T. Brooke Benjamin

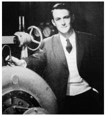
... a brilliant researcher .... original and elegant applied mathematics ... extraordinary physical insights ...
changed the way we now think about hydrodynamics
... about 6ft tall ... moved slowly ... smiled slightly ... often smoking a pipe ... every word counted ...
... he and his wife Natalia were a joyous couple with many friends ...

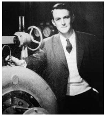
Mathematical Aspects of Classical Water Wave Theory from the Past 20 Year

- the fascination of what's difficult -

Brooke Benjamin Lecture

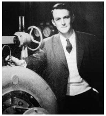
Oxford 26th November 2013





Classical Water Waves

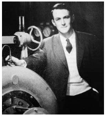
What distinguishes the water wave problems from others in hydrodynamics is that the unknown is the region occupied by the fluid. It is referred to as the free surface



Classical Water Waves

What distinguishes the water wave problems from others in hydrodynamics is that the unknown is the region occupied by the fluid. It is referred to as the free surface

For today Classical means two-dimensional irrotational flow under gravity of a heavy perfect liquid with a free surface,

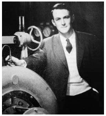


Classical Water Waves

What distinguishes the water wave problems from others in hydrodynamics is that the unknown is the region occupied by the fluid. It is referred to as the free surface

For today Classical means two-dimensional irrotational flow under gravity of a heavy perfect liquid with a free surface,

There is no surface tension and there are no floating bodies



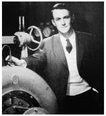
Classical Water Waves

What distinguishes the water wave problems from others in hydrodynamics is that the unknown is the region occupied by the fluid. It is referred to as the free surface

For today Classical means two-dimensional irrotational flow under gravity of a heavy perfect liquid with a free surface,

There is no surface tension and there are no floating bodies

Although there has been important recent progress on the initial-value problem for free boundaries and on fully 3-dimensional waves, that is not the concern here.



Classical Water Waves

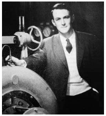
What distinguishes the water wave problems from others in hydrodynamics is that the unknown is the region occupied by the fluid. It is referred to as the free surface

For today Classical means two-dimensional irrotational flow under gravity of a heavy perfect liquid with a free surface,

There is no surface tension and there are no floating bodies

Although there has been important recent progress on the initial-value problem for free boundaries and on fully 3-dimensional waves, that is not the concern here.

2-Dimensional Irrotational Water Waves:



Classical Water Waves

What distinguishes the water wave problems from others in hydrodynamics is that the unknown is the region occupied by the fluid. It is referred to as the free surface

For today Classical means two-dimensional irrotational flow under gravity of a heavy perfect liquid with a free surface,

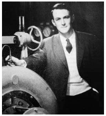
There is no surface tension and there are no floating bodies

Although there has been important recent progress on the initial-value problem for free boundaries and on fully 3-dimensional waves, that is not the concern here.

2-Dimensional Irrotational Water Waves:
In Eulerian co ordinates the velocity at a point (x, y, z) in the fluid at time t is given by the gradient of a scalar potential ϕ on $\mathbb{R}^2$

$$\vec{v}(x, y, z; t) = \nabla\phi(x, y; t)$$

which satisfy



Irrotational Water Waves

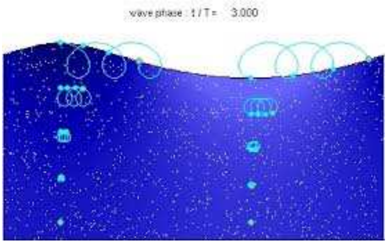
Infinite depth

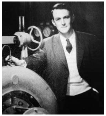
Wave interior

$$\Omega = \{(x, y) : y < \eta(x, t)\}$$

$$\Delta\phi(x, y; t) = 0$$

$$\nabla\phi(x, y; t) \to 0 \text{ as } y \to -\infty$$





Irrotational Water Waves

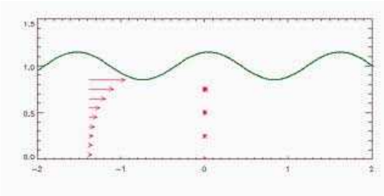
Boundary Conditions

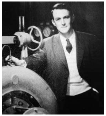
gravity g acts vertically down

Wave Surface

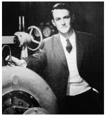
$$\mathcal{S} = \{(x, \eta(x,t)) : x \in \mathbb{R}\}$$

$$\left.\begin{aligned} \phi_t + \tfrac{1}{2}|\nabla\phi|^2 + gy &= 0 \\ \eta_t + \phi_x\eta_x - \phi_y &= 0 \end{aligned}\right\} \text{ on } \mathcal{S}$$



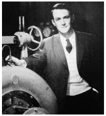


This is a very old problem, studied by Cauchy, Laplace, Lagrange, Poisson, Green, Airy, Stokes, Rayleigh ...



This is a very old problem, studied by Cauchy, Laplace, Lagrange, Poisson, Green, Airy, Stokes, Rayleigh ...

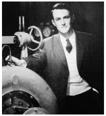
After he discovered experimentally the solitary wave, in 1844 John Scott Russell described



This is a very old problem, studied by Cauchy, Laplace, Lagrange, Poisson, Green, Airy, Stokes, Rayleigh ...

After he discovered experimentally the solitary wave, in 1844 John Scott Russell described

the greater part of the investigations of Poisson and Cauchy under the name of wave theory are rather to be regarded as mathematical exercises, not physical investigations



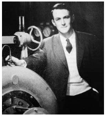
This is a very old problem, studied by Cauchy, Laplace, Lagrange, Poisson, Green, Airy, Stokes, Rayleigh ...

After he discovered experimentally the solitary wave, in 1844 John Scott Russell described

the greater part of the investigations of Poisson and Cauchy under the name of wave theory are rather to be regarded as mathematical exercises, not physical investigations

... since no one had predicted the wave it remained for

mathematicians to give an "*a priori* demonstration *a posteriori*"



This is a very old problem, studied by Cauchy, Laplace, Lagrange, Poisson, Green, Airy, Stokes, Rayleigh ...

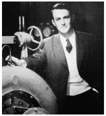
After he discovered experimentally the solitary wave, in 1844 John Scott Russell described

the greater part of the investigations of Poisson and Cauchy under the name of wave theory are rather to be regarded as mathematical exercises, not physical investigations

... since no one had predicted the wave it remained for

mathematicians to give an "*a priori* demonstration *a posteriori*"

Progress was very slow throughout the 19th century with approximation for certain scaling limits but the full equations remained essentially unexamined for more than one hundred years but progress had accelerated hugely in recent years



This is a very old problem, studied by Cauchy, Laplace, Lagrange, Poisson, Green, Airy, Stokes, Rayleigh ...

After he discovered experimentally the solitary wave, in 1844 John Scott Russell described

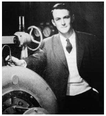
the greater part of the investigations of Poisson and Cauchy under the name of wave theory are rather to be regarded as mathematical exercises, not physical investigations

... since no one had predicted the wave it remained for

mathematicians to give an "*a priori* demonstration *a posteriori*"

Progress was very slow throughout the 19th century with approximation for certain scaling limits but the full equations remained essentially unexamined for more than one hundred years but progress had accelerated hugely in recent years

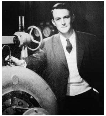
Alex Craik, a PhD student of Brooke's, has written extensively on the history of the water wave problem



Energetics

The total wave energy at time t is Kinetic + Potential:

$$\frac{1}{2}\int\int_{-\infty}^{\eta(x,t)}|\nabla\phi(x,y;t)dy|^2dx+\frac{g}{2}\int\eta^2(x;t)dx$$



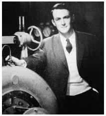
Energetics

The total wave energy at time t is Kinetic + Potential:

$$\frac{1}{2}\int\int_{-\infty}^{\eta(x,t)} |\nabla\phi(x,y;t)dy|^2 dx + \frac{g}{2}\int \eta^2(x;t)dx$$

Given periodic functions η and Φ of the single variable x let

$$\Omega = \{(x,y) : y < \eta(x)\}$$



Energetics

The total wave energy at time t is Kinetic + Potential:

$$\frac{1}{2}\int\int_{-\infty}^{\eta(x,t)} |\nabla\phi(x,y;t)dy|^2 dx + \frac{g}{2}\int \eta^2(x;t)dx$$

