MAT syllabus

Sequences defined iteratively and by formulae. Arithmetic and geometric progressions*. Their sums*. Convergence condition for infinite geometric progressions*.
* Part of full A-level Mathematics syllabus.

Revision

- A sequence a_n might be defined by a formula for the n^{th} term like $a_n = n^2 - n$.

- A sequence a_n might be defined with an relation like $a_{n+1} = f(a_n)$ for $n \geqslant 0$, if we're given the function $f(x)$ and also given a first term like $a_0 = 1$. (The "first term" might be a_0 if we feel like counting from zero).

- The sum of the first n terms of a sequence a_k can be written with the notation $\displaystyle\sum_{k=0}^{n-1} a_k$ (if the first term is a_0) or $\displaystyle\sum_{k=1}^{n} a_k$ (if the first term is a_1).

- An arithmetic sequence is one where the difference between terms is constant. The terms can be written as a, $a + d$, $a + 2d$, $a + 3d$, $\ldots$, where a is the first term and d is the common difference.

- The sum of the first n terms of an arithmetic sequence with first term a and common difference d is $\frac{n}{2}\left(2a+(n-1)d\right)$, which you can remember as "first term plus last term, times the number of terms, divided by two".

- A geometric sequence is one where the ratio between consecutive terms is constant. The terms can be written as a, ar, ar^2, ar^3, $\ldots$ where a is the first term and r is the common ratio.

- The sum of the first n terms of a geometric sequence with first term a and common ratio r is $\dfrac{a(1-r^n)}{1-r}$. One way to remember this is to remember what happens if we multiply the sum of the first n terms of a geometric series by $(1-r)$,

$$\begin{aligned}(1-r)(a+ar+\cdots+ar^{n-1}) =& (a-ar)+(ar-ar^2)+\cdots+(ar^{n-1}-ar^n)\\ =& a-ar^n.\end{aligned}$$

- For a geometric sequence a_n, the sum to infinity is written as $\sum_{k=0}^{\infty} a_k$. If the common ratio r satisfies $|r|<1$ then this is equal to $\dfrac{a}{1-r}$. If $|r|\geqslant 1$ then this sum to infinity does not converge (it does not approach any particular real number).

Warm-up

1. A sequence is defined by $a_n = n^2 - n$. What is a_3? What is a_{10}? Find $a_{n+1} - a_n$ in terms of n. Find $a_{n+1} - 2a_n + a_{n-1}$ in terms of n.

2. A sequence is defined by $a_0 = 1$ and $a_n = a_{n-1} + 3$ for $n \geqslant 1$. Find $a_0 + a_1 + \cdots + a_{10}$. Find a_{1000}.

3. A sequence is defined by $a_0 = 1$ and $a_n = \frac{a_{n-1}}{3}$ for $n \geqslant 1$. Find $a_0 + a_1 + \cdots + a_{10}$. Find a_{1000}. Does the sum of all the terms of this sequence converge? If it does, what is the sum to infinity?

4. A sequence is defined by $a_0 = 1$ and $a_n = 3a_{n-1} + 1$ for $n \geqslant 1$. A sequence b_n is defined by $b_n = A \times 3^n + B$ where A and B are real numbers. Find values for A and B such that $a_n = b_n$ for all $n \geqslant 0$.

5. A sequence is defined by $a_n = An^2 + Bn + C$ where A, B, and C are real numbers. Find A, B, and C in terms of a_0, a_1, and a_2. Hint: you'll need to solve 3 simultaneous equations.

6. Simplify $2^1 + 2^2 + 2^3 + \cdots + 2^n$ for $n \geqslant 1$.

7. Simplify $3^4 + 3^5 + 3^6 + \cdots + 3^n$ for $n \geqslant 4$.

8. When does the sum $1 + x^3 + x^6 + x^9 + x^{12} + ...$ converge? Simplify it in the case that it converges.

9. When does the sum $2 - x + \dfrac{x^2}{2} - \dfrac{x^3}{4} + \dots$ converge? Simplify it in the case that it converges.

10. Consider the sum of the first n terms of an arithmetic sequence $a_1, a_2, \dots, a_n$ with $a_1 = a$ and $a_2 = a + d$. Explain why the sum of the i^{th} term and the $(n+1-i)^{\text{th}}$ term doesn't depend on i, as long as $1 \leqslant i \leqslant n$. By considering separate cases where n is even or where n is odd, deduce that the sum of the first n terms of an arithmetic sequence is n times the average term.

11. Consider the sum of the first n terms of an arithmetic sequence with first term a and constant difference d. Consider the special case $d = 0$. Write down the sum in this case. Now consider the case $a = 0$. In this case, write the sum in terms of the triangle numbers $T_n = 1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1)$. Hence write down the sum of the first n terms of an arithmetic sequence. Check that this agrees with the formula above.

MAT questions

MAT 2016 Q1A

A sequence a_n has first term $a_1 = 1$, and subsequent terms defined by $a_{n+1} = la_n$ for $n \geqslant 1$. What is the product of the first 15 terms of the sequence?

(a) l^{14}, (b) $15 + l^{14}$, (c) $15l^{14}$, (d) l^{105}, (e) $15 + l^{105}$.

Hint: note that this question is asking for the *product* and not the *sum*. Also note that the first term is a_1 and not a_0, so the first 15 terms will be a_1, a_2, $\ldots$, a_{14}, a_{15}.

