Viscous Core-Annular Flows: A Laboratory Playground for Dispersive Hydrodynamics

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Oxford University, September 22, 2022

Dispersive Hydrodynamics Laboratory

Collaborators

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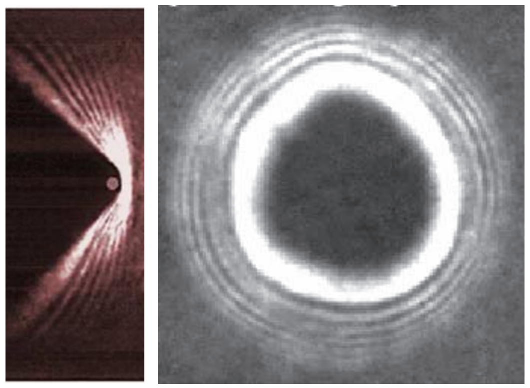
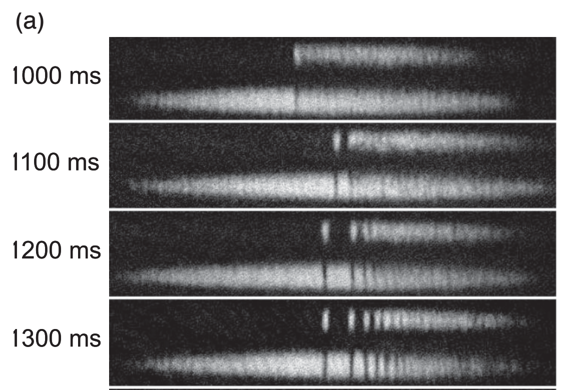
Dispersive Hydrodynamics

Superfluids

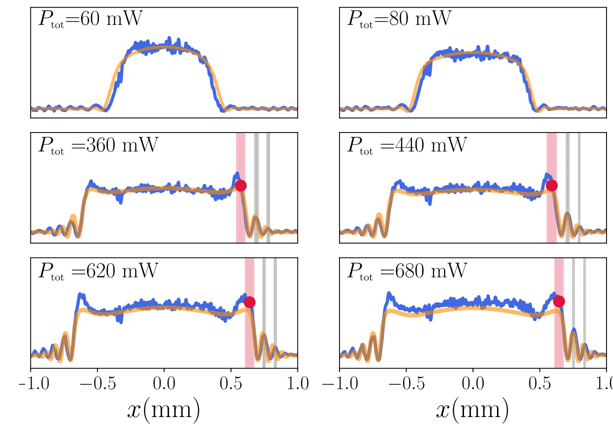


[Cornell JILA group 2004-2006]

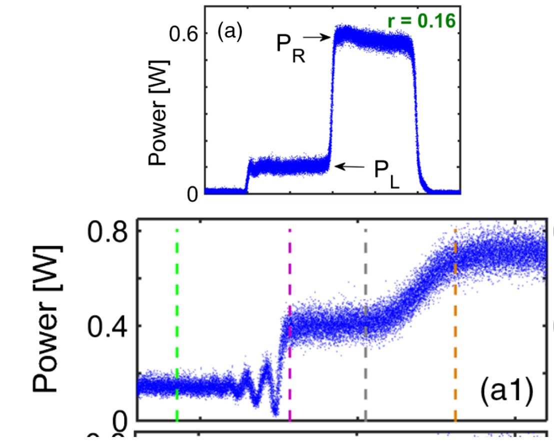


[Hamner *et al* 2011]

Nonlinear optics



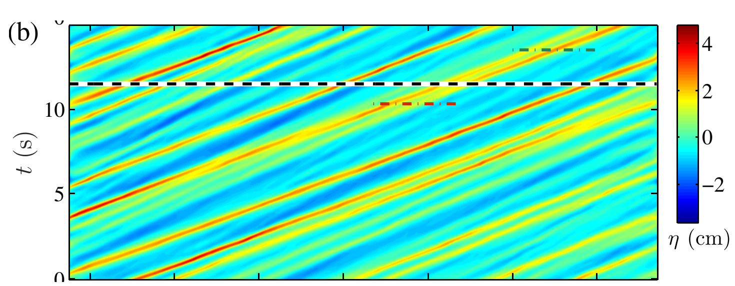
[Bienaimé *et al* 2021]



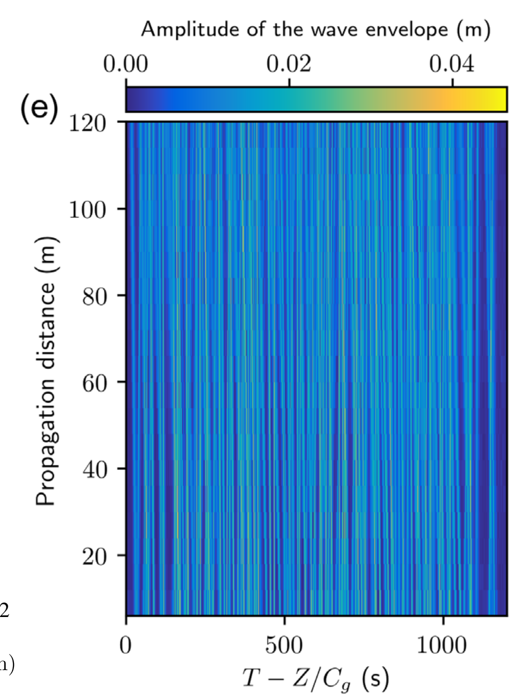
[Xu *et al* 2017]

Surface Waves

[Grawin 2004]

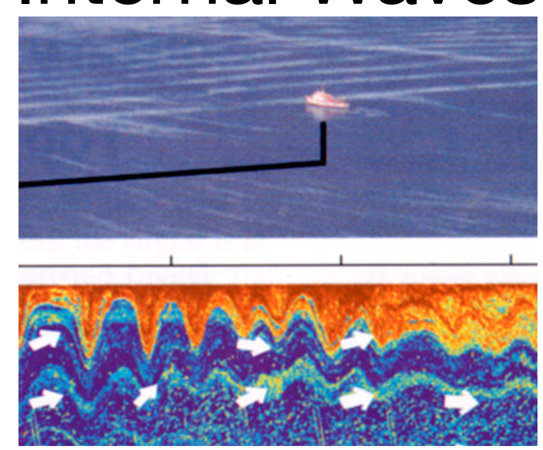


[Redor *et al* 2019]

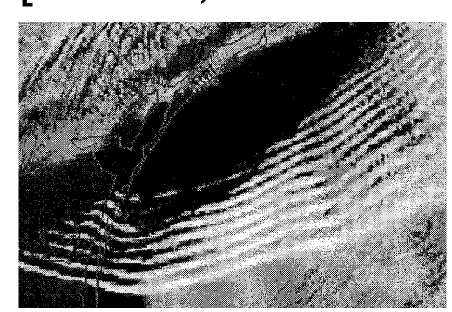


[Suret *et al* 2020]

Internal Waves

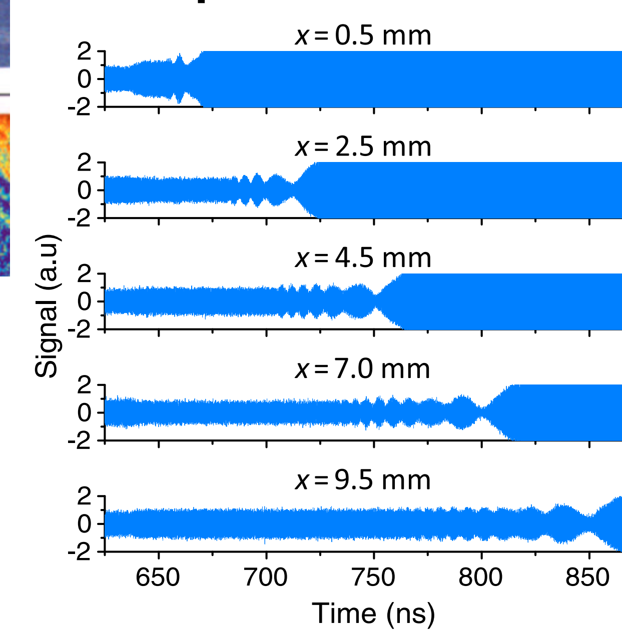


[Farmer, Armi 1999]



[Clarke 1997]

Magnetic Spin Waves



[Janantha *et al* 2017]

Dissipationless, Dispersive Fluid Dynamics

… an applied mathematician's perspective:

$$\underbrace{\mathbf{u}_t + \nabla \cdot \mathrm{F}(\mathbf{u})}_{\text{1st order system of conservation laws}} = \underbrace{\nabla \cdot \mathrm{D}[\mathbf{u}]}_{\text{higher order, dispersive operator}}, \quad \mathbf{u}(\mathbf{x}, t) \in \mathbb{R}^n$$

1st order system of conservation laws

higher order, dispersive operator

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{A} e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} + c.c., \quad |\mathbf{A}| \ll 1 \quad \Rightarrow \quad \omega = \omega(\mathbf{k}, \mathbf{u}_0) \in \mathbb{R}$$

real valued dispersion relation

Mathematical models of BEC, optics, shallow water waves, internal waves in ocean and atmosphere, cold plasma, ….