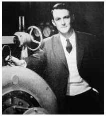
Given periodic functions η and Φ of the single variable x let

$$\Omega = \{(x,y) : y < \eta(x)\}$$

ϕ the solution of the <u>corresponding</u> Dirichlet problem

$$\left.\begin{array}{c} \Delta\phi(x,y) = 0 \\ \phi \to 0 \text{ as } y \to -\infty \end{array}\right\} \text{ on } \Omega$$

$$\phi(x,\eta(x)) = \Phi(x)$$



Energetics

The total wave energy at time t is Kinetic + Potential:

$$\frac{1}{2}\int\int_{-\infty}^{\eta(x,t)} |\nabla\phi(x,y;t)dy|^2 dx + \frac{g}{2}\int \eta^2(x;t)dx$$

Given periodic functions η and Φ of the single variable x let

$$\Omega = \{(x,y) : y < \eta(x)\}$$

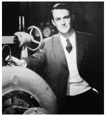
ϕ the solution of the <u>corresponding</u> Dirichlet problem

$$\left.\begin{array}{c} \Delta\phi(x,y) = 0 \\ \phi \to 0 \text{ as } y \to -\infty \end{array}\right\} \text{ on } \Omega$$

$$\phi(x,\eta(x)) = \Phi(x)$$

let

$$\mathcal{E}(\eta,\Phi) = \frac{1}{2}\int\int_{-\infty}^{\eta(x)} |\nabla\phi(x,y)dy|^2 dx + g\eta^2(x)dx$$

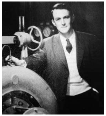


With this functional

$$\mathcal{E}(\eta, \Phi) := \frac{1}{2} \int \int_{-\infty}^{\eta(x)} |\nabla \phi(x, y) dy|^2 dx + g\eta^2(x) dx$$

and with the "variational" derivatives

$$\frac{\partial \mathcal{E}}{\partial \Phi} \text{ and } \frac{\partial \mathcal{E}}{\partial \eta}$$



With this functional

$$\mathcal{E}(\eta, \Phi) := \frac{1}{2} \int \int_{-\infty}^{\eta(x)} |\nabla \phi(x, y) dy|^2 dx + g\eta^2(x) dx$$

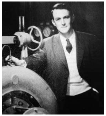
and with the “variational” derivatives

$$\frac{\partial \mathcal{E}}{\partial \Phi} \text{ and } \frac{\partial \mathcal{E}}{\partial \eta}$$

Zakharov (1968) observed that solutions (η, Φ) of

$$\boxed{\frac{\partial \eta}{\partial t} = \frac{\partial \mathcal{E}}{\partial \Phi}(\eta, \Phi); \qquad \frac{\partial \Phi}{\partial t} = -\frac{\partial \mathcal{E}}{\partial \eta}(\eta, \Phi)}$$

yields a water wave



Benjamin & Olver (1982) studied

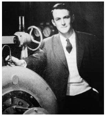
$$\frac{\partial \eta}{\partial t} = \frac{\partial \mathcal{E}}{\partial \Phi}(\eta, \Phi); \qquad \frac{\partial \Phi}{\partial t} = -\frac{\partial \mathcal{E}}{\partial \eta}(\eta, \Phi)$$

as a Hamiltonian system of classical type

$$\dot{x} = J\nabla\mathcal{E}(x), \quad x = (\eta, \Phi), \;\; J = \begin{pmatrix} 0, I \\ -I, 0 \end{pmatrix},$$

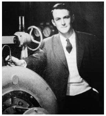
η, ϕ being the infinite dimensional canonical variables which they referred to as "coordinates" and "momentum" and in an Appendix gives the Hamiltonian formulation independent of coordinates.

Both Zakharov and Benjamin was conscious of the implications of the Hamiltonian formulation for stability



Spatially Periodic Waves

Normalised Period 2π



Spatially Periodic Waves

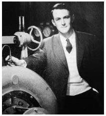
Normalised Period 2π

Based on conformal mapping theory any rectifiable periodic Jordan curve $\mathcal{S} = \{(x, \eta(x)) : x \in \mathbb{R}\}$ can be re-parametrised as

$$\mathcal{S} = \{(-\xi - \mathcal{C}w(\xi), w(\xi)) : \xi \in \mathbb{R}\}$$

where $\mathcal{C}w$ is the Hilbert transform of a periodic function w:

$$\mathcal{C}w(\xi) = pv\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{w(\sigma)\,d\sigma}{\tan\frac{1}{2}(\xi - \sigma)}$$



Spatially Periodic Waves

Normalised Period 2π

Based on conformal mapping theory any rectifiable periodic Jordan curve $\mathcal{S} = \{(x, \eta(x)) : x \in \mathbb{R}\}$ can be re-parametrised as

$$\mathcal{S} = \{(-\xi - \mathcal{C}w(\xi), w(\xi)) : \xi \in \mathbb{R}\}$$

where $\mathcal{C}w$ is the Hilbert transform of a periodic function w:

$$\mathcal{C}w(\xi) = pv\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{w(\sigma)\, d\sigma}{\tan \frac{1}{2}(\xi - \sigma)}$$

Dyachenko, Kuznetsov, Spector & Zakharov (1996) used complex function theory in a deep and beautiful way to reduce
Zakharov's awkward system to the following "simple" system:

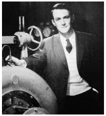
$$\dot{w}(1 + \mathcal{C}w') - \mathcal{C}\varphi' - w'\mathcal{C}\dot{w} = 0$$

$$\mathcal{C}\left(w'\dot{\varphi} - \dot{w}\varphi' + \lambda w w'\right) + (\dot{\varphi} + \lambda w)(1 + \mathcal{C}w') - \varphi'\mathcal{C}\dot{w} = 0$$

$\dot{} = \partial/\partial t, \qquad {}' = \partial/\partial x$

w = wave height φ = potential at surface:

$0 < \lambda$ = gravity after normalisating the wavelength as 2π



Spatially Periodic Waves

Normalised Period 2π

Based on conformal mapping theory any rectifiable periodic Jordan curve $\mathcal{S} = \{(x, \eta(x)) : x \in \mathbb{R}\}$ can be re-parametrised as

$$\mathcal{S} = \{(-\xi - \mathcal{C}w(\xi), w(\xi)) : \xi \in \mathbb{R}\}$$

where $\mathcal{C}w$ is the Hilbert transform of a periodic function w:

$$\mathcal{C}w(\xi) = pv\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{w(\sigma)\,d\sigma}{\tan\frac{1}{2}(\xi - \sigma)}$$

Dyachenko, Kuznetsov, Spector & Zakharov (1996) used complex function theory in a deep and beautiful way to reduce
Zakharov's awkward system to the following "simple" system:

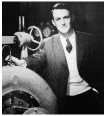
$$\dot{w}(1 + \mathcal{C}w') - \mathcal{C}\varphi' - w'\mathcal{C}\dot{w} = 0$$

$$\mathcal{C}\left(w'\dot{\varphi} - \dot{w}\varphi' + \lambda w w'\right) + (\dot{\varphi} + \lambda w)(1 + \mathcal{C}w') - \varphi'\mathcal{C}\dot{w} = 0$$

$\dot{} = \partial/\partial t, \qquad ' = \partial/\partial x$

w = wave height φ = potential at surface:

$0 < \lambda$ = gravity after normalisating the wavelength as 2π

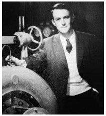


But it does not look like a Hamiltonian system any more!

