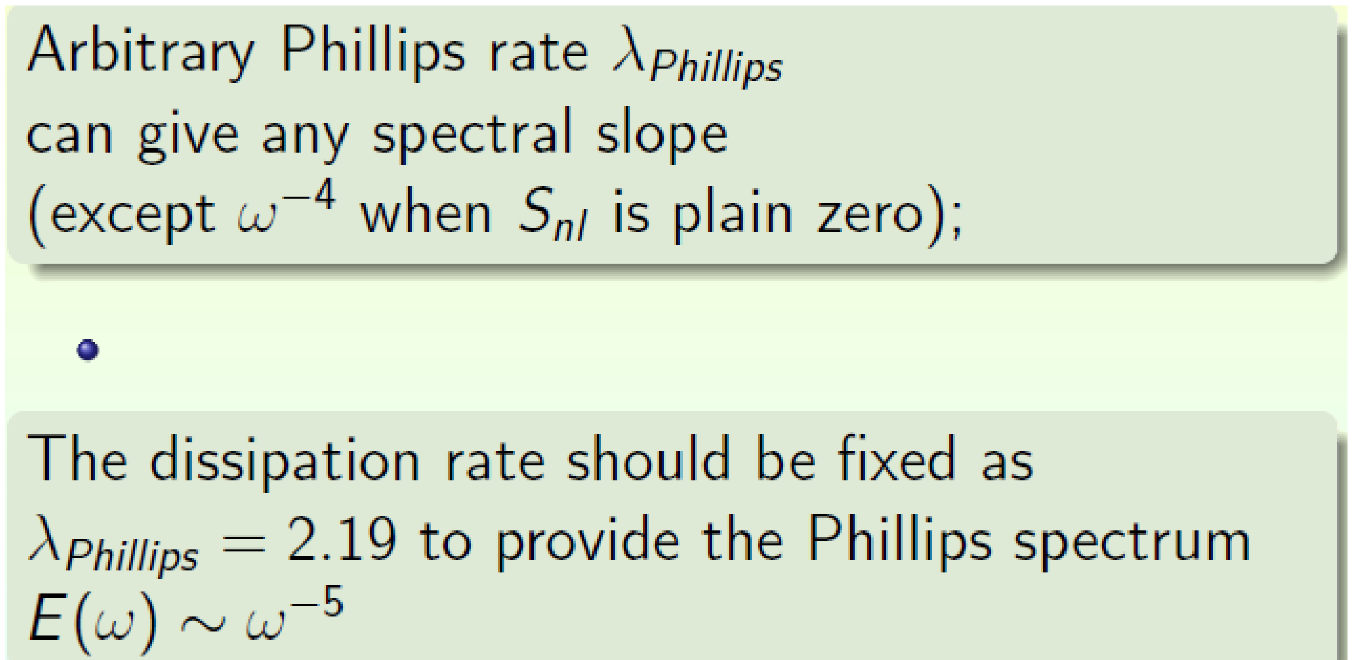


Figure 23:

Setup of numerical experiments. Isotropic swell under the new dissipation

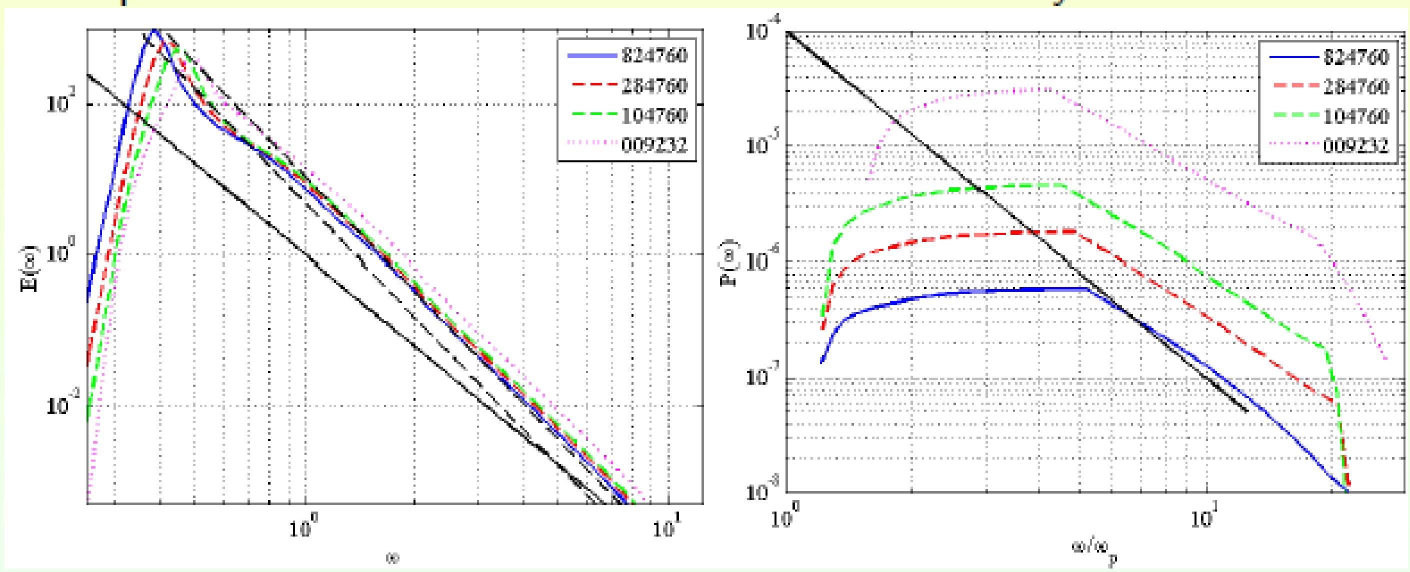
Frequency scaling

- $f_0 = 0.02$ Hz – low frequency limit;
- $f_p = 0.079$ Hz – initial spectral peak frequency (approx. 240 meters wavelength);
- $f_{diss} = 0.318$ Hz – low frequency dissipation cutoff (≈ 4 peak frequencies), or no dissipation cutoff;
- $f_h = 2$ Hz – the high frequency limit.