MAT 2016 Q1G

The sequence x_n, where $n \geqslant 0$, is defined by $x_0 = 1$ and

$$x_n = \sum_{k=0}^{n-1} x_k \qquad \text{for } n \geqslant 1.$$

The sum

$$\sum_{k=0}^{\infty} \frac{1}{x_k}$$

equals

(a) 1, (b) $\frac{6}{5}$, (c) $\frac{8}{5}$, (d) 3, (e) $\frac{27}{5}$.

Hint: work out a few of the values x_1, x_2, x_3, ... before trying to work out the sum to infinity.

MAT 2017 Q1C

A sequence (a_n) has the property that

$$a_{n+1} = \frac{a_n}{a_{n-1}}$$

for every $n \geqslant 2$. Given that $a_1 = 2$ and $a_2 = 6$, what is a_{2017}?

(a) $\frac{1}{6}$, (b) $\frac{2}{3}$, (c) $\frac{3}{2}$, (d) 2, (e) 3.

Hint: again, work out a few of the values x_1, x_2, x_3, $\ldots$.

MAT 2016 Q5

This question concerns the sum s_n defined by

$$s_n = 2 + 8 + 24 + \cdots + n2^n.$$

(i) Let $f(n) = (An+B)2^n+C$ for constants A, B and C yet to be determined, and suppose $s_n = f(n)$ for all $n \geqslant 1$. By setting $n = 1, 2, 3$, find equations that must be satisfied by A, B and C.

(ii) Solve the equations from part (i) to obtain values for A, B and C.

(iii) Using these values, show that if $s_k = f(k)$ for some $k \geqslant 1$ then $s_{k+1} = f(k+1)$.

You may now assume that $f(n) = s_n$ for all $n \geqslant 1$.

(iv) Find simplified expressions for the following sums:

$$t_n = n + 2(n-1) + 4(n-2) + 8(n-3) + \cdots + 2^{n-1}1,$$
$$u_n = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdots + \frac{n}{2^n}.$$

(v) Find the sum

$$\sum_{k=1}^{n} s_k.$$

Hints: At the start, take a moment to understand the definition of s_n. How do the numbers 8 and 24 relate to the $+n2^n$ part of the definition? What are the values of s_1 and s_2 and s_3? Be careful: s_2 is not 8.

In part (iii) we're being asked to investigate what happens when we go from s_k to s_{k+1}. From the definition at the top, what changes when we go from s_k to s_{k+1}? If we do that to $f(k)$, do we get to $f(k+1)$?

In part (iv), it would be good if we could find a link between t_n and s_n, perhaps by spotting a copy of the sum that defines s_n hiding in there. Then we want to find a link between u_n and s_n, or a link between u_n and t_n. If that doesn't work, we can go back to the idea in part (i) and try to find a general expression for the n^{th} term of t_n or u_n by guessing a function like $(An + B)2^n + C$ or maybe like $(An + B)2^{-n} + C$.

In part (v), we know an expression for s_k in terms of things like 2^k and $k2^k$. We know how to sum the first of those things, and the sum of the second thing there is oddly familiar from earlier in this question...

Extension

A future session of the Oxford MAT Livestream will be on “recursion”, and we’ll look at more expressions that are like $a_n = f(a_{n-1})$ but more complex.

The following material is included for your interest only, and not for MAT preparation.

There’s a general formula for sequences where the difference between terms is itself an arithmetic sequence. The sequences are sometimes called quadratic sequences, and they have $a_n = An^2 + Bn + C$ for some A, B, and C. You can probably guess what happens if the difference between terms of a sequence is itself a quadratic sequence.

In MAT 2016 Q5, we found a formula for the sum of the first n terms of the sequence $a_k = k2^k$ with one particular method (guess the formula, check the formula). Here’s a more direct proof; expand and sum and sum.

- Expand out each term into 2^ks, so that we’ve got

$$\sum_{k=1}^{n} k2^k = (2^1) + (2^2 + 2^2) + (2^3 + 2^3 + 2^3) + \cdots + (\underbrace{2^n + 2^n + \cdots + 2^n}_{n}).$$

- Regroup the terms and sum

$$\begin{aligned}\sum_{k=1}^{n} k2^k &= \left(2^1 + 2^2 + \cdots + 2^n\right) + (2^2 + 2^3 + \cdots + 2^n) + \cdots + (2^{n-1} + 2^n) + 2^n \\ &= 2^1(2^n - 1) + 2^2(2^{n-1} - 1) + \cdots + 2^{n-1}(2^2 - 1) + 2^n(2^1 - 1)\end{aligned}$$

- Expand these brackets, bring together all the 2^{n+1}s and notice that the remaining terms are another geometric series

$$\begin{aligned}\sum_{k=1}^{n} k2^k &= n2^{n+1} - 2^1 - 2^2 - 2^3 - \cdots - 2^n \\ &= n2^{n+1} - 2^1(2^n - 1) \\ &= (2n-2)2^n + 2\end{aligned}$$

You might be able to adapt this method to similar sums $\sum_{k=1}^{n} kx^k$ for other numbers x. (Watch out for a factor of $(x-1)$ from the geometric sums, which is 1 above when $x = 2$.)

For the sum to infinity, if $|x| < 1$, then we get $x + 2x^2 + 3x^3 + 4x^4 + \cdots = \dfrac{x}{(1-x)^2}$. This agrees with a different, more advanced calculation using calculus; the sum is

$$x + 2x^2 + 3x^3 + 4x^4 + \cdots = x\frac{\mathrm{d}}{\mathrm{d}x}\left(1 + x + x^2 + x^3 + \dots\right) = x\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{1-x}\right) = \frac{x}{(1-x)^2}$$

but this calculation uses the “chain rule” for differentiation, which is not on the MAT syllabus!