A Symplectic Form

For $(w, \varphi) \in M := W^{1,2}_{2\pi} \times W^{1,2}_{2\pi}$ let

$$
\begin{aligned}
\omega_{(w,\varphi)}\big((w_1, \varphi_1), (w_2, \varphi_2)\big) \\
= \int_{-\pi}^{\pi} (1 + \mathcal{C}w')(\varphi_2 w_1 - \varphi_1 w_2) \\
+ w'\big(\varphi_1 \mathcal{C} w_2 - \varphi_2 \mathcal{C} w_1\big) \\
- \varphi'\big(w_1 \mathcal{C} w_2 - w_2 \mathcal{C} w_1\big)\, d\xi
\end{aligned}
$$



A Symplectic Form

For $(w, \varphi) \in M := W^{1,2}_{2\pi} \times W^{1,2}_{2\pi}$ let

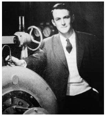
$$\begin{aligned}\omega_{(w,\varphi)}\big((w_1, \varphi_1), (w_2, \varphi_2)\big) \\ = \int_{-\pi}^{\pi} (1 + \mathcal{C}w')(\varphi_2 w_1 - \varphi_1 w_2) \\ + w'\big(\varphi_1 \mathcal{C} w_2 - \varphi_2 \mathcal{C} w_1\big) \\ - \varphi'\big(w_1 \mathcal{C} w_2 - w_2 \mathcal{C} w_1\big)\, d\xi\end{aligned}$$

This is a skew-symmetric bilinear form with

$$\omega = d\varpi$$

where ω is the 1-form

$$\varpi_{\varphi,w}(\hat{w}, \hat{\varphi}) = \int_{-\pi}^{\pi} \left\{\varphi(1 + \mathcal{C}w') + \mathcal{C}\big(\varphi w'\big)\right\} \hat{w}\, d\xi$$



A Symplectic Form

For $(w, \varphi) \in M := W^{1,2}_{2\pi} \times W^{1,2}_{2\pi}$ let

$$\omega_{(w,\varphi)}\big((w_1, \varphi_1), (w_2, \varphi_2)\big)$$
$$= \int_{-\pi}^{\pi} (1 + \mathcal{C}w')(\varphi_2 w_1 - \varphi_1 w_2) + w'\big(\varphi_1 \mathcal{C} w_2 - \varphi_2 \mathcal{C} w_1\big) - \varphi'\big(w_1 \mathcal{C} w_2 - w_2 \mathcal{C} w_1\big)\, d\xi$$

This is a skew-symmetric bilinear form with

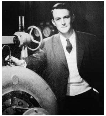
$$\omega = d\varpi$$

where ω is the 1-form

$$\varpi_{\varphi,w}(\hat{w}, \hat{\varphi}) = \int_{-\pi}^{\pi} \left\{\varphi(1 + \mathcal{C}w') + \mathcal{C}\big(\varphi w'\big)\right\} \hat{w} d\xi$$

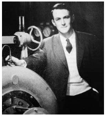
Hence ω is exact (and so closed)

From Riemann-Hilbert theory it is non-degenerate



Thus the Water-Wave Problem

involving a PDE with nonlinear boundary conditions on an unknown domain:



Thus the Water-Wave Problem

involving a PDE with nonlinear boundary conditions on an unknown domain:

$$\mathcal{S} = \{(x, \eta(x,t)) : x \in \mathbb{R}\}$$

$$\eta(x + 2\pi) = \eta(x,t) = \eta(x, t + 2\pi)$$

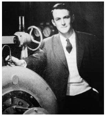
$$\left.\begin{array}{r} \phi_t + \frac{1}{2}|\nabla\phi|^2 + \lambda y = 0; \\ \phi_t + \phi_x\eta_x - \phi_y\eta_y = 0 \end{array}\right\} \text{ on } \mathcal{S}$$

For $(x, y; t)$ with $y < \eta(x,t)$

$$\Delta\phi(x, y; t) = 0$$

$$\nabla\phi(x, y; t) \to 0 \text{ as } y \to -\infty$$

$$\phi(x, y; t) = \phi(x + 2\pi, y; t)$$



Thus the Water-Wave Problem

involving a PDE with nonlinear boundary conditions on an unknown domain:

$$\mathcal{S} = \{(x, \eta(x,t)) : x \in \mathbb{R}\}$$

$$\eta(x + 2\pi) = \eta(x,t) = \eta(x, t + 2\pi)$$

$$\left.\begin{array}{r} \phi_t + \frac{1}{2}|\nabla\phi|^2 + \lambda y = 0; \\ \phi_t + \phi_x\eta_x - \phi_y\eta_y = 0 \end{array}\right\} \text{ on } \mathcal{S}$$

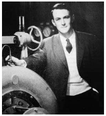
For $(x, y; t)$ with $y < \eta(x,t)$

$$\Delta\phi(x, y; t) = 0$$

$$\nabla\phi(x, y; t) \to 0 \text{ as } y \to -\infty$$

$$\phi(x, y; t) = \phi(x + 2\pi, y; t)$$

is equivalent to:



Thus the Water-Wave Problem

involving a PDE with nonlinear boundary conditions on an unknown domain:

$$\mathcal{S} = \{(x, \eta(x,t)) : x \in \mathbb{R}\}$$

$$\eta(x + 2\pi) = \eta(x,t) = \eta(x, t + 2\pi)$$

$$\left.\begin{aligned} \phi_t + \tfrac{1}{2}|\nabla\phi|^2 + \lambda y = 0; \\ \phi_t + \phi_x\eta_x - \phi_y\eta_y = 0 \end{aligned}\right\} \text{ on } \mathcal{S}$$

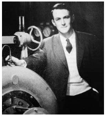
For $(x, y; t)$ with $y < \eta(x,t)$

$$\Delta\phi(x, y; t) = 0$$

$$\nabla\phi(x, y; t) \to 0 \text{ as } y \to -\infty$$

$$\phi(x, y; t) = \phi(x + 2\pi, y; t)$$

is equivalent to:
two equations



Thus the Water-Wave Problem

involving a PDE with nonlinear boundary conditions on an unknown domain:

$$\mathcal{S} = \{(x, \eta(x,t)) : x \in \mathbb{R}\}$$

$$\eta(x + 2\pi) = \eta(x,t) = \eta(x, t + 2\pi)$$

$$\left.\begin{array}{r} \phi_t + \frac{1}{2}|\nabla\phi|^2 + \lambda y = 0; \\ \phi_t + \phi_x\eta_x - \phi_y\eta_y = 0 \end{array}\right\} \text{ on } \mathcal{S}$$

For $(x, y; t)$ with $y < \eta(x,t)$

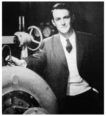
$$\Delta\phi(x, y; t) = 0$$

$$\nabla\phi(x, y; t) \to 0 \text{ as } y \to -\infty$$

$$\phi(x, y; t) = \phi(x + 2\pi, y; t)$$

is equivalent to:

two equations each with quadratic nonlinearities



Thus the Water-Wave Problem

involving a PDE with nonlinear boundary conditions on an unknown domain:

$$\mathcal{S} = \{(x, \eta(x,t)) : x \in \mathbb{R}\}$$

$$\eta(x+2\pi) = \eta(x,t) = \eta(x, t+2\pi)$$

$$\left.\begin{array}{r} \phi_t + \frac{1}{2}|\nabla\phi|^2 + \lambda y = 0; \\ \phi_t + \phi_x\eta_x - \phi_y\eta_y = 0 \end{array}\right\} \text{ on } \mathcal{S}$$

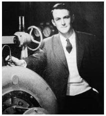
For $(x, y; t)$ with $y < \eta(x,t)$

$$\Delta\phi(x,y;t) = 0$$

$$\nabla\phi(x,y;t) \to 0 \text{ as } y \to -\infty$$

$$\phi(x,y;t) = \phi(x+2\pi, y; t)$$

is equivalent to:
two equations each with quadratic nonlinearities for two real-valued functions



Thus the Water-Wave Problem

involving a PDE with nonlinear boundary conditions on an unknown domain:

$$\mathcal{S} = \{(x, \eta(x,t)) : x \in \mathbb{R}\}$$

$$\eta(x+2\pi) = \eta(x,t) = \eta(x, t+2\pi)$$

$$\left.\begin{array}{r} \phi_t + \frac{1}{2}|\nabla\phi|^2 + \lambda y = 0; \\ \phi_t + \phi_x\eta_x - \phi_y\eta_y = 0 \end{array}\right\} \text{ on } \mathcal{S}$$

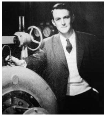
For $(x, y; t)$ with $y < \eta(x,t)$

$$\Delta\phi(x,y;t) = 0$$

$$\nabla\phi(x,y;t) \to 0 \text{ as } y \to -\infty$$

$$\phi(x,y;t) = \phi(x+2\pi, y; t)$$

is equivalent to:
two equations each with quadratic nonlinearities for two real-valued functions of one space and one time variable



Thus the Water-Wave Problem

involving a PDE with nonlinear boundary conditions on an unknown domain:

$$\mathcal{S} = \{(x, \eta(x,t)) : x \in \mathbb{R}\}$$

$$\eta(x + 2\pi) = \eta(x,t) = \eta(x, t + 2\pi)$$

$$\left.\begin{array}{r} \phi_t + \frac{1}{2}|\nabla\phi|^2 + \lambda y = 0; \\ \phi_t + \phi_x\eta_x - \phi_y\eta_y = 0 \end{array}\right\} \text{ on } \mathcal{S}$$