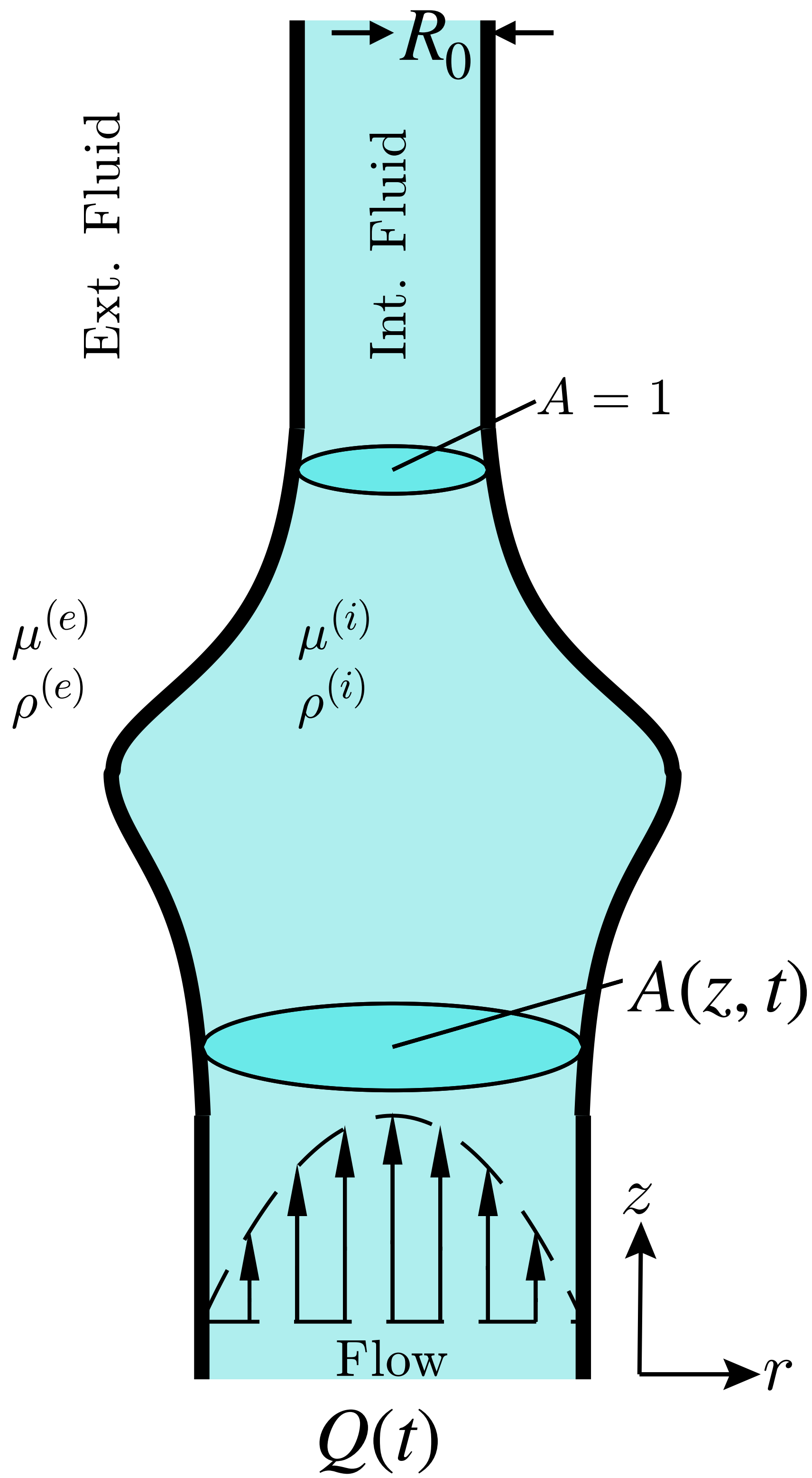
Dispersive regularization of conservation laws

Dispersive Hydrodynamics Programme

- Goal: accurate description of dispersive hydrodynamics
- Mathematical approaches:
  - ✦ PDE (functional) analysis
  - ✦ Integrable systems
  - ✦ Modulation theory
- Problems:
  - ✦ Physical applications
  - ✦ Randomness (soliton gas)
  - ✦ Boundary value problems
  - ✦ Non-convexity
  - ✦ Multiple dimensions

Dispersive Hydrodynamics Program
Isaac Newton Institute, Cambridge, UK
July 4–December 16

Ex: Viscous Core-Annular Flow



Physical setting:

two viscous fluids,
inner fluid forms axisymmetric conduit

Exterior fluid:

$\mu^{(e)}$ - viscosity
$\rho^{(e)}$ - density

Interior fluid:

$\mu^{(i)}$ - viscosity
$\rho^{(i)}$ - density

Key relations:

$\rho^{(i)} < \rho^{(e)}$ - buoyant flow
$\mu^{(i)} \ll \mu^{(e)}$

$$\mathrm{Re} = \frac{T_{\mathrm{momentum}}}{T_{\mathrm{inertial}}} \approx 10^{-1} \qquad \mathrm{Pe} = \frac{T_{\mathrm{mass}}}{T_{\mathrm{inertial}}} \approx 10^{5}$$

Multiscale Model Hierarchy

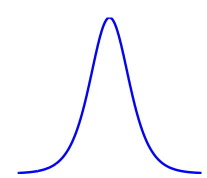
Physical requirements: (1) Miscibility (2) Buoyancy (3) High viscosity contrast (4) Stokes regime (5) Weak mass diffusion

Microscopic continuum model: two-fluid Stokes equations

$$\nabla \cdot \mathbf{u} = 0$$

$$\nabla p = \mu \Delta \mathbf{u}$$

Mesoscopic long wave interfacial model: conduit equation



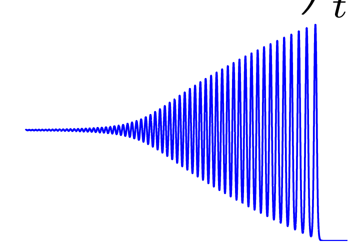
$$A_t + (A^2)_z - \left(A^2 (A^{-1} A_t)_z\right)_z = 0$$

Macroscopic modulated nonlinear wavetrains: Whitham equations

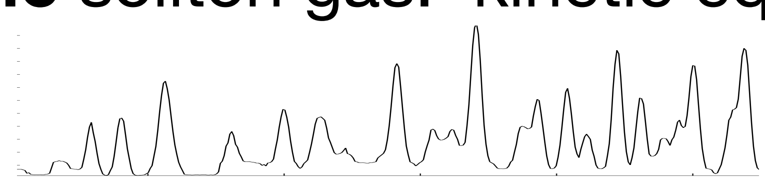
$$(\overline{A})_t + \left(\overline{A^2} - 2k\omega \overline{A_\theta^2}\right)_z = 0$$

$$\left(\overline{A^{-1}} + k^2 \overline{A^{-2} A_\theta^2}\right)_t - \left(2\overline{\ln A}\right)_z = 0$$

$$k_t + \omega_z = 0$$



Megascopic soliton gas: kinetic equation



$$f_t + (Sf)_x = 0$$

$$S(a) = s(a) + \int_0^\infty K(a,a') f(a') [S(a) - S(a')] da'$$

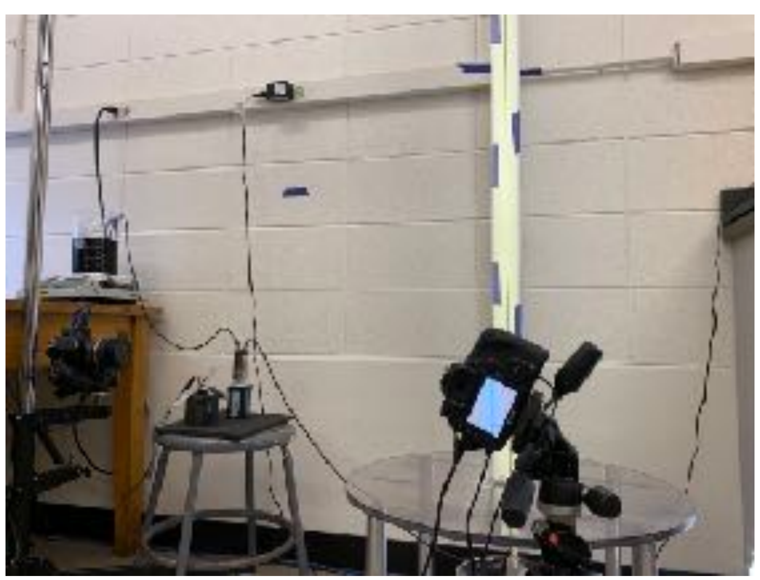
Conduit Equation

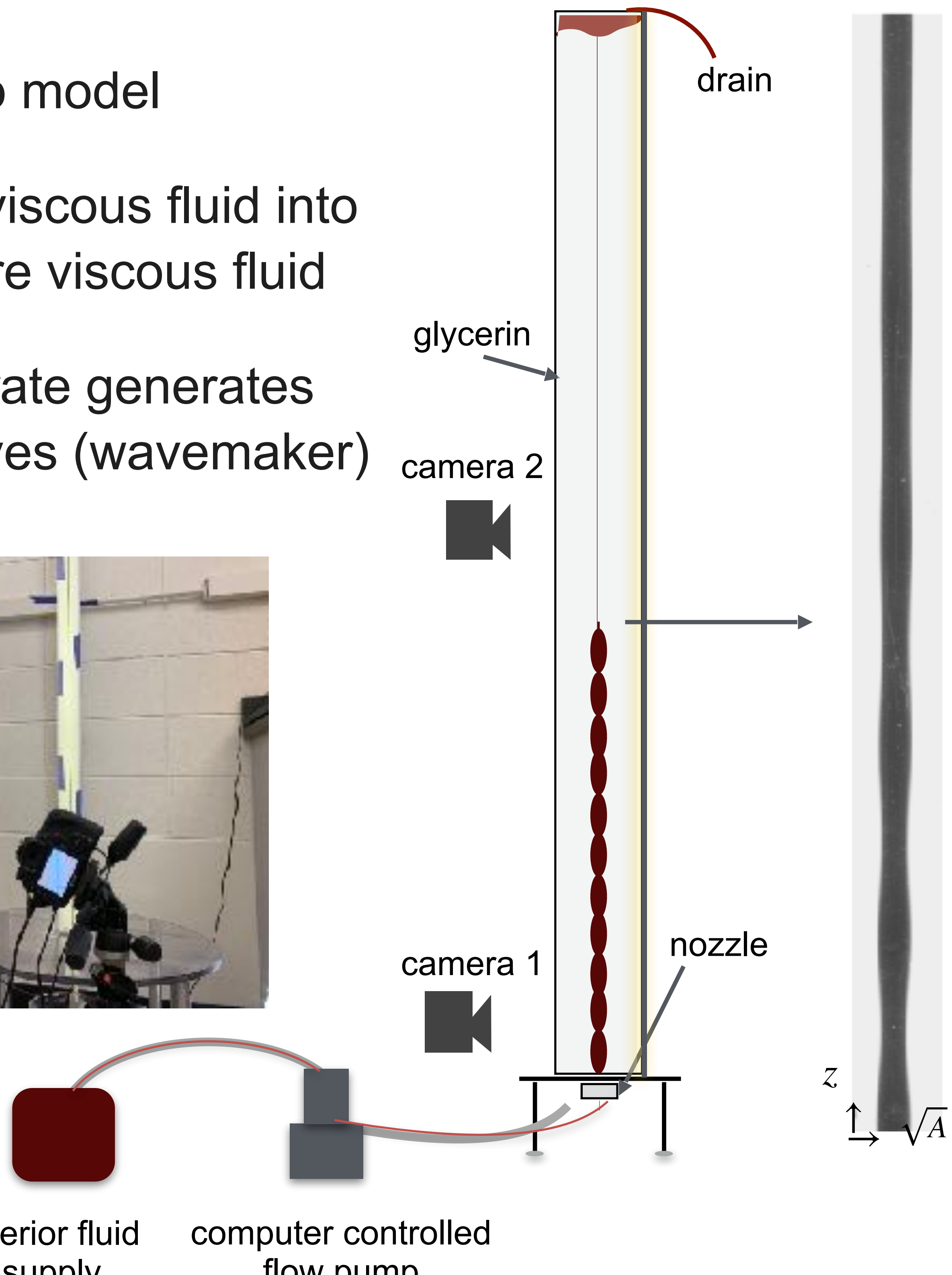
$$A_t + \left(A^2\right)_z = \left(A^2 \left(A^{-1} A_t / A^2\right)_z\right)_z$$

