Super-identical pseudospectra

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Plan

1. Introduction to pseudospectra
2. Do pseudospectra determine matrix behavior? (Joint work with Maxime Fortier Bourque, J. London Math. Soc., 2009)

Part I: Introduction to pseudospectra

General problem

Let A be a complex $N \times N$ matrix, thought of as acting on ℓ^2_N.
Determine the evolution of $\|A^n\|$ (or of $\|e^{tA}\|$).

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First remarks: Let $\rho(A) =$ spectral radius of A.

- $\|A^n\| \geq \rho(A)^n$ for all n, with equality if A normal.
- $\|A^n\|^{1/n} \to \rho(A)$ as $n \to \infty$.

Cautionary example:

Let A be the $N \times N$ matrix

$$A = \begin{pmatrix} 0 & 1 & & & \\ 1/4 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1/4 & 0 & 1 \\ & & & 1/4 & 0 \end{pmatrix}$$

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Let's actually compute $\|A^n\|$ ...

Cautionary example (continued)

We'll consider the case $N = 32$.

n	$\|A^n\|$
1	
2	
4	
10	
20	
40	
100	
200	
400	
1000	
10000	

Cautionary example (continued)

We'll consider the case $N = 32$.

n	$\|A^n\|$
1	1.24
2	
4	
10	
20	
40	
100	
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400	
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10000	

Cautionary example (continued)

We'll consider the case $N = 32$.

n	$\|A^n\|$
1	1.24
2	1.55
4	
10	
20	
40	
100	
200	
400	
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10000	

Cautionary example (continued)

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n	$\|A^n\|$
1	1.24
2	1.55
4	2.41
10	
20	
40	
100	
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10000	

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n	$\|A^n\|$
1	1.24
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4	2.41
10	8.98
20	
40	
100	
200	
400	
1000	
10000	

Cautionary example (continued)

We'll consider the case $N = 32$.

n	$\|A^n\|$
1	1.24
2	1.55
4	2.41
10	8.98
20	78.44
40	
100	
200	
400	
1000	
10000	

Cautionary example (continued)

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n	$\|A^n\|$
1	1.24
2	1.55
4	2.41
10	8.98
20	78.44
40	4442.09
100	
200	
400	
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400	544597.34
1000	36339.67
10000	

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4	2.41
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20	78.44
40	4442.09
100	485866.04
200	1043599.98
400	544597.34
1000	36339.67
10000	6.63×10^{-14}

Resolvents

Question: How to estimate $\max_{n\geq 0} \|A^n\|$ in general?

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Kreiss matrix theorem (Leveque–Trefethen 1984, Spijker 1991)

$$\max_{n\geq 0}\|A^n\| \geq \sup_{|z|>1}(|z|-1)\|(A-zI)^{-1}\|$$

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Moral: It's useful to look at $\|(A-zI)^{-1}\|$. But how to compute it?

Singular values

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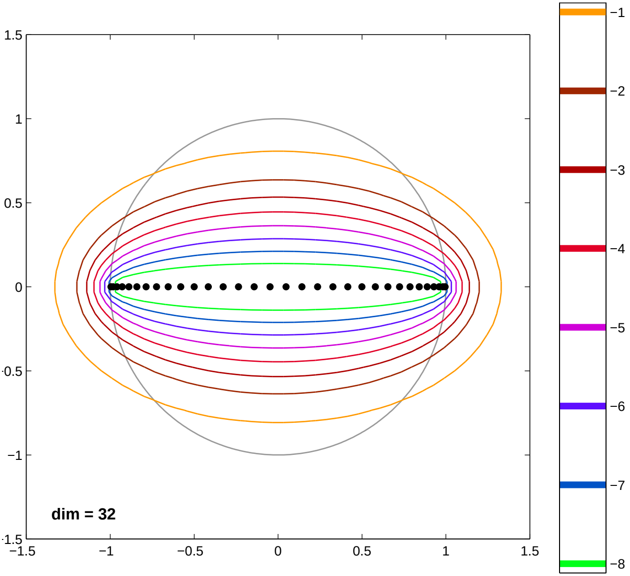
Basic trick for computing resolvent norms

$$\|(A - zI)^{-1}\| = 1/s_{\min}(A - zI)$$

Level curves of $\|(A - zI)^{-1}\|$ for $A = \text{tridiag}(32, \frac{1}{4}, 0, 1)$