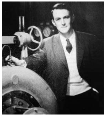
For $(x, y; t)$ with $y < \eta(x,t)$

$$\Delta\phi(x, y; t) = 0$$

$$\nabla\phi(x, y; t) \to 0 \text{ as } y \to -\infty$$

$$\phi(x, y; t) = \phi(x + 2\pi, y; t)$$

is equivalent to:
two equations each with quadratic nonlinearities for two real-valued functions of one space and one time variable on a fixed domain:



The Hamiltonian System

defined by

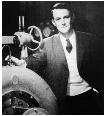
$$\mathcal{E}(w,\varphi) = \frac{1}{2}\int_{-\pi}^{\pi} \hat{\varphi}\mathcal{C}\varphi' + \lambda w^2(1+\mathcal{C}w')\,d\xi$$

with the skew form

$$\begin{aligned}\omega_{(w,\varphi)}&((w_1,\varphi_1),(w_2,\varphi_2))\\ &= \int_{-\pi}^{\pi}(1+\mathcal{C}w')(\varphi_2 w_1 - \varphi_1 w_2)\\ &\quad + w'\left(\varphi_1\mathcal{C}w_2 - \varphi_2\mathcal{C}w_1\right)\\ &\quad - \varphi'\left(w_1\mathcal{C}w_2 - w_2\mathcal{C}w_1\right)d\xi\end{aligned}$$

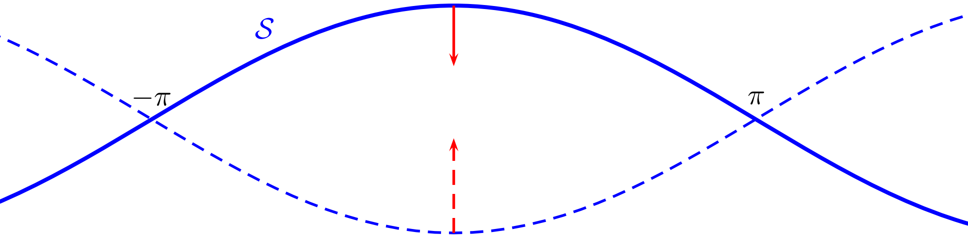
for x-periodic functions $(\phi(x,t), w(x,t))$ of real variables leads to the equations

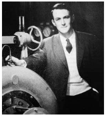
$$\boxed{\begin{aligned}\dot{w}(1+\mathcal{C}w') - \mathcal{C}\varphi' - w'\mathcal{C}\dot{w} = 0\\ \mathcal{C}\left(w'\dot{\varphi} - \dot{w}\varphi' + \lambda w w'\right) + (\dot{\varphi}+\lambda w)(1+\mathcal{C}w') - \varphi'\mathcal{C}\dot{w} = 0\end{aligned}}$$



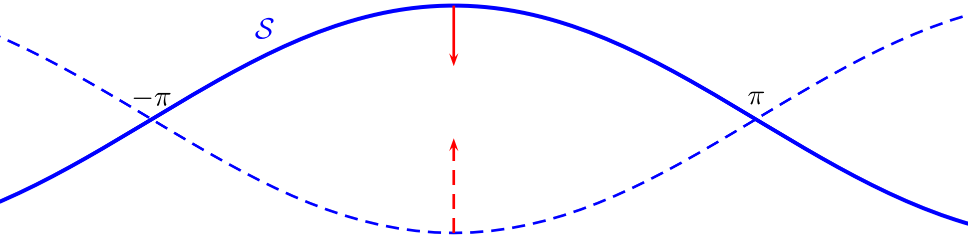
a tidy version of the water-wave problem!!

Standing Waves





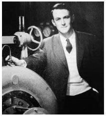
Standing Waves



The first person to consider the initial-value problem for water waves was Siméon Denis Poisson (1781–1840)

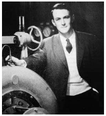
In the process he considered the standing waves – "le clapotis" he called them

– which offer a good example of how the Hamiltonian approach helps organise a fiendishly difficult problem



Linearized Standing Wave Problem

Standing Waves have normalised spatial period 2π and temporal period T

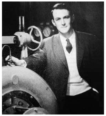


Linearized Standing Wave Problem

Standing Waves have normalised spatial period 2π and temporal period T

The velocity potential ϕ on the lower half plane $\{(x, t) :\in \mathbb{R}^2 : y < 0\}$:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \qquad x,\ t \in \mathbb{R},\ \ y < 0,$$



Linearized Standing Wave Problem

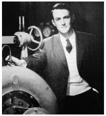
Standing Waves have normalised spatial period 2π and temporal period T

The velocity potential ϕ on the lower half plane $\{(x,t) :\in \mathbb{R}^2 : y < 0\}$:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \qquad x,\ t \in \mathbb{R}, \quad y < 0,$$

Boundary Conditions

$$\begin{aligned}
\phi(x+2\pi, y; t) = \phi(x, y; t) = \phi(x, y; t+T), &\quad x,\ t \in \mathbb{R}, \quad y < 0, \\
\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial y} = 0, &\quad y = 0 \\
\phi(-x, y; t) = \phi(x, y; t) = -\phi(x, y; -t), &\quad x,\ t \in \mathbb{R}, \quad y < 0 \\
\nabla\phi(x, y; t) \to (0, 0), &\quad y \to -\infty
\end{aligned}$$



Linearized Standing Wave Problem

Standing Waves have normalised spatial period 2π and temporal period T

The velocity potential ϕ on the lower half plane $\{(x,t) :\in \mathbb{R}^2 : y < 0\}$:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \qquad x,\ t \in \mathbb{R}, \quad y < 0,$$

Boundary Conditions

$$\phi(x+2\pi, y; t) = \phi(x, y; t) = \phi(x, y; t+T), \quad x,\ t \in \mathbb{R}, \quad y < 0,$$

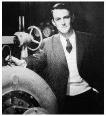
$$\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial y} = 0, \qquad y = 0$$

$$\phi(-x, y; t) = \phi(x, y; t) = -\phi(x, y; -t), \quad x,\ t \in \mathbb{R}, \quad y < 0$$

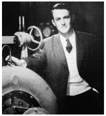
$$\nabla \phi(x, y; t) \to (0, 0), \qquad y \to -\infty$$

The wave Elevation η:

$$g\eta(x,t) = -\frac{\partial \phi}{\partial t}(x, 0, t)$$



In 1818 Poisson observed that when $\lambda := gT^2/2\pi\Lambda$ is irrational there are no non-constant solutions

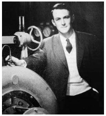


In 1818 Poisson observed that when $\lambda := gT^2/2\pi\Lambda$ is irrational there are no non-constant solutions

However, when $\lambda \in \mathbb{Q}$, for every m, n with $\frac{n^2}{m} = \lambda$

$$\phi(x, y, t) = \sin\left(\frac{2n\pi t}{T}\right) \cos\left(\frac{2m\pi x}{\Lambda}\right) \exp\left(\frac{2m\pi y}{\Lambda}\right)$$

is a solution



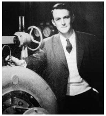
In 1818 Poisson observed that when $\lambda := gT^2/2\pi\Lambda$ is irrational there are no non-constant solutions

However, when $\lambda \in \mathbb{Q}$, for every m, n with $\frac{n^2}{m} = \lambda$

$$\phi(x, y, t) = \sin\left(\frac{2n\pi t}{T}\right)\cos\left(\frac{2m\pi x}{\Lambda}\right)\exp\left(\frac{2m\pi y}{\Lambda}\right)$$

is a solution

The eigenvalues λ of the linearized problem are $\mathbb{Q} \cap [0, \infty)$ which is dense and each eigenspace is infinite-dimensional



In 1818 Poisson observed that when $\lambda := gT^2/2\pi\Lambda$ is irrational there are no non-constant solutions

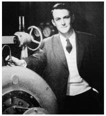
However, when $\lambda \in \mathbb{Q}$, for every m, n with $\frac{n^2}{m} = \lambda$

$$\phi(x, y, t) = \sin\Big(\frac{2n\pi t}{T}\Big) \cos\Big(\frac{2m\pi x}{\Lambda}\Big) \exp\Big(\frac{2m\pi y}{\Lambda}\Big)$$

is a solution

The eigenvalues λ of the linearized problem are $\mathbb{Q} \cap [0, \infty)$ which is dense and each eigenspace is infinite-dimensional

This is called complete resonance. It is a huge nightmare because inversion of the linearised problem involves small-divisor problems when attacking the nonlinear problem.



