MAT syllabus

Sequences defined iteratively and by formulae. Arithmetic and geometric progressions*. Their sums*. Convergence condition for infinite geometric progressions*.
* Part of full A-level Mathematics syllabus.

Revision

- A sequence a_n might be defined by a formula for the n^{th} term like $a_n = n^2 - n$.
- A sequence might be defined with an relation like $x_{n+1} = f(x_n)$ for $n \geqslant 0$, if we're given the function $f(x)$ and also given a first term like $a_0 = 1$. (The "first term" might be a_0 if we feel like counting from zero).
- The sum of the first n terms of a sequence a_k can be written with the notation $\sum_{k=0}^{n-1} a_k$.
- An arithmetic sequence is one where the difference between terms is constant. The terms can be written as $a, \quad a+d, \quad a+2d, \quad a+3d, \ldots$, where a is the first term and d is the common difference.
- The sum of the first n terms of an arithmetic sequence with first term a and common difference d is $\frac{n}{2}(2a + (n-1)d)$, which you can read as "first term plus last term, times the number of terms, divided by two". If we recognise the quantity "first term plus last term, divided by two" as the "average of the terms", then this gives an idea of why the formula is true.
- A geometric sequence is one where the ratio between consecutive terms is constant. The terms can be written as $a, \quad ar, \quad ar^2, \quad ar^3, \ldots$, where a is the first term and r is the common ratio.
- The sum of the first n terms of a geometric sequence with first term a and common ratio r is $\frac{a(1-r^n)}{1-r}$. One way to remember this is to remember what happens if we multiply the sum of a geometric series by $(1-r)$,

$$\begin{aligned}(1-r)(a + ar + \cdots + ar^{n-1}) =& (a - ar) + (ar - ar^2) + \cdots + (ar^{n-1} - ar^n) \\ =& a - ar^n.\end{aligned}$$

- For a geometric sequence a_n, the sum to infinity is written as $\sum_{k=0}^{\infty} a_k$. If the common ratio r satisfies $|r| < 1$ then this is equal to $\frac{a}{1-r}$. If $|r| \geqslant 1$ then this sum to infinity does not converge (it does not approach any particular real number).

Warm-up

- A sequence is defined by $a_n = n^2 - n$. What is a_3? What is a_{10}? Find $a_{n+1} - a_n$ in terms of n. Find $a_{n+1} - 2a_n + a_{n-1}$ in terms of n.

- A sequence is defined by $a_0 = 1$ and $a_n = a_{n-1} + 3$ for $n \geqslant 1$. Find $a_0 + a_1 + \cdots + a_{10}$. Find a_{1000}.

- A sequence is defined by $a_0 = 1$ and $a_n = \frac{a_{n-1}}{3}$ for $n \geqslant 1$. Find $a_0 + a_1 + \cdots + a_{10}$. Find a_{1000}. Does the sum of all the terms of this sequence converge? If it does, what is the sum to infinity?

- A sequence is defined by $a_0 = 1$ and $a_n = 3a_{n-1} + 1$ for $n \geqslant 1$. A sequence b_n is defined by $b_n = A \times 3^n + B$ where A and B are real numbers. Find values for A and B such that $a_n = b_n$ for all $n \geqslant 0$.

- A sequence is defined by $a_n = An^2 + Bn + C$ where A, B, and C are real numbers. Find A, B, and C in terms of a_0, a_1, and a_2.

- Simplify $2^1 + 2^2 + 2^3 + \cdots + 2^n$ for $n \geqslant 1$.

- Simplify $3^4 + 3^5 + 3^6 + \cdots + 3^n$ for $n \geqslant 4$.

- When does the sum $1 + x^3 + x^6 + x^9 + x^{12} + ...$ converge? Simplify it in the case that it converges.

- When does the sum $2 - x + \dfrac{x^2}{2} - \dfrac{x^3}{4} + \ldots$ converge? Simplify it in the case that it converges.

- Consider the sum of the first n terms of an arithmetic sequence $a_1, a_2, \ldots, a_n$ with $a_1 = a$ and $a_2 = a + d$. Explain why the sum of the i^{th} term and the $(n+1-i)^{\text{th}}$ term doesn't depend on i, as long as $1 \leqslant i \leqslant n$. By considering the cases (i) n is even, and (ii) n is odd, separately, deduce that the sum of the first n terms of an arithmetic sequence is n times the average term (the first term plus the last term, divided by 2).

- Consider the sum of the first n terms of an arithmetic sequence with first term a and constant difference d. Consider the special case $d = 0$. Write down the sum in this case. Now consider the case $a = 0$. In this case, write the sum in terms of the triangle numbers $T_n = 1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1)$. Hence write down the sum of the first n terms of an arithmetic sequence. Check that this agrees with the formula above.

MAT questions

MAT 2016 Q1A

A sequence a_n has first term $a_1 = 1$, and subsequent terms defined by $a_{n+1} = la_n$ for $n \geqslant 1$. What is the product of the first 15 terms of the sequence?

(a) l^{14}, (b) $15 + l^{14}$, (c) $15l^{14}$, (d) l^{105}, (e) $15 + l^{105}$.

Hint: note that this question is asking for the *product* and not the *sum*. Also note that the first term is a_1 not a_0, so the first 15 terms will be $a_1, a_2, \ldots, a_{14}, a_{15}$.

