MAT syllabus

Sequences defined iteratively and by formulae. Arithmetic and geometric progressions*. Their sums*. Convergence condition for infinite geometric progressions*.  
* Part of full A-level Mathematics syllabus.

Revision

- A sequence $a_n$ might be defined by a formula for the $n^{th}$ term like $a_n = n^2 - n$.
- A sequence might be defined with a relation like $x_{n+1} = f(x_n)$ for $n \geq 0$, if we’re given the function $f(x)$ and also given a first term like $a_0 = 1$. (The “first term” might be $a_0$ if we feel like counting from zero).
- The sum of the first $n$ terms of a sequence $a_k$ can be written with the notation $\sum_{k=0}^{n-1} a_k$.
- An arithmetic sequence is one where the difference between terms is constant. The terms can be written as $a, \quad a+d, \quad a+2d, \quad a+3d, \ldots$, where $a$ is the first term and $d$ is the common difference.
- The sum of the first $n$ terms of an arithmetic sequence with first term $a$ and common difference $d$ is $\frac{n}{2}(2a + (n-1)d)$, which you can read as “first term plus last term, times the number of terms, divided by two”. If we recognise the quantity “first term plus last term, divided by two” as the “average of the terms”, then this gives an idea of why the formula is true.
- A geometric sequence is one where the ratio between consecutive terms is constant. The terms can be written as $a, \quad ar, \quad ar^2, \quad ar^3, \ldots$, where $a$ is the first term and $r$ is the common ratio.
- The sum of the first $n$ terms of a geometric sequence with first term $a$ and common ratio $r$ is $\frac{a(1-r^n)}{1-r}$. One way to remember this is to remember what happens if we multiply the sum of a geometric series by $(1-r)$,

$$
(1-r)(a + ar + \cdots + ar^{n-1}) = (a - ar) + (ar - ar^2) + \cdots + (ar^{n-1} - ar^n) = a - ar^n.
$$

- For a geometric sequence $a_n$, the sum to infinity is written as $\sum_{k=0}^{\infty} a_k$. If the common ratio $r$ satisfies $|r| < 1$ then this is equal to $\frac{a}{1-r}$. If $|r| \geq 1$ then this sum to infinity does not converge (it does not approach any particular real number).

For solutions see [www.maths.ox.ac.uk/r/matlive](http://www.maths.ox.ac.uk/r/matlive)
Warm-up

• A sequence is defined by $a_n = n^2 - n$. What is $a_3$? What is $a_{10}$? Find $a_{n+1} - a_n$ in terms of $n$. Find $a_{n+1} - 2a_n + a_{n-1}$ in terms of $n$.

• A sequence is defined by $a_0 = 1$ and $a_n = a_{n-1} + 3$ for $n \geq 1$. Find $a_0 + a_1 + \cdots + a_{10}$. Find $a_{1000}$.

• A sequence is defined by $a_0 = 1$ and $a_n = \frac{2a_{n-1}}{3}$ for $n \geq 1$. Find $a_0 + a_1 + \cdots + a_{10}$. Find $a_{1000}$. Does the sum of all the terms of this sequence converge? If it does, what is the sum to infinity?

• A sequence is defined by $a_0 = 1$ and $a_n = 3a_{n-1} + 1$ for $n \geq 1$. A sequence $b_n$ is defined by $b_n = A \times 3^n + B$ where $A$ and $B$ are real numbers. Find values for $A$ and $B$ such that $a_n = b_n$ for all $n \geq 0$.

• A sequence is defined by $a_n = An^2 + Bn + C$ where $A$, $B$, and $C$ are real numbers. Find $A$, $B$, and $C$ in terms of $a_0$, $a_1$, and $a_2$.

• Simplify $2^1 + 2^2 + 2^3 + \cdots + 2^n$ for $n \geq 1$.

• Simplify $3^4 + 3^5 + 3^6 + \cdots + 3^n$ for $n \geq 4$.

• When does the sum $1 + x^3 + x^6 + x^9 + x^{12} + \cdots$ converge? Simplify it in the case that it converges.