In 1818 Poisson observed that when $\lambda := gT^2/2\pi\Lambda$ is irrational there are no non-constant solutions

However, when $\lambda \in \mathbb{Q}$, for every m, n with $\frac{n^2}{m} = \lambda$

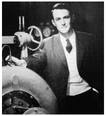
$$\phi(x, y, t) = \sin\left(\frac{2n\pi t}{T}\right)\cos\left(\frac{2m\pi x}{\Lambda}\right)\exp\left(\frac{2m\pi y}{\Lambda}\right)$$

is a solution

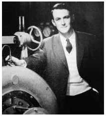
The eigenvalues λ of the linearized problem are $\mathbb{Q} \cap [0, \infty)$ which is dense and each eigenspace is infinite-dimensional

This is called complete resonance. It is a huge nightmare because inversion of the linearised problem involves small-divisor problems when attacking the nonlinear problem.

However following work by Amick and others, in the past decade this Hamiltonian formulation combined with the Nash-Moser approach (from differential geometry) has led to non-trivial solutions of the full nonlinear problem for a measurable set of λ which is dense at 1 (Plotnikov & Iooss)



Stokes Waves

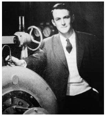


Stokes Waves

With $\phi(x,t) = \phi(x - ct)$ and $w(x,t) = w(x - ct)$ the system simplifies dramatically to $\phi' = cw'$ and an equation for w only:

$$\mathcal{C}w' = \lambda\big(w + w\mathcal{C}w' + \mathcal{C}(ww')\big) \qquad (*)$$

here the wave speed c has been absorbed in λ



Stokes Waves

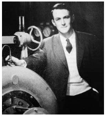
With $\phi(x,t) = \phi(x-ct)$ and $w(x,t) = w(x-ct)$ the system simplifies dramatically to $\phi' = cw'$ and an equation for w only:

$$\mathcal{C}w' = \lambda\big(w + w\mathcal{C}w' + \mathcal{C}(ww')\big) \qquad (*)$$

here the wave speed c has been absorbed in λ

Note that $w \mapsto \mathcal{C}w'$ is first-order, non-negative-definite, self-adjoint which is densely defined on $L^2_{2\pi}$ by

$$\mathcal{C}(e^{ik})' = |k|e^{ik}, \quad k \in \mathbb{Z}$$



Stokes Waves

With $\phi(x,t) = \phi(x - ct)$ and $w(x,t) = w(x - ct)$ the system simplifies dramatically to $\phi' = cw'$ and an equation for w only:

$$\mathcal{C}w' = \lambda\big(w + w\mathcal{C}w' + \mathcal{C}(ww')\big) \qquad (*)$$

here the wave speed c has been absorbed in λ

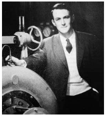
Note that $w \mapsto \mathcal{C}w'$ is first-order, non-negative-definite, self-adjoint which is densely defined on $L^2_{2\pi}$ by

$$\mathcal{C}(e^{ik})' = |k|e^{ik}, \quad k \in \mathbb{Z}$$

Therefore

$$\mathcal{C}w' = \sqrt{-\frac{\partial^2 w}{\partial \xi^2}}$$

behaves like an elliptic differential operator
but significantly it lacks a maximum principle



Stokes Waves

With $\phi(x,t) = \phi(x-ct)$ and $w(x,t) = w(x-ct)$ the system simplifies dramatically to $\phi' = cw'$ and an equation for w only:

$$\mathcal{C}w' = \lambda\big(w + w\mathcal{C}w' + \mathcal{C}(ww')\big) \qquad (*)$$

here the wave speed c has been absorbed in λ

Note that $w \mapsto \mathcal{C}w'$ is first-order, non-negative-definite, self-adjoint which is densely defined on $L^2_{2\pi}$ by

$$\mathcal{C}(e^{ik})' = |k|e^{ik}, \quad k \in \mathbb{Z}$$

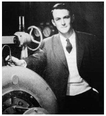
Therefore

$$\mathcal{C}w' = \sqrt{-\frac{\partial^2 w}{\partial \xi^2}}$$

behaves like an elliptic differential operator
but significantly it lacks a maximum principle

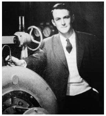
Note also that $(*)$ is the Euler-Lagrange equation of

$$\int_{-\pi}^{\pi} w\mathcal{C}w' - \lambda\big(w^2(1 + \mathcal{C}w')\big)\, d\xi,$$



The Stokes Wave Equation

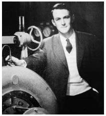
Equation $(*)$ is quadratic in w and has beautiful structure:



The Stokes Wave Equation

Equation $(*)$ is quadratic in w and has beautiful structure: apart from being variational it can be re-written

$$(1-2\lambda w)\mathcal{C}w' = \lambda(w + \mathcal{C}(ww') - w\mathcal{C}w')$$



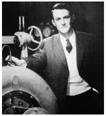
The Stokes Wave Equation

Equation $(*)$ is quadratic in w and has beautiful structure:

apart from being variational it can be re-written

$$(1 - 2\lambda w)\mathcal{C}w' = \lambda(w + \mathcal{C}(ww') - w\mathcal{C}w')$$

where miraculously $\mathcal{C}(ww') - w\mathcal{C}w'$ is always negative and always smoother than w'.



The Stokes Wave Equation

Equation $(*)$ is quadratic in w and has beautiful structure:

apart from being variational it can be re-written

$$(1 - 2\lambda w)\mathcal{C}w' = \lambda(w + \mathcal{C}(ww') - w\mathcal{C}w')$$

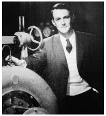
where miraculously $\mathcal{C}(ww') - w\mathcal{C}w'$ is always negative and always smoother than w'.

When $1 - 2\lambda w > 0$ it becomes

$$\mathcal{C}w' = \lambda\left(\frac{w + \mathcal{C}(ww') - w\mathcal{C}w'}{1 - 2\lambda w}\right)$$

and an elementary application of bifurcation theory yields small amplitude waves with λ close to 1

This problem vexed Lord Rayleigh and was finally settled by Nekrasov and Levi Civita in the 1920s



The Stokes Wave Equation

Equation $(*)$ is quadratic in w and has beautiful structure:

apart from being variational it can be re-written

$$(1 - 2\lambda w)\mathcal{C}w' = \lambda(w + \mathcal{C}(ww') - w\mathcal{C}w')$$

where miraculously $\mathcal{C}(ww') - w\mathcal{C}w'$ is always negative and always smoother than w'.

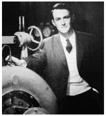
When $1 - 2\lambda w > 0$ it becomes

$$\mathcal{C}w' = \lambda\left(\frac{w + \mathcal{C}(ww') - w\mathcal{C}w'}{1 - 2\lambda w}\right)$$

and an elementary application of bifurcation theory yields small amplitude waves with λ close to 1

This problem vexed Lord Rayleigh and was finally settled by Nekrasov and Levi Civita in the 1920s

The only difficulty arises when $1 - 2\lambda w$ has zeros



The Stokes Wave Equation

Equation $(*)$ is quadratic in w and has beautiful structure:

apart from being variational it can be re-written

$$(1-2\lambda w)\mathcal{C}w' = \lambda(w + \mathcal{C}(ww') - w\mathcal{C}w')$$

where miraculously $\mathcal{C}(ww') - w\mathcal{C}w'$ is always negative and always smoother than w'.

When $1-2\lambda w > 0$ it becomes

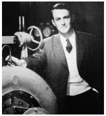
$$\mathcal{C}w' = \lambda\left(\frac{w + \mathcal{C}(ww') - w\mathcal{C}w'}{1-2\lambda w}\right)$$

and an elementary application of bifurcation theory yields small amplitude waves with λ close to 1

This problem vexed Lord Rayleigh and was finally settled by Nekrasov and Levi Civita in the 1920s

The only difficulty arises when $1-2\lambda w$ has zeros

This corresponds to Stokes wave of extreme form



History

For small-amplitude travelling waves (∗) was derived by Babenko (1987) who was realised that its variational structure was important.