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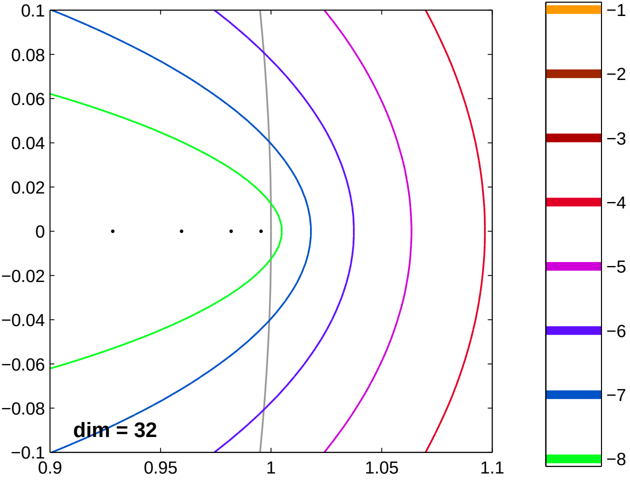

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Definition of the ϵ-pseudospectrum of A

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Pseudospectra have applications in many fields, including:

atmospheric science	magnetohydrodynamics
control theory	Markov chains
ecology	non-hermitian quantum mechanics
hydrodynamic stability	numerical solutions of odes/pdes
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Reference:

L. N. Trefethen, M. Embree, *Spectra and Pseudospectra*, Princeton University Press, 2005.

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Question: Suppose that A, B have identical pseudospectra, i.e.

$$\|(A - zI)^{-1}\| = \|(B - zI)^{-1}\| \quad \text{for all } z \in \mathbb{C}.$$

- Must we have $\|A^n\| = \|B^n\|$ for all n?
- Must A, B be unitarily equivalent?

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Answers:

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- **No**, if $N \geq 4$.

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- **No**, if $N \geq 4$.
- **Good news:** we always have $1/2 \leq \|A\|/\|B\| \leq 2$.
- **Bad news:** given submultiplicative sequences (α_n) and (β_n), there exist A, B with identical pseudospectra such that

$$\|A^n\| = \alpha_n \quad \text{and} \quad \|B^n\| = \beta_n \qquad (2 \leq n \leq (N-3)/2).$$

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Definition:

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$$s_k(A - zI) = s_k(B - zI) \quad \text{for all } z \in \mathbb{C} \textbf{ and all } k.$$

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- Does this condition imply that $\|A^n\| = \|B^n\|$ for all n?
- Does it imply that A, B are unitarily equivalent?

Some reformulations

By definition, A, B have super-identical pseudospectra iff

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This is also equivalent to the same condition, but with $1 \leq k \leq N$.

Some easy consequences

Theorem 1

Let F be a uniqueness set for polynomials in $z, \overline{z}$ of bidegree N, N. Then A, B have super-identical pseudospectra iff

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Theorem 2

If A, B have super-identical pseudospectra then, for every polynomial p,

$$\frac{1}{\sqrt{N}} \leq \frac{\|p(A)\|}{\|p(B)\|} \leq \sqrt{N}.$$

An example

For $0 < \alpha < \beta \leq \pi/4$, define

$$A := \begin{pmatrix} 0 & \sec\alpha & 0 & 1 \\ 0 & 0 & \sec\beta\csc\beta & 0 \\ 0 & 0 & 0 & \csc\alpha \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and let B be the same matrix with the roles of α, β interchanged. Then A, B have super-identical pseudospectra, but

$$\|A^2\|/\|B^2\| = \cos\alpha/\cos\beta \neq 1.$$

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Consequences:

- In Theorem 2, cannot replace $\sqrt{N}$ by 1 (but maybe by $\sqrt{2}$?).
- Super-identical pseudospectra $\not\Rightarrow$ unitary equivalence.

Super-identical pseudospectra and unitary equivalence

Theorem 3 (Discreteness theorem)

'Almost every' equivalence class wrt super-identical pseudospectra is a union of a finite number of unitary equivalence classes. The number is bounded by a constant depending only on N.

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Idea: recall that A, B have super-identical pseudospectra iff

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Also, by a theorem of Specht, A, B are unitarily equivalent iff

$$\mathrm{tr}(w(A, A^*)) = \mathrm{tr}(w(B, B^*)) \quad \text{for all words } w.$$

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Algebraicity theorem

Given a word w, the polynomial $\operatorname{tr}(w(X,Y))$ is algebraic over the algebra generated by $\{\operatorname{tr}([(X-zI)(Y-\overline{z}I)]^k) : k\geq 0,\ z\in\mathbb{C}\}$.

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THANK YOU!