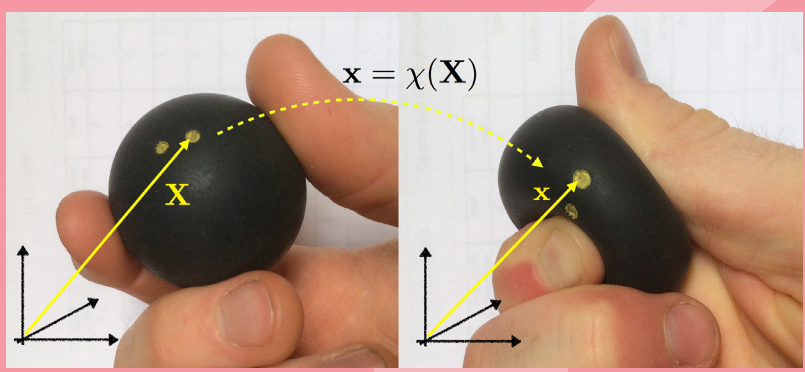
Oxford Mathematics Alphabet

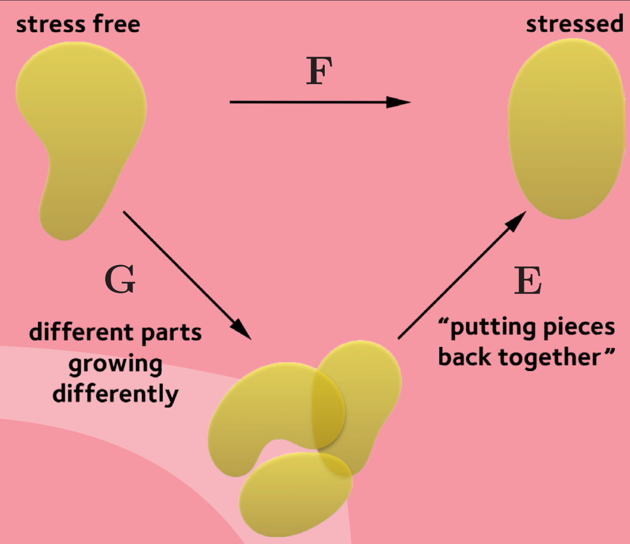
G *is for* growth tensor

A tree branch, a ram's horn, your hand – how have these distinct and consistent shapes come about? The growth and form of a biological entity is a complex matter that involves integrated activities across a number of length scales. Viewed at the scale of tissues, or large clusters of cells, understanding growth and form is a problem well suited for continuum mechanics and mathematical modelling.



Within continuum mechanics, elasticity studies the deformation of objects that quickly return to their original shape once applied pressures are removed. For example, when you squeeze a rubber ball it deforms; when you let go it returns to a sphere. A starting point for describing mathematically the behaviour of an elastic object is to define a map from an unstressed reference state (e.g. the rubber ball as it sits in your hand) to a current state (the squeezed ball). Mathematically, if we denote material points in the reference state with position vector $\mathbf{X}$ and those in the current state with position vector $\mathbf{x}$, then there exists a one-to-one map χ between the two.

We then define the *deformation gradient tensor* as the gradient of this map, that is $\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}$. $\mathbf{F}$ is a second order tensor (it maps vectors to vectors) that describes the local deformation (stretching, compressing, shearing) of the material.



To incorporate growth, we decompose $\mathbf{F}$ as shown above. This decomposition forms the key theoretical component of the field of morphoelasticity, a term coined by Oxford mathematician Alain Goriely.

An important aspect of morphoelasticity is the notion of *differential growth* – different rates of growth in different locations or directions. You can observe the effects of this with a stalk of rhubarb. If you peel the rhubarb, you will notice that it curls, as the outer layer wants to be shorter than the inner layer – this is due to a non-homogeneous growth tensor!

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For more about growth tensors visit www.maths.ox.ac.uk/r/alphabet