History

For small-amplitude travelling waves $(*)$ was derived by Babenko (1987) who was realised that its variational structure was important.

But he did not grasped its global significance and was unaware of the connection to Dyachenko, Kuznetsov, Spector & Zakharov

History

For small-amplitude travelling waves ($*$) was derived by Babenko (1987) who was realised that its variational structure was important.
But he did not grasped its global significance and was unaware of the connection to Dyachenko, Kuznetsov, Spector & Zakharov

Independently Plotnikov (1992) and Balk (1996) rediscovered ($*$) for travelling waves

History

For small-amplitude travelling waves $(*)$ was derived by Babenko (1987) who was realised that its variational structure was important.
But he did not grasped its global significance and was unaware of the connection to Dyachenko, Kuznetsov, Spector & Zakharov

Independently Plotnikov (1992) and Balk (1996) rediscovered $(*)$ for travelling waves

Plotnikov introduced Morse index calculations in his brilliant study of the non-uniqueness question for solitary waves

History

For small-amplitude travelling waves ($*$) was derived by Babenko (1987) who was realised that its variational structure was important.
But he did not grasped its global significance and was unaware of the connection to Dyachenko, Kuznetsov, Spector & Zakharov

Independently Plotnikov (1992) and Balk (1996) rediscovered ($*$) for travelling waves

Plotnikov introduced Morse index calculations in his brilliant study of the non-uniqueness question for solitary waves

Buffoni & Séré and Dancer added a great deal to existence theory using this variational structure. However there remain many open questions.

History

For small-amplitude travelling waves (∗) was derived by Babenko (1987) who was realised that its variational structure was important.
But he did not grasped its global significance and was unaware of the connection to Dyachenko, Kuznetsov, Spector & Zakharov

Independently Plotnikov (1992) and Balk (1996) rediscovered (∗) for travelling waves

Plotnikov introduced Morse index calculations in his brilliant study of the non-uniqueness question for solitary waves

Buffoni & Séré and Dancer added a great deal to existence theory using this variational structure. However there remain many open questions.

In the last ten years the theory has been extended to a very general class of free boundary problems Shargorodsky

History

For small-amplitude travelling waves (∗) was derived by Babenko (1987) who was realised that its variational structure was important.
But he did not grasped its global significance and was unaware of the connection to Dyachenko, Kuznetsov, Spector & Zakharov

Independently Plotnikov (1992) and Balk (1996) rediscovered (∗) for travelling waves

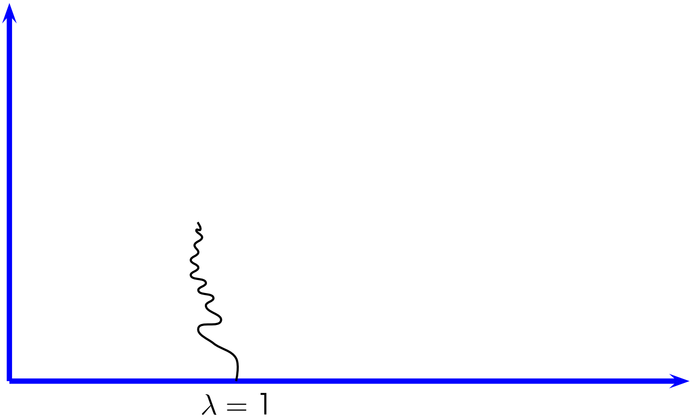
Plotnikov introduced Morse index calculations in his brilliant study of the non-uniqueness question for solitary waves

Buffoni & Séré and Dancer added a great deal to existence theory using this variational structure. However there remain many open questions.

In the last ten years the theory has been extended to a very general class of free boundary problems Shargorodsky

Remarkably, despite its very special form nothing has emerged that makes Stokes Waves special in that much wider class of free-boundary problems

Numerical Evidence Suggests a Global Bifurcation Picture Like This:

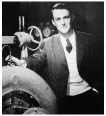


Moreover:

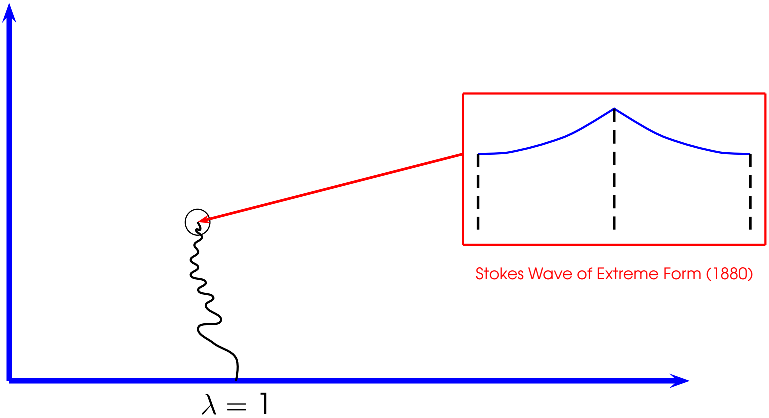
energy oscillates

maximum slope oscillates

number of inflection points increases without bound



Numerical Evidence Suggests a Global Bifurcation Picture Like This:



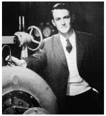
Moreover:

energy oscillates

maximum slope oscillates

number of inflection points increases without bound

as the extreme wave ($(1 - 2\lambda w = 0)$) is approached



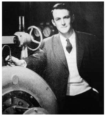
Morse Index $\mathcal{M}(w)$

The Morse index $\mathcal{M}(w)$ of a critical point w is the number of eigenvalues $\mu < 0$ of $D^2\mathcal{J}(w)$:

$$D^2\mathcal{J}(w)\,\phi\ = \mu\phi, \quad \mu < 0$$

where $D^2\mathcal{J}[w]$ is the linearisation of $(*)$ at (λ, w):

$$D^2\mathcal{J}(w)\,\phi\ = \mathcal{C}\phi' - \lambda\big(\phi + \phi\mathcal{C}w' + w\mathcal{C}\phi' + \mathcal{C}(w\phi)'\big)$$



Morse Index $\mathcal{M}(w)$

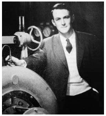
The Morse index $\mathcal{M}(w)$ of a critical point w is the number of eigenvalues $\mu < 0$ of $D^2\mathcal{J}(w)$:

$$D^2\mathcal{J}(w)\,\phi \;= \mu\phi, \quad \mu < 0$$

where $D^2\mathcal{J}[w]$ is the linearisation of $(*)$ at (λ, w):

$$D^2\mathcal{J}(w)\,\phi \;= \mathcal{C}\phi' - \lambda\big(\phi + \phi\mathcal{C}w' + w\mathcal{C}\phi' + \mathcal{C}(w\phi)'\big)$$

The Morse Index may be infinite if $1 - 2\lambda w$ has zeros (corresponding to an extreme wave)



Morse Index $\mathcal{M}(w)$

The Morse index $\mathcal{M}(w)$ of a critical point w is the number of eigenvalues $\mu < 0$ of $D^2\mathcal{J}(w)$:

$$D^2\mathcal{J}(w)\,\phi = \mu\phi, \quad \mu < 0$$

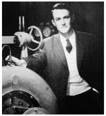
where $D^2\mathcal{J}[w]$ is the linearisation of $(*)$ at (λ, w):

$$D^2\mathcal{J}(w)\,\phi = \mathcal{C}\phi' - \lambda\big(\phi + \phi\mathcal{C}w' + w\mathcal{C}\phi' + \mathcal{C}(w\phi)'\big)$$

The Morse Index may be infinite if $1 - 2\lambda w$ has zeros (corresponding to an extreme wave)

Plotnikov showed that this is equivalent to a simpler eigenvalue problem

$$\mathcal{C}\psi' - q[\lambda, w]\psi = \mu\psi$$



Morse Index $\mathcal{M}(w)$

The Morse index $\mathcal{M}(w)$ of a critical point w is the number of eigenvalues $\mu < 0$ of $D^2\mathcal{J}(w)$:

$$D^2\mathcal{J}(w)\,\phi \;= \mu\phi, \quad \mu < 0$$

where $D^2\mathcal{J}[w]$ is the linearisation of $(*)$ at (λ, w):

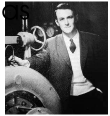
$$D^2\mathcal{J}(w)\,\phi \;= \mathcal{C}\phi' - \lambda\big(\phi + \phi\mathcal{C}w' + w\mathcal{C}\phi' + \mathcal{C}(w\phi)'\big)$$

The Morse Index may be infinite if $1 - 2\lambda w$ has zeros (corresponding to an extreme wave)

Plotnikov showed that this is equivalent to a simpler eigenvalue problem

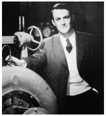
$$\mathcal{C}\psi' - q[\lambda, w]\psi = \mu\psi$$

Moreover the "Plotnikov potential" $q[\lambda, w]$ becomes singular as $\min\{1 - 2\lambda w\} \searrow 0$



Plotnikov's Theorem

Suppose that a sequence $\{(\lambda_k, w_k)\}$ of solutions of $(*)$ has $1 - 2\lambda_k w_k \neq 0$ and the Morse indices $\{\mathcal{M}(w_k)\}$ are bounded.