• When does the sum $2 + x + \frac{x^2}{2} - \frac{x^3}{4} + \cdots$ converge? Simplify it in the case that it converges.

• Consider the sum of the first $n$ terms of an arithmetic sequence $a_1$, $a_2$, $\ldots$, $a_n$ with $a_1 = a$ and $a_2 = a + d$. Explain why the sum of the $i^{th}$ term and the $(n + 1 - i)^{th}$ term doesn’t depend on $i$, as long as $1 \leq i \leq n$. By considering the cases (i) $n$ is even, and (ii) $n$ is odd, separately, deduce that the sum of the first $n$ terms of an arithmetic sequence is $n$ times the average term (the first term plus the last term, divided by 2).

• Consider the sum of the first $n$ terms of an arithmetic sequence with first term $a$ and constant difference $d$. Consider the special case $d = 0$. Write down the sum in this case. Now consider the case $a = 0$. In this case, write the sum in terms of the triangle numbers $T_n = 1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n + 1)$. Hence write down the sum of the first $n$ terms of an arithmetic sequence. Check that this agrees with the formula above.
MAT questions

MAT 2016 Q1A
A sequence $a_n$ has first term $a_1 = 1$, and subsequent terms defined by $a_{n+1} = la_n$ for $n \geq 1$. What is the product of the first 15 terms of the sequence?

(a) $l^{14}$, (b) $15 + l^{14}$, (c) $15l^{14}$, (d) $l^{105}$, (e) $15 + l^{105}$.

Hint: note that this question is asking for the product and not the sum. Also note that the first term is $a_1$ not $a_0$, so the first 15 terms will be $a_1, a_2, \ldots, a_{14}, a_{15}$.

MAT 2016 Q1G
The sequence $x_n$, where $n \geq 0$, is defined by $x_0 = 1$ and

$$x_n = \sum_{k=0}^{n-1} x_k \quad \text{for } n \geq 1.$$ 

The sum

$$\sum_{k=0}^{\infty} \frac{1}{x_k}$$

equals

(a) 1, (b) $\frac{6}{5}$, (c) $\frac{8}{5}$, (d) 3, (e) $\frac{27}{5}$.

Hint: work out a few of the values $x_1, x_2, x_3, \ldots$ before trying to work out the sum to infinity.

MAT 2017 Q1C
A sequence $(a_n)$ has the property that

$$a_{n+1} = \frac{a_n}{a_{n-1}}$$

for every $n \geq 2$. Given that $a_1 = 2$ and $a_2 = 6$, what is $a_{2017}$?

(a) $\frac{1}{6}$, (b) $\frac{2}{3}$, (c) $\frac{3}{2}$, (d) 2, (e) 3.

Hint: again, work out a few of the values $x_1, x_2, x_3, \ldots$.
MAT 2016 Q5
This question concerns the sum \( s_n \) defined by
\[
 s_n = 2 + 8 + 24 + \cdots + n2^n.
\]

(i) Let \( f(n) = (An + B)2^n + C \) for constants \( A, B \) and \( C \) yet to be determined, and suppose \( s_n = f(n) \) for all \( n \geq 1 \). By setting \( n = 1, 2, 3 \), find equations that must be satisfied by \( A, B \) and \( C \).

(ii) Solve the equations from part (i) to obtain values for \( A, B \) and \( C \).

(iii) Using these values, show that if \( s_k = f(k) \) for some \( k \geq 1 \) then \( s_{k+1} = f(k+1) \).
You may now assume that \( f(n) = s_n \) for all \( n \geq 1 \).

(iv) Find simplified expressions for the following sums:
\[
 t_n = n + 2(n-1) + 4(n-2) + 8(n-3) + \cdots + 2^{n-1}1,
\]
\[
 u_n = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdots + \frac{n}{2^n}.
\]

(v) Find the sum
\[
 \sum_{k=1}^{n} s_k.
\]

Hints: At the start, take a moment to understand the definition of \( s_n \). How do the numbers 8 and 24 relate to the \(+ n2^n\) part of the definition? What are the values of \( s_1 \) and \( s_2 \) and \( s_3 \)?