- $A(x, t)$: conduit cross-sectional area
- Long wave approximation of Stokes equations w/ no amplitude restriction: scalar analog of Serre-Green-Naghdi, Choi-Camassa, …
- Nonlocal, nonlinear dispersion: generalizes BBM equation
- Well-studied model, e.g.,
  - Asymptotic solitary wave stability [Simpson, Weinstein *SIMA* 2008]
  - Stability of periodic traveling waves [Maiden, Hoefer *Proc Roy Soc A* 2016; Johnson, Perkins *SIMA* 2020]
  - Generalizes to models of magma [Scott, Stevenson 1984] and channelized glacier water flow [Stubblefield, Spiegelman, Creyts 2020]

Conduit experiment

- Accessible lab model
- Inject lighter, viscous fluid into column of more viscous fluid
- Variable flow rate generates interfacial waves (wavemaker)





Initial, boundary value problem:

$$A(0,t) = \begin{cases} A(0,t+T), & t > 0 \\ 1, & t \le 0 \end{cases}$$

Flow rate $Q(t) = Q_0 A(0,t)^2$

Nondimensionalization:

Vertical length scale:

$$L = \frac{R_0}{\sqrt{8\epsilon}} \sim 0.2 - 0.3 \quad \text{[cm]}$$

Speed scale:

$$U = \frac{g R_0^2 \Delta}{8\mu^{(i)}} \sim 0.3 - 0.5 \quad \text{[cm/s]}$$

Vertical long time scale:

$$T = \sqrt{\frac{8}{\epsilon}} \frac{\mu^{(i)}}{g R_0 \Delta} \sim 0.4 - 1.0 \quad \text{[s]}$$

Example Dynamics/Solutions

solitary waves (solitons)

soliton gas

periodic traveling waves

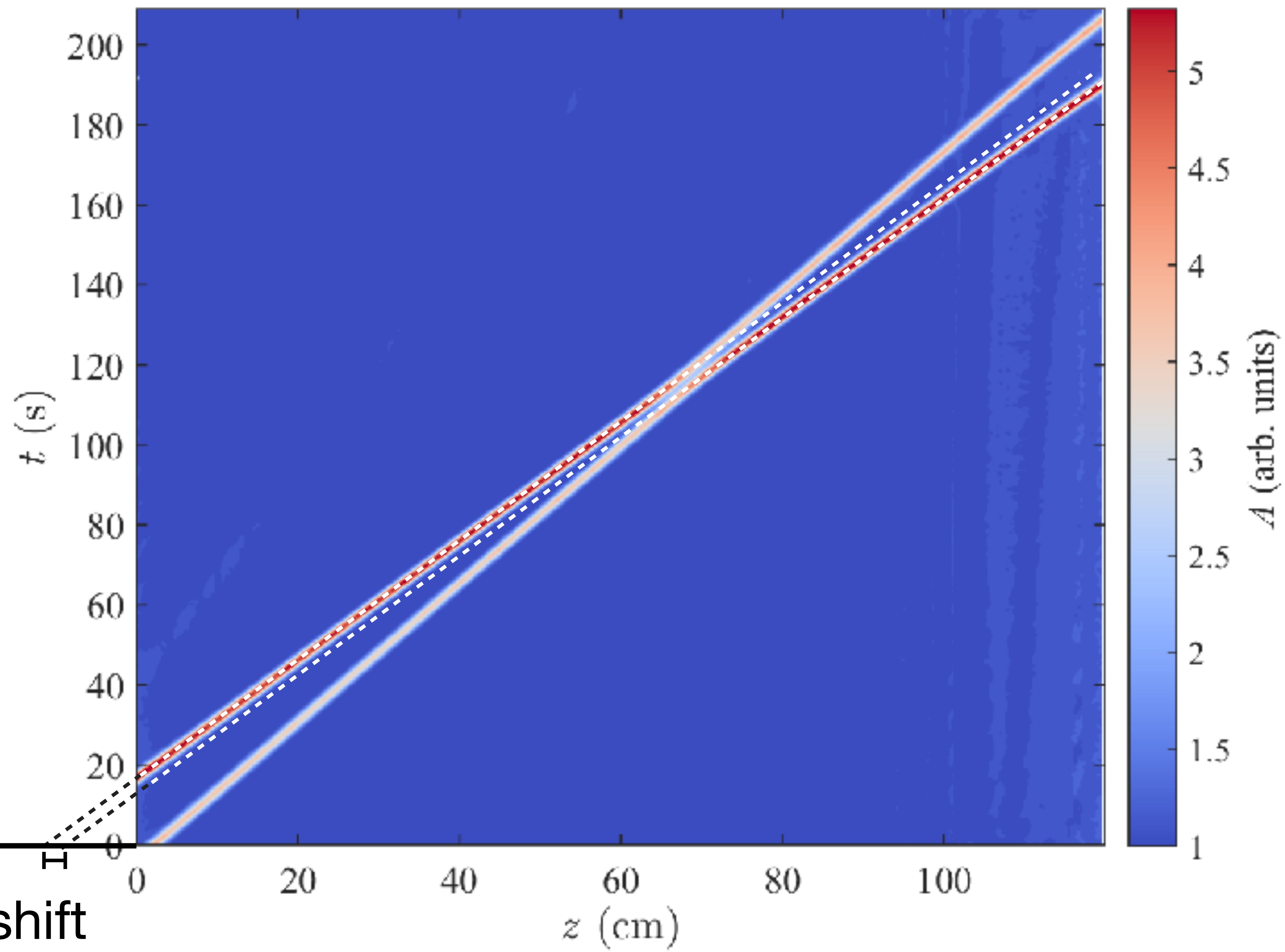
dispersive shock waves

breathers

Conduit Solitary Waves

Lowman, Hoefer, El, *J Fluid Mech* 2014

Two Solitons in a Viscous Fluid Conduit



Phase shift

KdV 2-Soliton Lax Categories

COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL. XXI, 467–490 (1968)

Integrals of Nonlinear Equations of Evolution and Solitary Waves*

PETER D. LAX

Sec 1:

$$L_t = BL - LB = [B, L] \iff u_t = K(u)$$

$$L = \partial_{xx} + \frac{1}{6}u, \quad B = 24\partial_{xxx} + 3u\partial_x + 3\partial_x u, \quad K(u) = -uu_x - u_{xxx}$$

Sec 2:

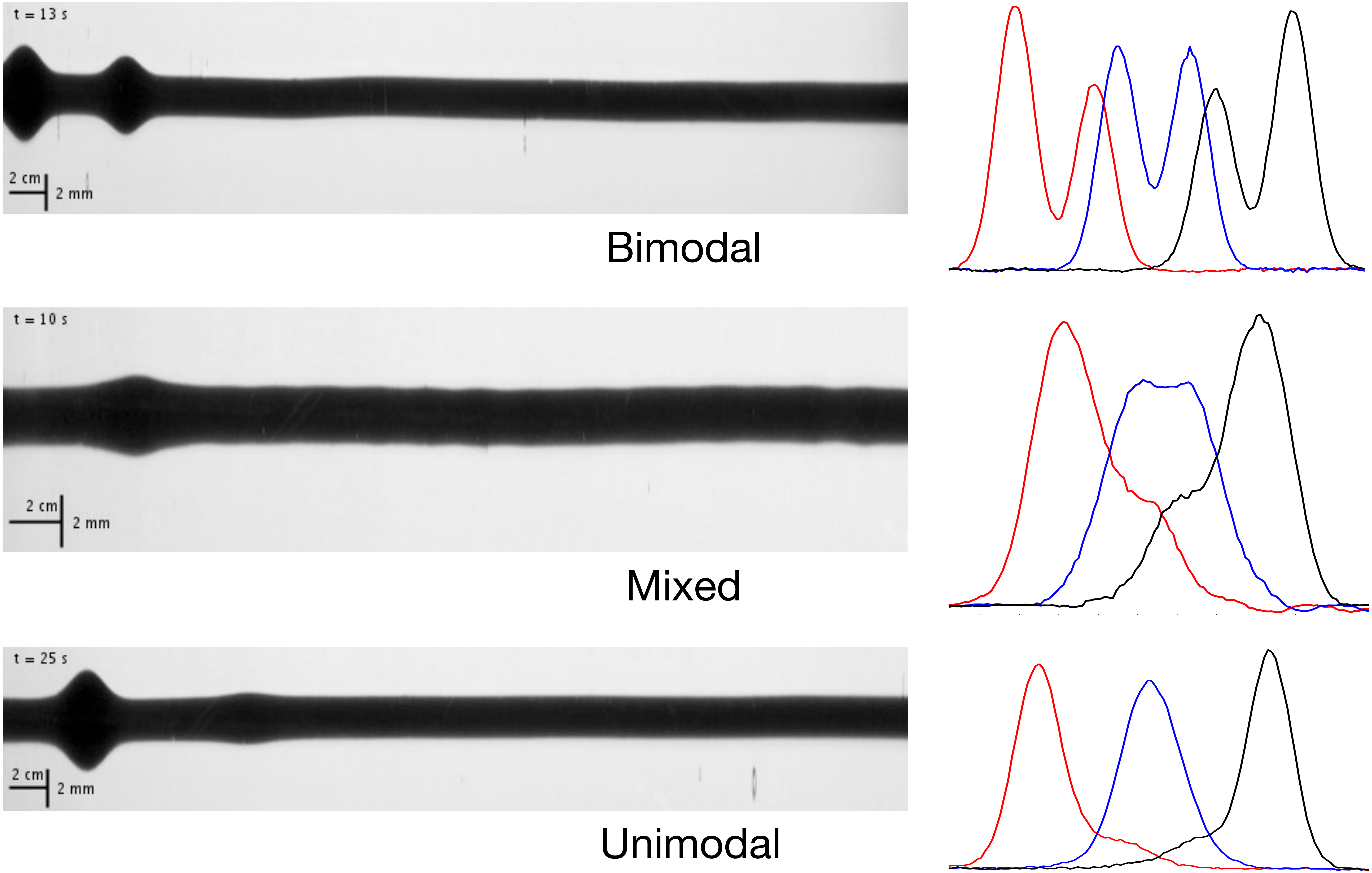
THEOREM 2.1. *For any pair of speeds c_1 and c_2 there exists a double wave, i.e., a solution $d(x, t)$ of the KdV equation such that*