Plotnikov's Theorem

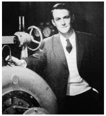
Suppose that a sequence $\{(\lambda_k, w_k)\}$ of solutions of $(*)$ has $1 - 2\lambda_k w_k \neq 0$ and the Morse indices $\{\mathcal{M}(w_k)\}$ are bounded.

Then for some $\alpha > 0$

$$\boxed{1 - 2\lambda_k w_k \geqslant \alpha, \quad k \in \mathbb{N}}$$

None of these solutions is close to being an extreme wave.

As the extreme Stokes wave is approached it becomes more and more unstable in the sense of Morse indices



Plotnikov's Theorem

Suppose that a sequence $\{(\lambda_k, w_k)\}$ of solutions of $(*)$ has $1 - 2\lambda_k w_k \neq 0$ and the Morse indices $\{\mathcal{M}(w_k)\}$ are bounded.

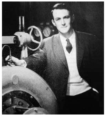
Then for some $\alpha > 0$

$$\boxed{1 - 2\lambda_k w_k \geqslant \alpha, \quad k \in \mathbb{N}}$$

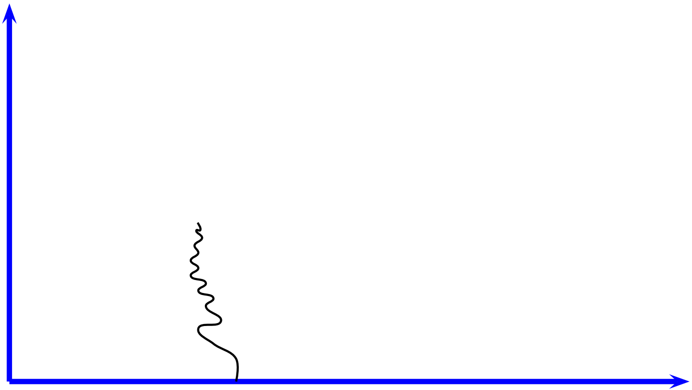
None of these solutions is close to being an extreme wave.

As the extreme Stokes wave is approached it becomes more and more unstable in the sense of Morse indices

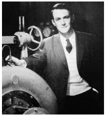
Recently Shargorodsky has quantified the relation between the size of the Morse index and the size of α



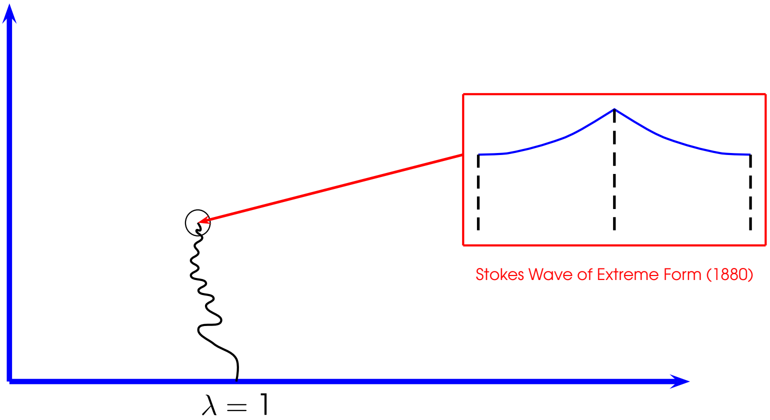
Primary Branch



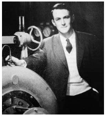
$\lambda = 1$



Primary Branch



Moreover:
energy oscillates
maximum slope oscillates
number of inflection points increases without bound
Morse index grows without bound as the extreme wave is approached



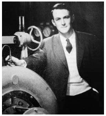
Open Question

By methods of topological degree theory the global branch of Stokes waves "terminates" at Stokes extreme wave. So Plotnikov's result implies that there are solutions arbitrarily large Morse index

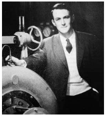
Despite this and the attractive form of $\mathcal{J}$, a global variational theory of existence remains undiscovered

The question is:

For all large $n \in \mathbb{N}$ does there exist a solution with Morse index n?



What Does Plotnikov's Result Mean?



What Does Plotnikov's Result Mean?

In abstract terms Babenko's equation for travelling waves is

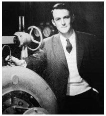
$$\mathcal{C}w' = \lambda \nabla \Phi(w) \tag{‡}$$

where

$$\Phi = \frac{1}{2}\int_{-\pi}^{\pi} \left\{ w^2(1 + \mathcal{C}w') \right\} dx$$

and it can be proved that there is a curve of solutions $\{(\lambda_s, w_s) : s \in [0, \infty)\}$ with $\mathcal{M}(w_s) \to \infty$ as $s \to \infty$.

Moreover the parametrisation $s \mapsto (\lambda_s, w_s)$ is real-analytic.



What Does Plotnikov's Result Mean?

In abstract terms Babenko's equation for travelling waves is

$$\mathcal{C}w' = \lambda \nabla \Phi(w) \tag{‡}$$

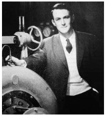
where

$$\Phi = \frac{1}{2}\int_{-\pi}^{\pi} \left\{ w^2(1 + \mathcal{C}w') \right\} dx$$

and it can be proved that there is a curve of solutions $\{(\lambda_s, w_s) : s \in [0, \infty)\}$ with $\mathcal{M}(w_s) \to \infty$ as $s \to \infty$.

Moreover the parametrisation $s \mapsto (\lambda_s, w_s)$ is real-analytic.

Suppose that the Morse Index changes as s passes through s^*



What Does Plotnikov's Result Mean?

In abstract terms Babenko's equation for travelling waves is

$$\mathcal{C}w' = \lambda \nabla \Phi(w) \qquad (\ddagger)$$

where

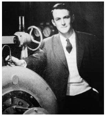
$$\Phi = \frac{1}{2}\int_{-\pi}^{\pi} \left\{ w^2(1+\mathcal{C}w') \right\} dx$$

and it can be proved that there is a curve of solutions $\{(\lambda_s, w_s) : s \in [0,\infty)\}$ with $\mathcal{M}(w_s) \to \infty$ as $s \to \infty$.

Moreover the parametrisation $s \mapsto (\lambda_s, w_s)$ is real-analytic.

Suppose that the Morse Index changes as s passes through s^*

- The variational structure of $(\ddagger)$



What Does Plotnikov's Result Mean?

In abstract terms Babenko's equation for travelling waves is

$$\mathcal{C}w' = \lambda \nabla \Phi(w) \tag{‡}$$

where

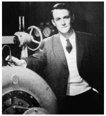
$$\Phi = \frac{1}{2}\int_{-\pi}^{\pi} \left\{ w^2(1+\mathcal{C}w') \right\} dx$$

and it can be proved that there is a curve of solutions $\{(\lambda_s, w_s) : s \in [0,\infty)\}$ with $\mathcal{M}(w_s) \to \infty$ as $s \to \infty$.

Moreover the parametrisation $s \mapsto (\lambda_s, w_s)$ is real-analytic.

Suppose that the Morse Index changes as s passes through s^*

- The variational structure of (‡)
- The fact that λ is a multiplier on the right



What Does Plotnikov's Result Mean?

In abstract terms Babenko's equation for travelling waves is

$$\mathcal{C}w' = \lambda \nabla \Phi(w) \qquad (\ddagger)$$

where

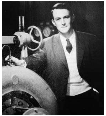
$$\Phi = \frac{1}{2}\int_{-\pi}^{\pi} \left\{ w^2(1+\mathcal{C}w') \right\} dx$$

and it can be proved that there is a curve of solutions $\{(\lambda_s, w_s) : s \in [0,\infty)\}$ with $\mathcal{M}(w_s) \to \infty$ as $s \to \infty$.