MAT 2016 Q1G

The sequence x_n, where $n \geqslant 0$, is defined by $x_0 = 1$ and

$$x_n = \sum_{k=0}^{n-1} x_k \qquad \text{for } n \geqslant 1.$$

The sum

$$\sum_{k=0}^{\infty} \frac{1}{x_k}$$

equals

(a) 1, (b) $\frac{6}{5}$, (c) $\frac{8}{5}$, (d) 3, (e) $\frac{27}{5}$.

Hint: work out a few of the values $x_1, x_2, x_3, \ldots$ before trying to work out the sum to infinity.

MAT 2017 Q1C

A sequence (a_n) has the property that

$$a_{n+1} = \frac{a_n}{a_{n-1}}$$

for every $n \geqslant 2$. Given that $a_1 = 2$ and $a_2 = 6$, what is a_{2017}?

(a) $\frac{1}{6}$, (b) $\frac{2}{3}$, (c) $\frac{3}{2}$, (d) 2, (e) 3.

Hint: again, work out a few of the values $x_1, x_2, x_3, \ldots$.

MAT 2016 Q5

This question concerns the sum s_n defined by

$$s_n = 2 + 8 + 24 + \cdots + n2^n.$$

(i) Let $f(n) = (An+B)2^n+C$ for constants A, B and C yet to be determined, and suppose $s_n = f(n)$ for all $n \geqslant 1$. By setting $n = 1, 2, 3$, find equations that must be satisfied by A, B and C.

(ii) Solve the equations from part (i) to obtain values for A, B and C.

(iii) Using these values, show that if $s_k = f(k)$ for some $k \geqslant 1$ then $s_{k+1} = f(k+1)$.

You may now assume that $f(n) = s_n$ for all $n \geqslant 1$.

(iv) Find simplified expressions for the following sums:

$$t_n = n + 2(n-1) + 4(n-2) + 8(n-3) + \cdots + 2^{n-1}1,$$
$$u_n = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdots + \frac{n}{2^n}.$$

(v) Find the sum

$$\sum_{k=1}^{n} s_k.$$

Hints: At the start, take a moment to understand the definition of s_n. How do the numbers 8 and 24 relate to the $+n2^n$ part of the definition? What are the values of s_1 and s_2 and s_3?

In part (iii) we're being asked to investigate what happens when we go from s_k to s_{k+1}. From the definition at the top, what changes when we go from s_k to s_{k+1}? If we do that to $f(k)$, do we get to $f(k+1)$?

In part (iv), it would be good if we could find a link between t_n and s_n, perhaps by spotting a copy of the sum that defines s_n hiding in there. Then we want to find a link between u_n and s_n, or a link between u_n and t_n. If that doesn't work, we can go back to the idea in part (i) and try to find a general expression for the n^{th} term of t_n or u_n by guessing a function like $(An + B)2^n + C$ or $(An + B)2^{-n} + C$.

In part (v), we know an expression for s_k in terms of things like 2^k and $k2^k$. We know how to sum the first of those things, and the sum of the second thing there is oddly familiar from earlier in this question...

Extension

A future session of the Oxford MAT Livestream will be on "recursion", and we'll look at more expressions that are like $a_n = f(a_{n-1})$ but more complex.

The following material is included for your interest only, and not for MAT preparation.

There's a general formula for sequences where the difference between terms is itself an arithmetic sequence. The sequences are sometimes called quadratic sequences, and they have $a_n = An^2 + Bn + C$ for some A, B, and C. You can probably guess what happens if the difference between terms of a sequence is itself a quadratic sequence.

In MAT 2016 Q5, we found a formula for the sum of the first n terms of the sequence $a_k = k2^k$ with one particular method (guess the formula, check the formula). Here's a more direct proof; expand and sum and sum.

- Expand out each term into 2^ks, so that we've got

$$\sum_{k=1}^{n} k2^k = (2^1) + (2^2 + 2^2) + (2^3 + 2^3 + 2^3) + \cdots + (\underbrace{2^n + 2^n + \cdots + 2^n}_{n}).$$

- Regroup the terms and sum

$$\begin{aligned}\sum_{k=1}^{n} k2^k &= \left(2^1 + 2^2 + \cdots + 2^n\right) + (2^2 + 2^3 + \cdots + 2^n) + \cdots + (2^{n-1} + 2^n) + 2^n \\ &= 2^1(2^n - 1) + 2^2(2^{n-1} - 1) + \cdots + 2^{n-1}(2^2 - 1) + 2^n(2^1 - 1)\end{aligned}$$

- Expand these brackets, bring together all the 2^{n+1}s and notice that the remaining terms are another geometric series

$$\begin{aligned}\sum_{k=1}^{n} k2^k &= n2^{n+1} - 2^1 - 2^2 - 2^3 - \cdots - 2^n \\ &= n2^{n+1} - 2^1(2^n - 1) \\ &= (2n-2)2^n + 2\end{aligned}$$

You might be able to adapt this method to similar sums $\sum_{k=1}^{n} kx^k$ for other numbers x. (Watch out for factors of $(x-1)$ from the geometric sums, which are 1 above when $x = 2$.)

For the sum to infinity, if $|x| < 1$, then we get $x + 2x^2 + 3x^3 + 4x^4 + \cdots = \dfrac{x}{(x-1)^2}$. This agrees with a different, more advanced calculation using calculus; the sum is

$$x + 2x^2 + 3x^3 + 4x^4 + \cdots = x\frac{\mathrm{d}}{\mathrm{d}x}\left(1 + x + x^2 + x^3 + \dots\right) = x\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{1-x}\right) = \frac{x}{(1-x)^2}$$

but this uses the "chain rule" for differentiation, which is not on the MAT syllabus!