$$d(x, t) - s(x - c_1 t - \theta_1^{\pm}\ ;\ c_1) - s(x - c_2 t - \theta_2^{\pm}\ ;\ c_2) \tag{2.36}$$

tends to zero uniformly as $t \to \pm\infty$.

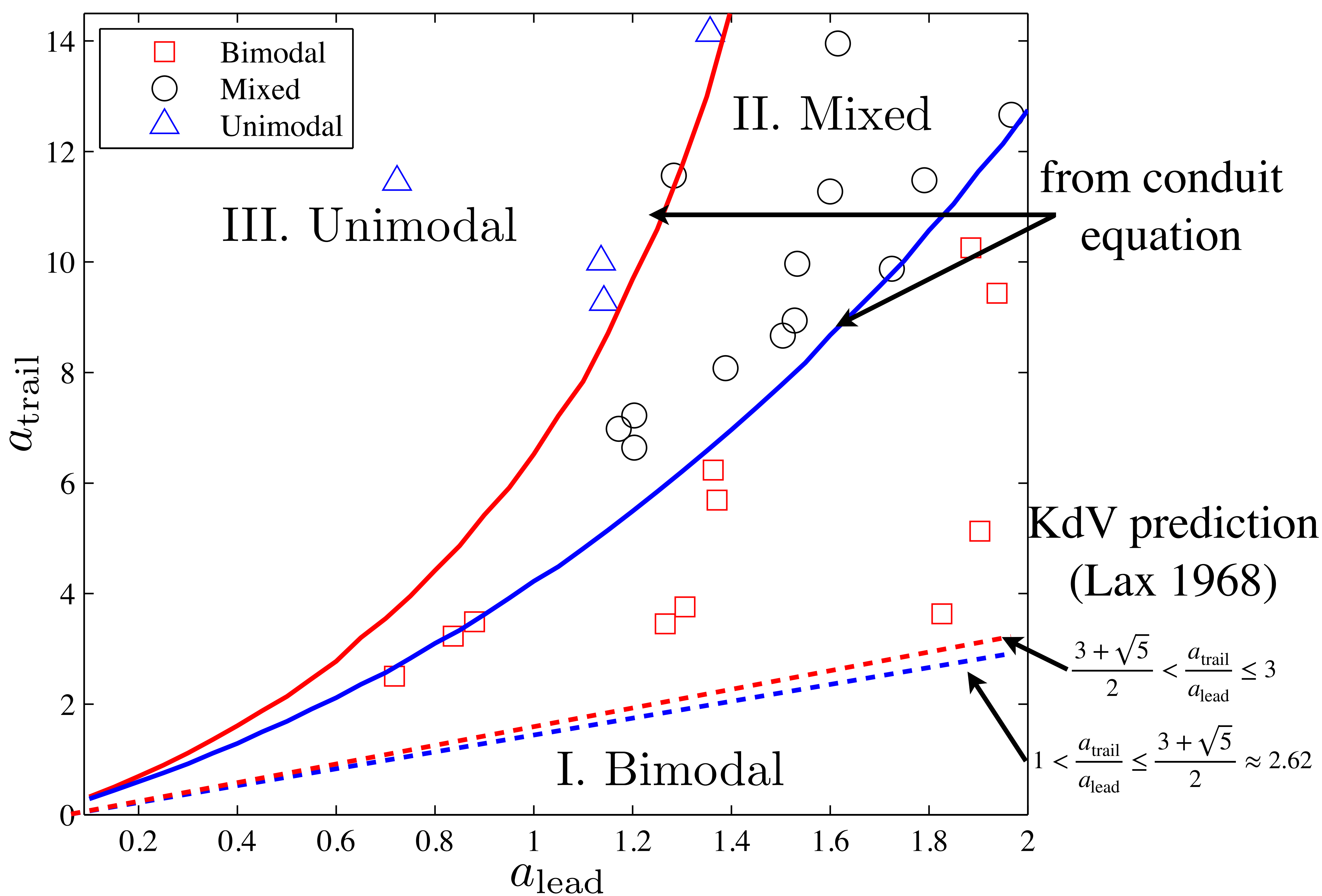
bimodal $\frac{c_1}{c_2} < \frac{3 + \sqrt{5}}{2}$ bi-uni-bi-uni-bimodal $\frac{3 + \sqrt{5}}{2} < \frac{c_1}{c_2} < 3$

bi-uni-bimodal $3 < \frac{c_1}{c_2}$



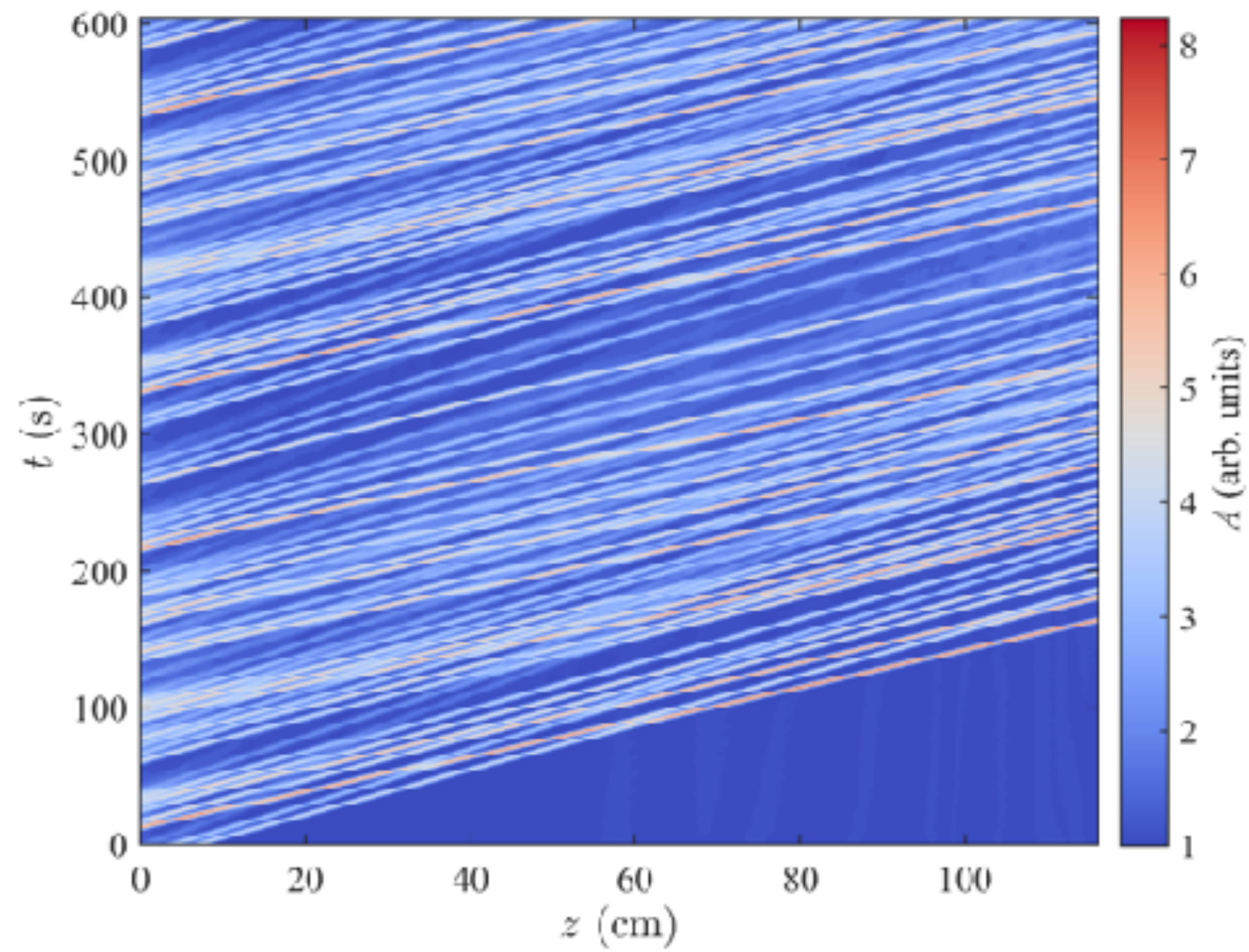
Soliton interaction geometries [Lowman, Hoefer, El *JFM* 2014]

Interaction Type Observations



Soliton Gas

pairwise soliton interactions characterize a soliton gas



Ex: KdV $u_t + 6uu_x + u_{xxx} = 0$

$$\mathscr{L}v = \left(-\partial_{xx} - u\right) v = \lambda v \quad \lambda_i = -\eta_i^2$$

$$u(x,t) = 2\eta_i^2 \operatorname{sech}^2\left[\eta_i(x - 4\eta_i^2 t - x_0)\right]$$