Phillips' dissipation $R = 0$ – dissipation linear in spectral flux of energy (self-similar solutions);

Figure 24:

Domains of constant and decaying fluxes are co-existing. The Phillips tail close to ω^{-5} is realized for ~ 10 days of evolution



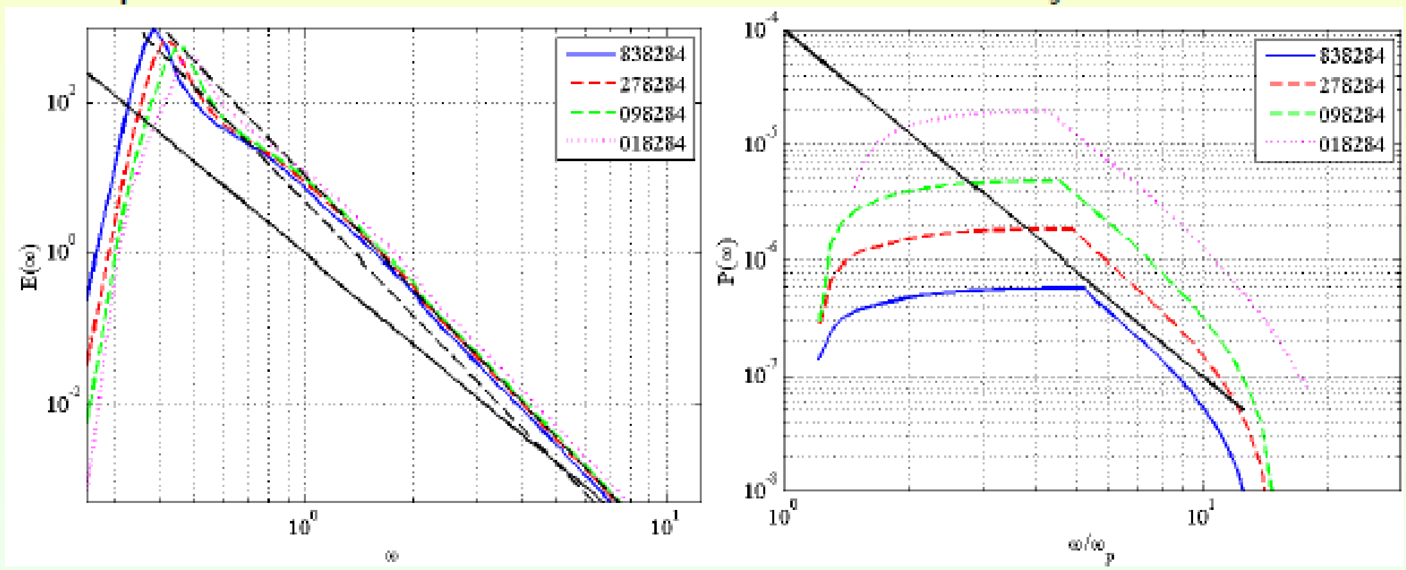
Phillips' spectra and fluxes for $C_{Phillips} = 1.22$ a little bit lower than theoretically perfect $C_{Phillips} = 2.2$

Left – dashed line extreme Phillips' constants $\alpha_p = 0.0081$ and $\alpha_p = 0.018$ (see *Dynamics and Modelling of Ocean Waves by Komen et al. 1994*); Right – straight line ω^{-5}

31 / 33

Figure 25: $C_{Phillips} = 1.22$

Domains of constant and decaying fluxes are co-existing. The Phillips tail close to ω^{-5} is realized for ~ 10 days of evolution



Phillips' spectra and fluxes for $C_{Phillips} = 2.75$ a little bit higher than theoretically perfect $C_{Phillips} = 2.2$

Left – dashed line extreme Phillips' constants $\alpha_p = 0.0081$ and $\alpha_p = 0.018$ (see *Dynamics and Modelling of Ocean Waves by Komen et al. 1994*); Right – straight line ω^{-5}

Figure 26: $C_{Phillips} = 2.75$

Summary

1. Both analytic and computation arguments show that S_{nl}, the four-wave nonlinear interaction, is the dominating process in the wind-driven sea physics. This fact makes possible to develop the semianalytical weak-turbulent theory of the wind-driven sea, based on the conservative Hasselmann equation.

2. This theory describes pretty well the wind-driven sea in the energy-containing range of scales

$$k_p < k < 10\,k_p$$

S_{in}, the wind-input term has to be found by comparison with observation data.

3. The proper choice of the dissipation function S_{diss} makes possible to extent the weak-turbulent theory for much broader range of scales.

What is missed in this talk?

I did not discuss the numerous deterministic numerical experiments, which try to describe the wind-driven sea from the "first principle", i.e. without use of the statistical approach. This is a subject for another lecture. The results obtained so far, basically support the weak-turbulent theory.