Moreover the parametrisation $s \mapsto (\lambda_s, w_s)$ is real-analytic.

Suppose that the Morse Index changes as s passes through s^*

- The variational structure of $(\ddagger)$
- The fact that λ is a multiplier on the right
- The Morse Index changes as s passes through s^*



What Does Plotnikov's Result Mean?

In abstract terms Babenko's equation for travelling waves is

$$\mathcal{C}w' = \lambda \nabla \Phi(w) \qquad (\ddagger)$$

where

$$\Phi = \frac{1}{2}\int_{-\pi}^{\pi} \left\{ w^2(1+\mathcal{C}w') \right\} dx$$

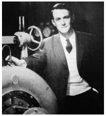
and it can be proved that there is a curve of solutions $\{(\lambda_s, w_s) : s \in [0,\infty)\}$ with $\mathcal{M}(w_s) \to \infty$ as $s \to \infty$.

Moreover the parametrisation $s \mapsto (\lambda_s, w_s)$ is real-analytic.

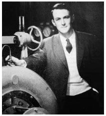
Suppose that the Morse Index changes as s passes through s^*

- The variational structure of $(\ddagger)$
- The fact that λ is a multiplier on the right
- The Morse Index changes as s passes through s^*

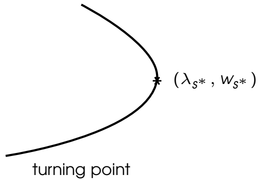
means that

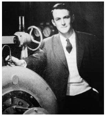


at least one of two things happens

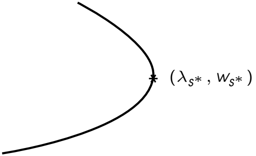


at least one of two things happens

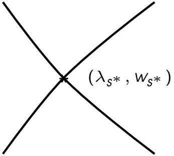




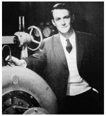
at least one of two things happens



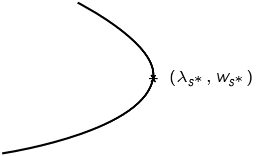
turning point



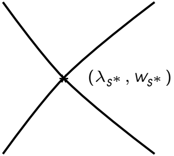
crossing



at least one of two things happens

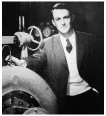


turning point

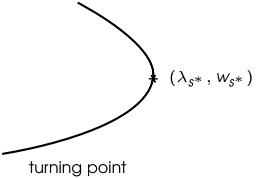


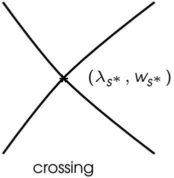
crossing

The numerical evidence is that the first happens each time the Morse index changes



at least one of two things happens

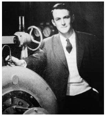




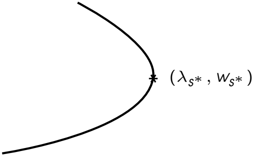
The numerical evidence is that the first happens each time the Morse index changes

We do not have a proof

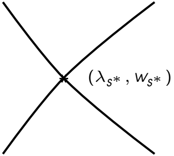
This seems to be a very hard problem



at least one of two things happens



turning point



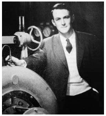
crossing

The numerical evidence is that the first happens each time the Morse index changes

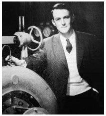
We do not have a proof

This seems to be a very hard problem

But there's more we can say ...

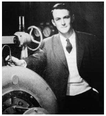


.. if you look at things differently



.. if you look at things differently

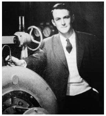
The solutions $\{(\lambda_s, w_s) : s \in [0, \infty)\}$ with $\mathcal{M}(w_s) \to \infty$ as $s \to \infty$ are 2π-periodic.



.. if you look at things differently

The solutions $\{(\lambda_s, w_s) : s \in [0, \infty)\}$ with $\mathcal{M}(w_s) \to \infty$ as $s \to \infty$ are 2π-periodic.

So they are also $2n\pi$-periodic for any n

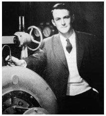


.. if you look at things differently

The solutions $\{(\lambda_s, w_s) : s \in [0, \infty)\}$ with $\mathcal{M}(w_s) \to \infty$ as $s \to \infty$ are 2π-periodic.

So they are also $2n\pi$-periodic for any n

Let $\mathcal{M}_n(w)$ denote their Morse index in that new setting



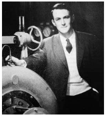
.. if you look at things differently

The solutions $\{(\lambda_s, w_s) : s \in [0, \infty)\}$ with $\mathcal{M}(w_s) \to \infty$ as $s \to \infty$ are 2π-periodic.

So they are also $2n\pi$-periodic for any n

Let $\mathcal{M}_n(w)$ denote their Morse index in that new setting

For p sufficiently large prime $\mathcal{M}_p(w_s)$ changes as s passes through $s_p^* \neq s^*$ close to s^*



.. if you look at things differently

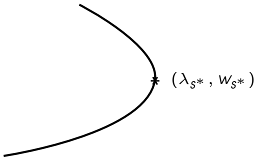
The solutions $\{(\lambda_s, w_s) : s \in [0, \infty)\}$ with $\mathcal{M}(w_s) \to \infty$ as $s \to \infty$ are 2π-periodic.

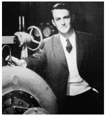
So they are also $2n\pi$-periodic for any n

Let $\mathcal{M}_n(w)$ denote their Morse index in that new setting

For p sufficiently large prime $\mathcal{M}_p(w_s)$ changes as s passes through $s_p^* \neq s^*$ close to s^*

Thus near s^* there is a bifurcation from the primary branch of solution with minimal period $2p\pi$





.. if you look at things differently

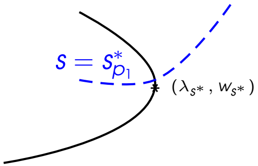
The solutions $\{(\lambda_s, w_s) : s \in [0, \infty)\}$ with $\mathcal{M}(w_s) \to \infty$ as $s \to \infty$ are 2π-periodic.

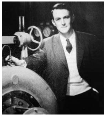
So they are also $2n\pi$-periodic for any n

Let $\mathcal{M}_n(w)$ denote their Morse index in that new setting

For p sufficiently large prime $\mathcal{M}_p(w_s)$ changes as s passes through $s_p^* \neq s^*$ close to s^*

Thus near s^* there is a bifurcation from the primary branch of solution with minimal period $2p\pi$





.. if you look at things differently

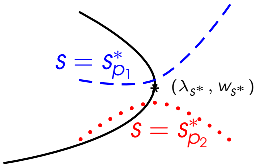
The solutions $\{(\lambda_s, w_s) : s \in [0, \infty)\}$ with $\mathcal{M}(w_s) \to \infty$ as $s \to \infty$ are 2π-periodic.

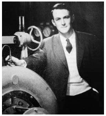
So they are also $2n\pi$-periodic for any n

Let $\mathcal{M}_n(w)$ denote their Morse index in that new setting

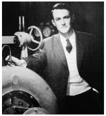
For p sufficiently large prime $\mathcal{M}_p(w_s)$ changes as s passes through $s_p^* \neq s^*$ close to s^*

Thus near s^* there is a bifurcation from the primary branch of solution with minimal period $2p\pi$



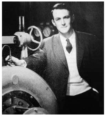


That period-multiplying (sub-harmonic) bifurcation occurs near the turning points on the primary branch was observed numerically by Chen & Saffman (1980)



That period-multiplying (sub-harmonic) bifurcation occurs near the turning points on the primary branch was observed numerically by Chen & Saffman (1980)

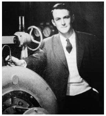
Brooke knew their result and would I think have loved this subtle proof



That period-multiplying (sub-harmonic) bifurcation occurs near the turning points on the primary branch was observed numerically by Chen & Saffman (1980)

Brooke knew their result and would I think have loved this subtle proof

but alas once again the mathematicians had discovered an a priori proof a postiori



That period-multiplying (sub-harmonic) bifurcation occurs near the turning points on the primary branch was observed numerically by Chen & Saffman (1980)

Brooke knew their result and would I think have loved this subtle proof

but alas once again the mathematicians had discovered an a priori proof a postiori

I think I had better stop here

Thank You