2 soliton phase shift

$$\Delta(\eta_i, \eta_j) = \frac{\operatorname{sgn}(\eta_i - \eta_j)}{\eta_i} \ln\left|\frac{\eta_i + \eta_j}{\eta_i - \eta_j}\right|$$

soliton interaction geometry effects higher order moments of a soliton gas

[Pelinovsky *et al Phys Lett A* 2013]

kinetic equation for density of states $f(\eta)$

[El *Phys Lett A* 2003]

$$f_t + (sf)_x = 0$$

$$s(\eta) = 4\eta^2 + \int_\Gamma \Delta(\eta,\mu) f(\mu)[s(\eta) - s(\mu)]\, \mathrm{d}\mu$$

The Wavemaker Problem

Mao, Hoefer *arXiv* 2022; Hoefer, Mao, Mantzavinos *in preparation* 2022

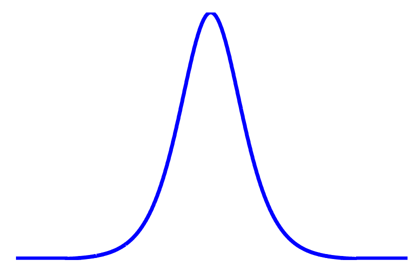
Modeling Experiment

IBVP

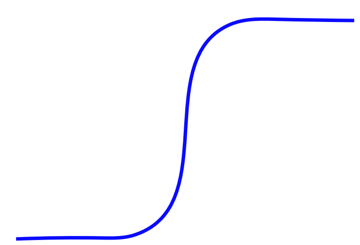
$$A_t + 2AA_z - AA_{zzt} + A_t A_{zz} = 0$$

$$A(z,0) = 1,\ z > 0, \quad A(0,t) = q(t),\ t > 0, \quad q(0) = 0$$

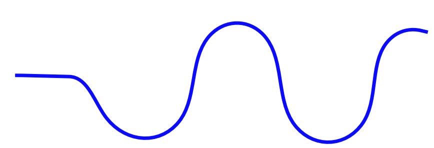
solitary wave

$q(t) =$ 

dispersive shock wave

$q(t) =$ 

periodic traveling wave

$q(t) =$ 

Linear Waves

linearized problem $\omega_0 > 0$

$$A(z,t) = 1 + u(z,2t), \quad |u| \ll 1 \quad \Rightarrow \quad u_\tau + u_z - u_{zz\tau} = 0$$

$$u(z,0) = 0,\ z > 0, \quad u(0,\tau) = -\sin(\omega_0 \tau),\ \tau > 0$$

linear dispersion

$$u(z,\tau) = e^{i(kz-\omega\tau)} \Rightarrow \omega(k) = \frac{k}{1+k^2}, \quad k_\pm(\omega) = \frac{1 \pm \sqrt{1-\omega^2}}{\omega}$$

expect traveling wave with $k_\pm(\omega_0)$ but which branch?

causality, radiation condition

$$\omega = \omega_0 + i\epsilon,\ 0 < \epsilon \ll \omega_0 \Rightarrow u(0,\tau) = e^{-i\omega_0\tau}e^{\epsilon\tau}$$

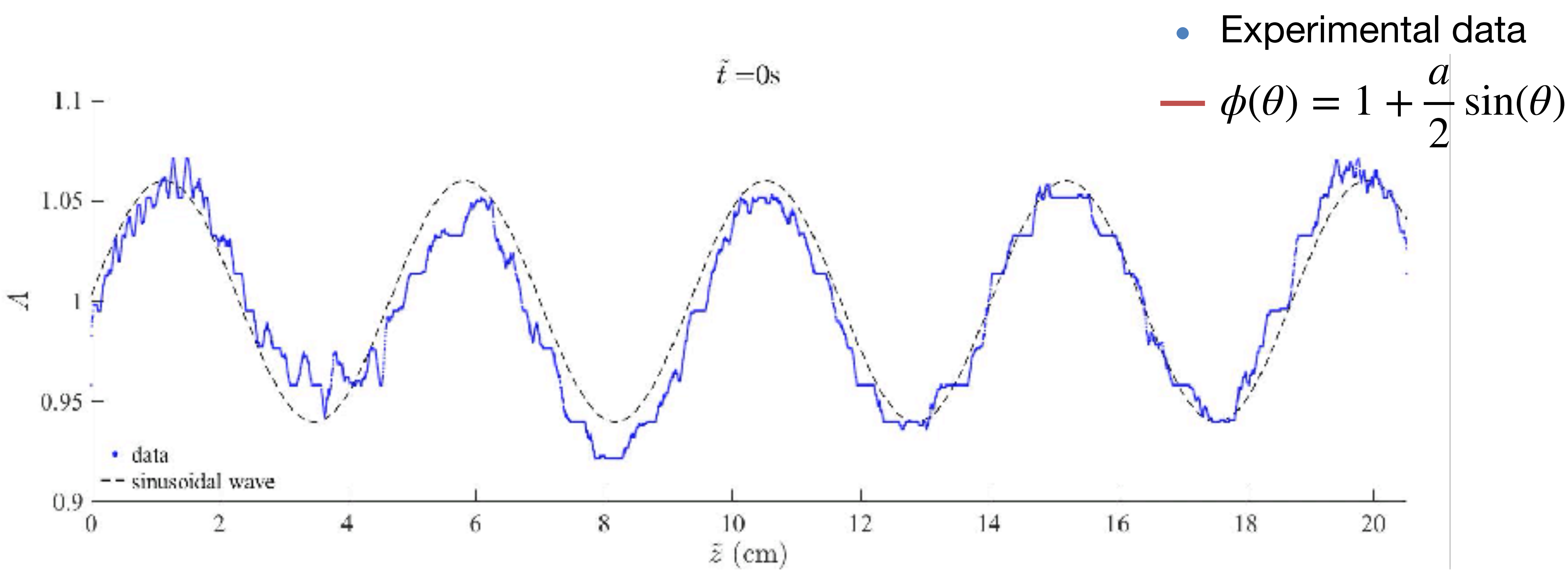
$$k(\omega_0 + i\epsilon) \sim k(\omega_0) + i\epsilon \frac{\mathrm{d}k}{\mathrm{d}\omega}(\omega_0) = k(\omega_0) + \frac{i\epsilon}{c_g(\omega_0)}$$

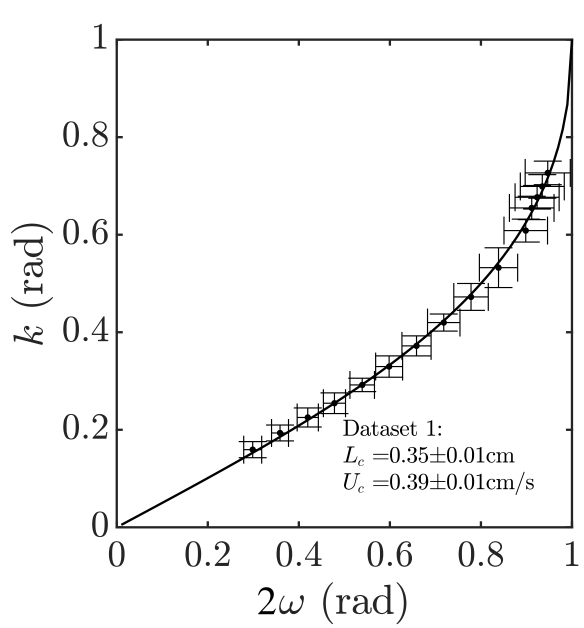
$$u(z,\tau) = e^{\epsilon(\tau - z/c_g)}e^{i(kz-\omega_0\tau)}, \quad u(\infty,\tau) = 0 \Rightarrow c_g(\omega_0) > 0$$

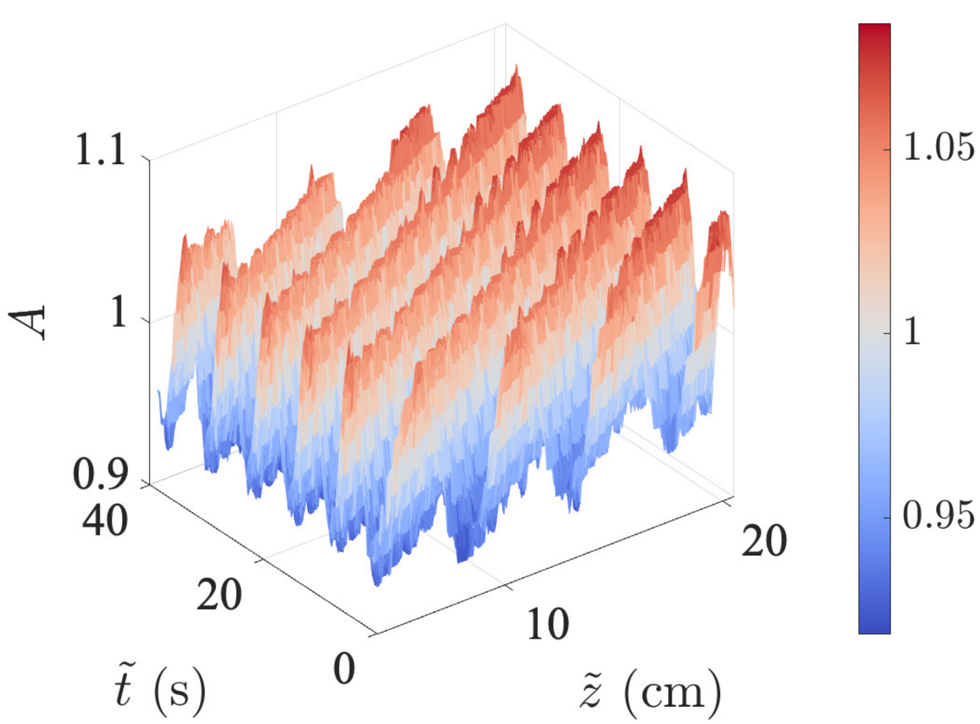
$\omega_0 < \frac{1}{2}$

$$1\Big/\frac{\mathrm{d}k_\pm}{\mathrm{d}\omega}(\omega_0) = \frac{1}{2}\left(1 - 4\omega_0^2 \mp \sqrt{1-4\omega_0^2}\right) \Rightarrow k = k_-(\omega_0)$$

