Vortex singularities in Ginzburg-Landau type problems

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The purpose of this course is to analyse vortex singularities appearing in Ginzburg-Landau type problems. For that, we consider the following variational model:

$$E_\varepsilon(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2\,dx, \quad u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2,$$

where $\varepsilon > 0$ is a small parameter. We are interested in the asymptotic behaviour as $\varepsilon \to 0$ of critical points u_ε of E_ε that are solutions to the system of PDEs:

$$-\Delta u_\varepsilon = \frac{1}{\varepsilon^2} u_\varepsilon (1-|u_\varepsilon|^2) \quad \text{in} \quad \Omega.$$

As $\varepsilon \to 0$, it is expected that u_ε converges to a so-called $\mathbb{S}^1$-valued canonical harmonic map, whose prototype is the following complex function:

$$u_*(z) = \left(\frac{z-a_1}{|z-a_1|}\right)^{d_1} \cdots \left(\frac{z-a_N}{|z-a_N|}\right)^{d_N},$$

where $a_j \in \Omega$ are the vortex singularities of winding number $d_j \in \mathbb{Z}$. These vortices correspond to zeros of u_ε around which the functional E_ε concentrates and blows up as $|\log \varepsilon|$ in the limit $\varepsilon \to 0$. Our aim is to present a variational approach in proving this concentration phenomenon of E_ε around vortices.

Organisation. I will start by introducing the problem: a quick physical motivation, the objects we focus on (vortices, jacobian, winding number...) and the main results we want to present (concentration of the jacobian of u_ε and of E_ε). To prove these results, I will review some basic facts of Functional Analysis, Calculus of Variations and Degree Theory, in particular, some properties of the jacobian, winding number, co-area formula, Γ-convergence etc. Then we will prove the main results.

Tentative schedule. Fridays May 14, May 21 and May 28 at 10am-noon.

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