In part (iii) we’re being asked to investigate what happens when we go from \( s_k \) to \( s_{k+1} \). From the definition at the top, what changes when we go from \( s_k \) to \( s_{k+1} \)? If we do that to \( f(k) \), do we get to \( f(k+1) \)?

In part (iv), it would be good if we could find a link between \( t_n \) and \( s_n \), perhaps by spotting a copy of the sum that defines \( s_n \) hiding in there. Then we want to find a link between \( u_n \) and \( s_n \), or a link between \( u_n \) and \( t_n \). If that doesn’t work, we can go back to the idea in part (i) and try to find a general expression for the \( n^{th} \) term of \( t_n \) or \( u_n \) by guessing a function like \((An + B)2^n + C\) or \((An + B)2^{-n} + C\).

In part (v), we know an expression for \( s_k \) in terms of things like \( 2^k \) and \( k2^k \). We know how to sum the first of those things, and the sum of the second thing there is oddly familiar from earlier in this question...

For solutions see www.maths.ox.ac.uk/r/matlive
Extension

A future session of the Oxford MAT Livestream will be on “recursion”, and we’ll look at more expressions that are like \( a_n = f(a_{n-1}) \) but more complex.

The following material is included for your interest only, and not for MAT preparation.

There’s a general formula for sequences where the difference between terms is itself an arithmetic sequence. The sequences are sometimes called quadratic sequences, and they have \( a_n = An^2 + Bn + C \) for some \( A, B, \) and \( C \). You can probably guess what happens if the difference between terms of a sequence is itself a quadratic sequence.

In MAT 2016 Q5, we found a formula for the sum of the first \( n \) terms of the sequence \( a_k = k2^k \) with one particular method (guess the formula, check the formula). Here’s a more direct proof; expand and sum and sum.

- Expand out each term into \( 2^k \) s, so that we’ve got
  \[
  \sum_{k=1}^{n} k2^k = (2^1) + (2^2 + 2^2) + (2^3 + 2^3 + 2^3) + \cdots + \left( \frac{2^n + 2^n + \cdots + 2^n}{n} \right).
  \]

- Regroup the terms and sum
  \[
  \sum_{k=1}^{n} k2^k = (2^1 + 2^2 + \cdots + 2^n) + (2^2 + 2^3 + \cdots + 2^n) + \cdots + (2^{n-1} + 2^n) + 2^n
  = 2^1(2^n - 1) + 2^2(2^{n-1} - 1) + \cdots + 2^{n-1}(2 - 1) + 2^n(2^1 - 1)
  \]

- Expand these brackets, bring together all the \( 2^n \) s and notice that the remaining terms are another geometric series
  \[
  \sum_{k=1}^{n} k2^k = n2^{n+1} - 2^1 - 2^2 - 2^3 - \cdots - 2^n
  = n2^{n+1} - 2(2^n - 1)
  = (2n - 2)2^n + 2
  \]

You might be able to adapt this method to similar sums \( \sum_{k=1}^{n} kx^k \) for other numbers \( x \). (Watch out for factors of \( (x - 1) \) from the geometric sums, which are 1 above when \( x = 2 \).)

For the sum to infinity, if \( |x| < 1 \), then we get \( x + 2x^2 + 3x^3 + 4x^4 + \cdots = \frac{x}{(x - 1)^2} \). This agrees with a different, more advanced calculation using calculus; the sum is

\[
  x + 2x^2 + 3x^3 + 4x^4 + \cdots = x \frac{d}{dx} \left( 1 + x + x^2 + x^3 + \cdots \right) = x \frac{d}{dx} \left( \frac{1}{1 - x} \right) = \frac{x}{(1 - x)^2}
\]

but this uses the “chain rule” for differentiation, which is not on the MAT syllabus!

For solutions see [www.maths.ox.ac.uk/r/matlive](http://www.maths.ox.ac.uk/r/matlive)