Propagating Linear Waves







measurement

$a = 0.12$

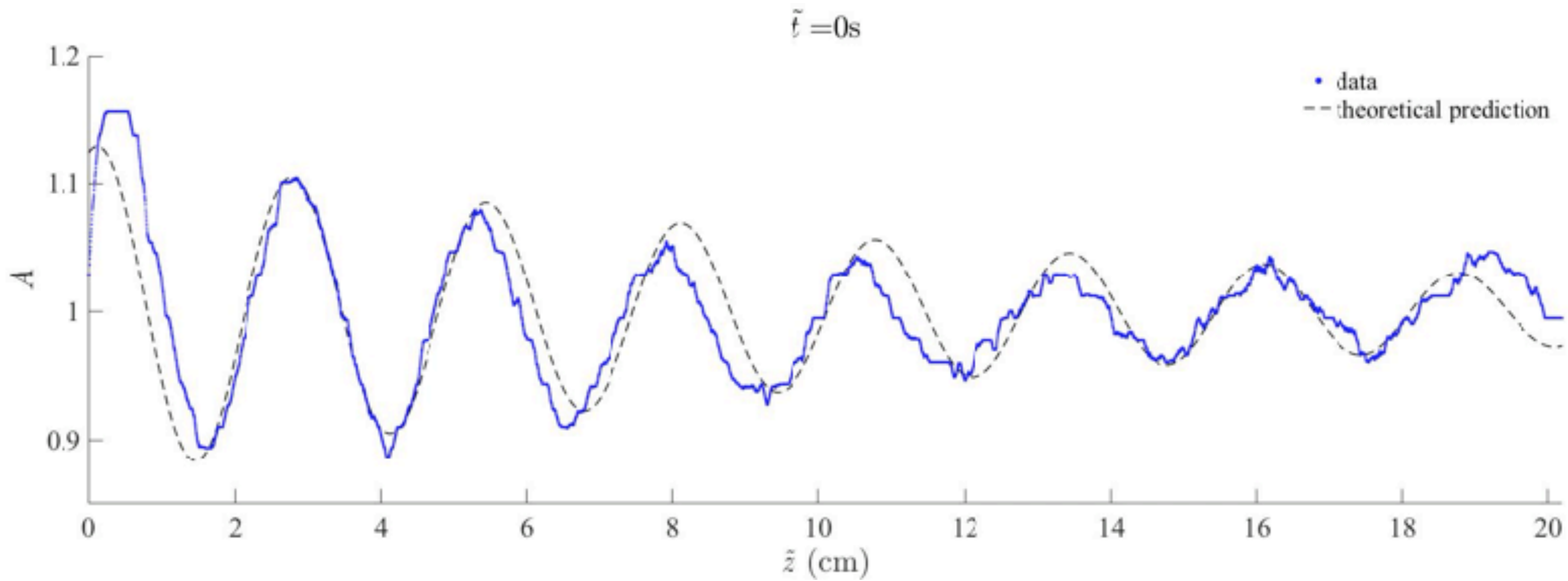
$\omega_0 = 0.374 \pm 0.001$ rad

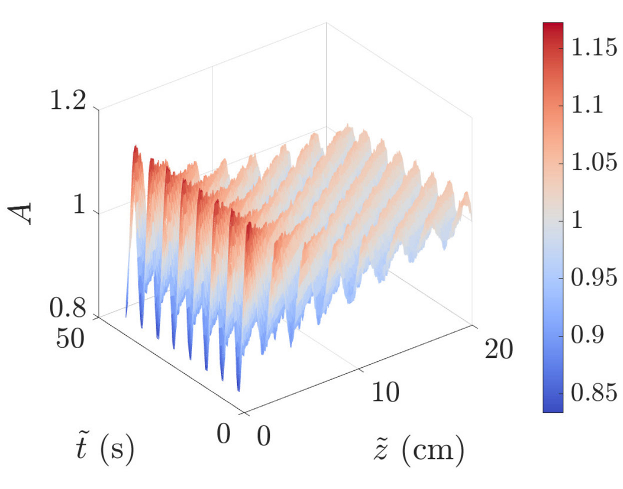
$k = 0.46 \pm 0.01$ rad

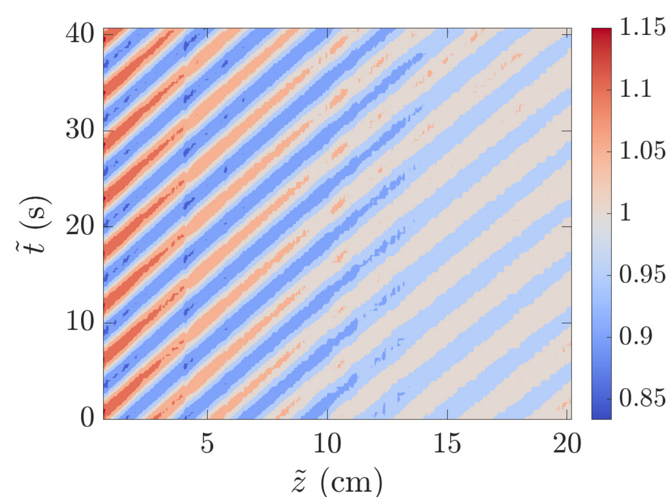
linear theory

$k(\omega_0) = 0.450 \pm 0.002$ rad

Spatially Decaying Linear Waves







observed *upshift* in critical frequency explained by full 2-Stokes dispersion

measurement

$a = 0.26 \pm 0.01$

$\omega_0 = 0.49 \pm 0.03$ rad

$k = (0.81 \pm 0.01) + i\,(0.027 \pm 0.001)$ rad

linear theory

$$k(\omega_0) = \begin{cases} 0.96 + i\,0.27 & \omega_0 = 0.52 \\ 0.82 & \omega_0 = 0.49 \text{ rad} \\ 0.66 & \omega_0 = 0.46 \end{cases}$$

Modulation Theory

$$u_\tau + u_z - u_{zz\tau} = 0$$

seek slowly varying, periodic traveling wave

$$u(z,\tau) = a(Z,T)\cos(\Theta/\epsilon) + \epsilon u_1(\Theta, Z, T) + \cdots, \quad \epsilon \ll 1$$

$$\Theta_Z = k(Z,T), \quad \Theta_T = -\omega(Z,T), \quad Z = \epsilon z, \quad T = \epsilon\tau$$

compatibility

$$\Theta_{ZT} = \Theta_{TZ} \;\Rightarrow\; k_T + \omega_Z = 0$$

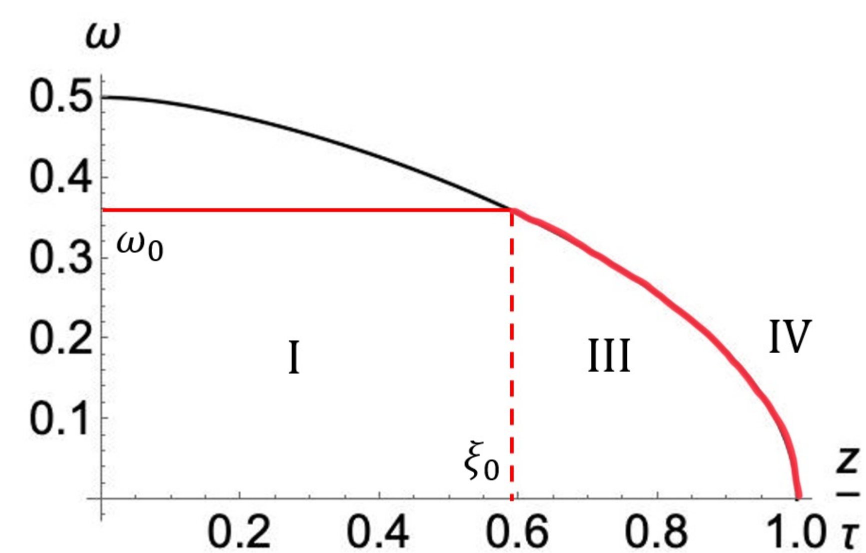
$\mathcal{O}(1)$ and $\mathcal{O}(\epsilon)$

$$\omega(Z,T) = \frac{k(Z,T)}{1 + k(Z,T)^2} \qquad \left(a^2\right)_T + \left(\omega_k(k)\,a^2\right)_Z = 0$$

self-similar solution $0 < \omega_0 < \frac{1}{2}$

$$k(Z,T) = \begin{cases} k_0 = k_-(\omega_0) & 0 < \xi \le c_g(k_0) \\ \sqrt{\dfrac{-1-2\xi+\sqrt{1+8\xi}}{2\xi}} & c_g(k_0) < \xi \le 1 \\ 0 & 1 < \xi \end{cases}$$

$$a(Z,T) = \frac{f(Z,T)}{\sqrt{T}}$$



but does not predict $f(Z,T)$ nor $\omega_0 > \frac{1}{2}$

Long Time Asymptotics

$$u_\tau + u_z - u_{\tau zz} = 0, \quad z \geq 0, \tau \geq 0,$$
$$u(z,0) = 0, \quad z \geq 0,$$
$$u(0,\tau) = \sin(-\omega_0 \tau), \quad \tau \geq 0.$$

solution via Unified transform

$$u(z,\tau) = \frac{1}{2\pi}\int_{\mathscr{C}} F(k;\omega_0) e^{\tau\phi(k;\xi)} \mathrm{d}k$$

$$F(k;\omega_0) = \frac{\omega_0(1-k^2)}{k^2(1-2\omega_0^2) - \omega_0^2(1+k^4)}$$
$$\phi(k;\xi) = ik\xi - i\frac{k}{1+k^2}$$

Vasan, Deconinck *Disc Cont Dyn Sys A* 2013;
Fokas *A unified approach to boundary value problems* SIAM 2008

$\mathscr{C}$: a small contour around $k = i$

Depends on both ω_0 and $\xi = z/\tau$
Essential singularity: $k = i$

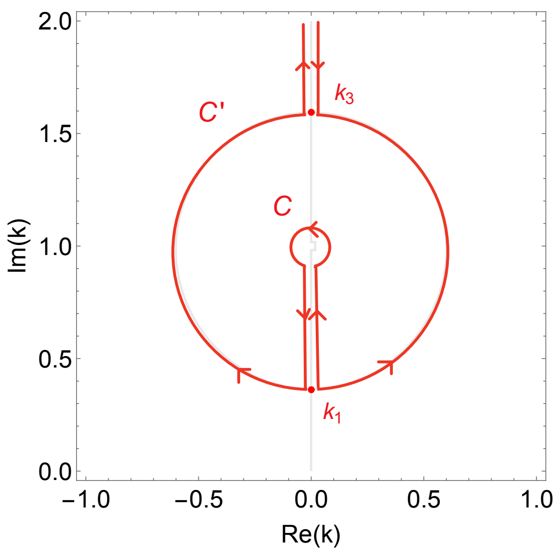
use the method of steepest descent

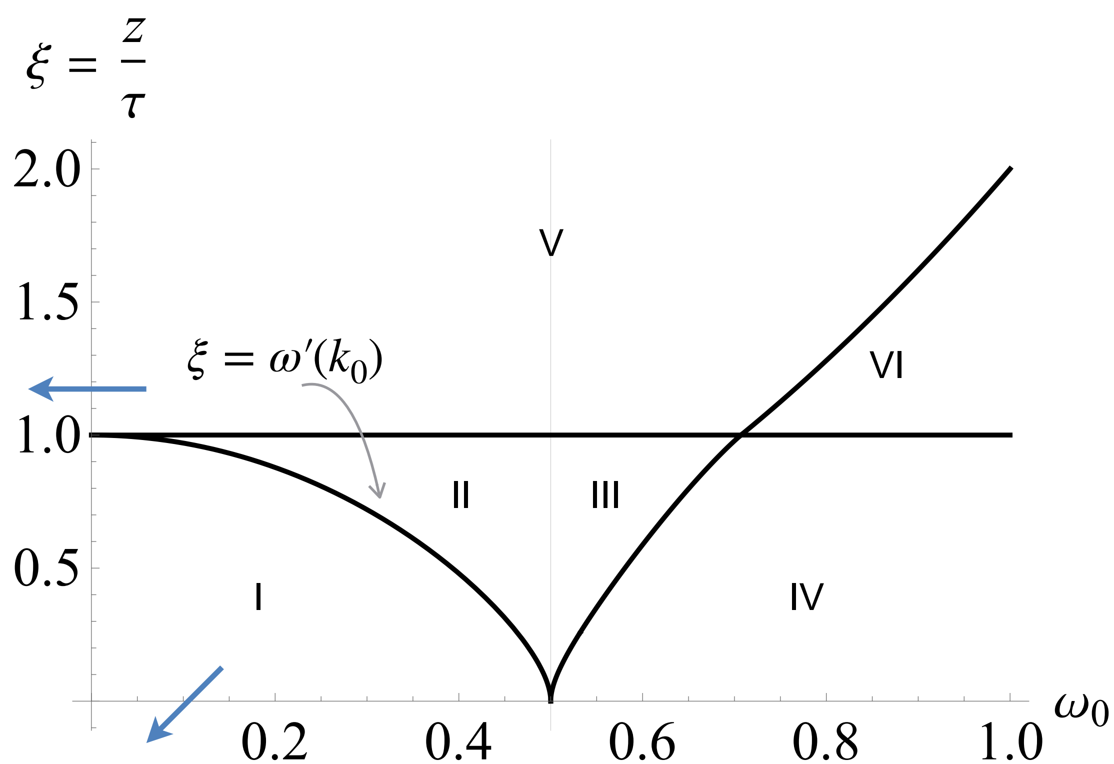
4 poles of F: $\kappa_j = \dfrac{\pm 1 \pm \sqrt{1 - 4\omega_0^2}}{\omega_0}$

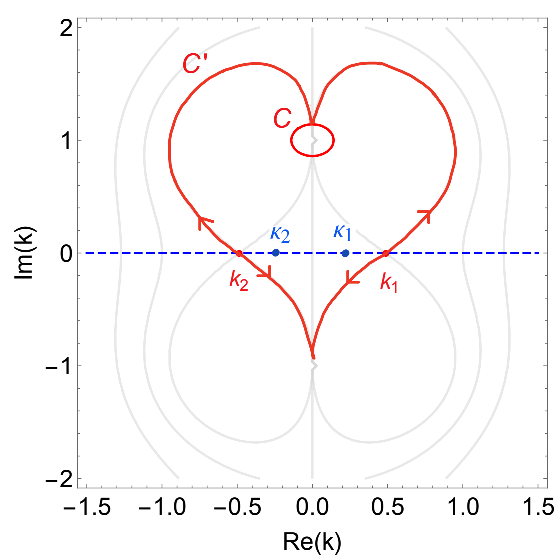
4 saddles $\phi_k = 0$: $k_j = \pm\sqrt{-1 - \dfrac{1 \pm \sqrt{1 + 8\xi}}{2\xi}}$

Contour Deformation

0 poles
2 imaginary saddles

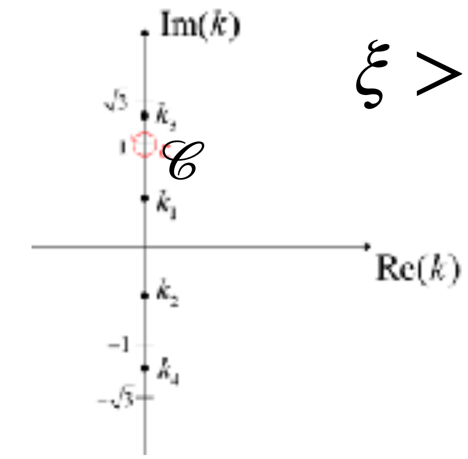




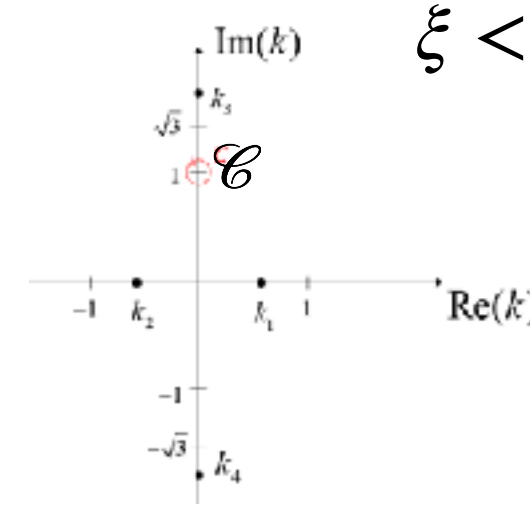


2 real poles
2 real saddles

$\xi > 1$

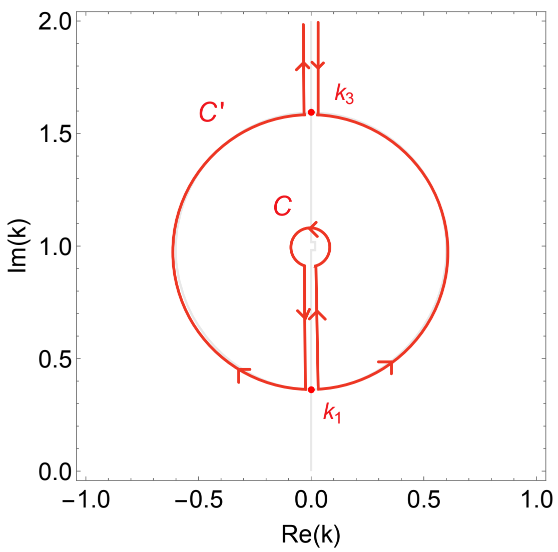


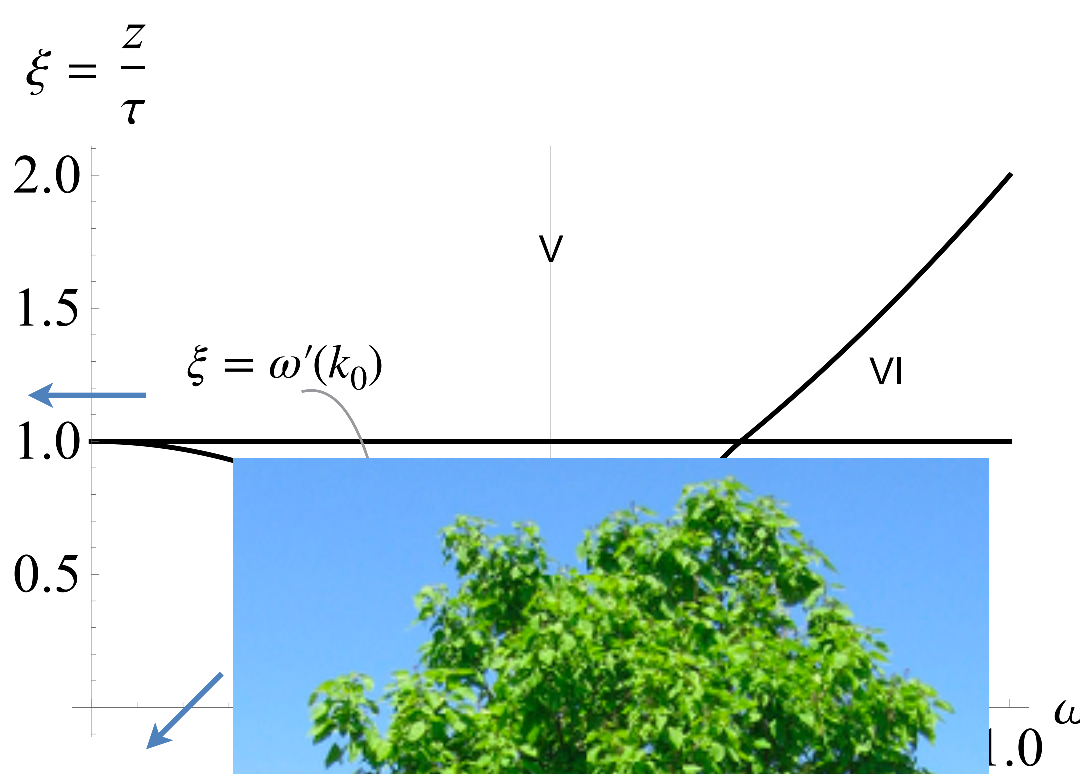
$\xi < 1$



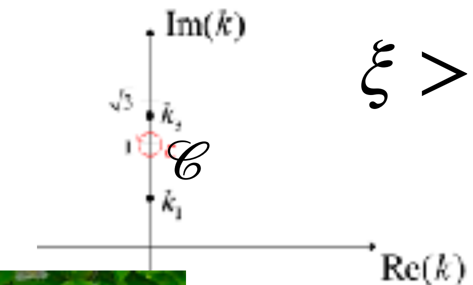
Contour Deformation

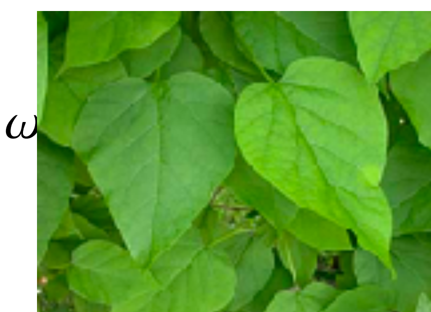
0 poles
2 imaginary saddles



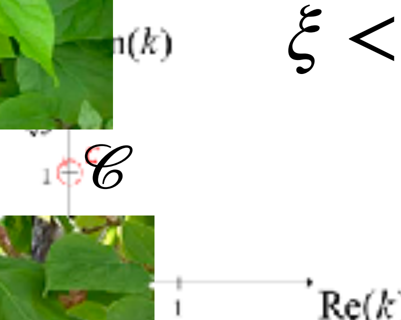


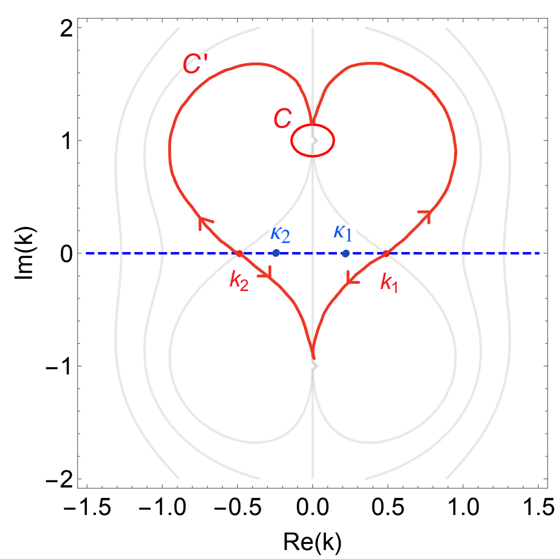
$\xi > 1$





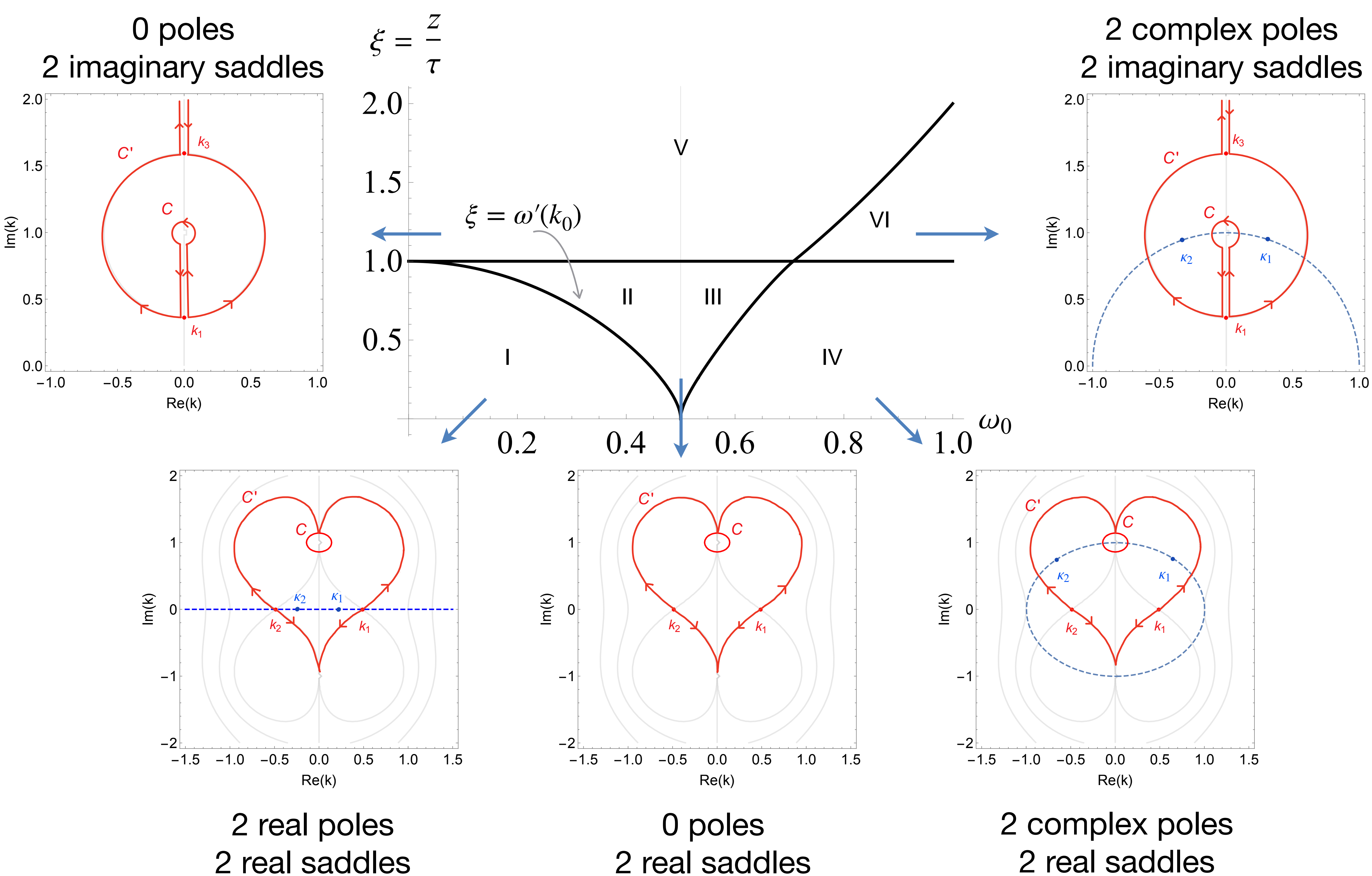
$\xi < 1$



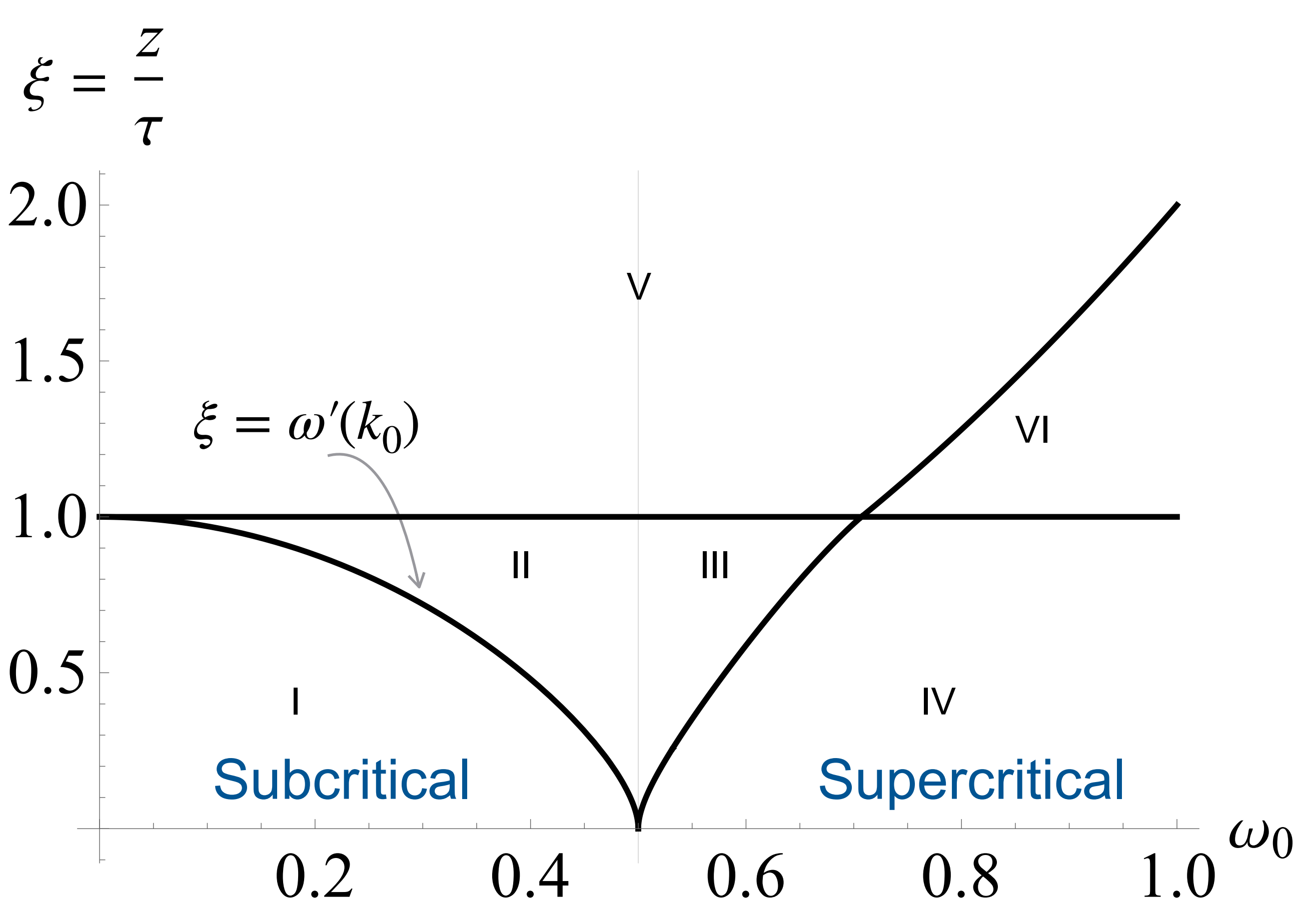


2 real poles
2 real saddles

Contour Deformation



Approximate Solution



Critical frequency: $\omega_c = 1/2$

I Sinusoidal plane wave solution

$$u(z,\tau) = \sin\left[\left(k(\omega_0)\xi - \omega_0\right)\tau\right] + \mathcal{O}(\tau^{-1/2}),$$

$$k(\omega_0) = \frac{1 - \sqrt{1 - 4\omega_0^2}}{2\omega_0}$$

II, III Algebraically decaying oscillatory solution of $\mathcal{O}(\tau^{-1/2})$

IV Damped sinusoidal wave

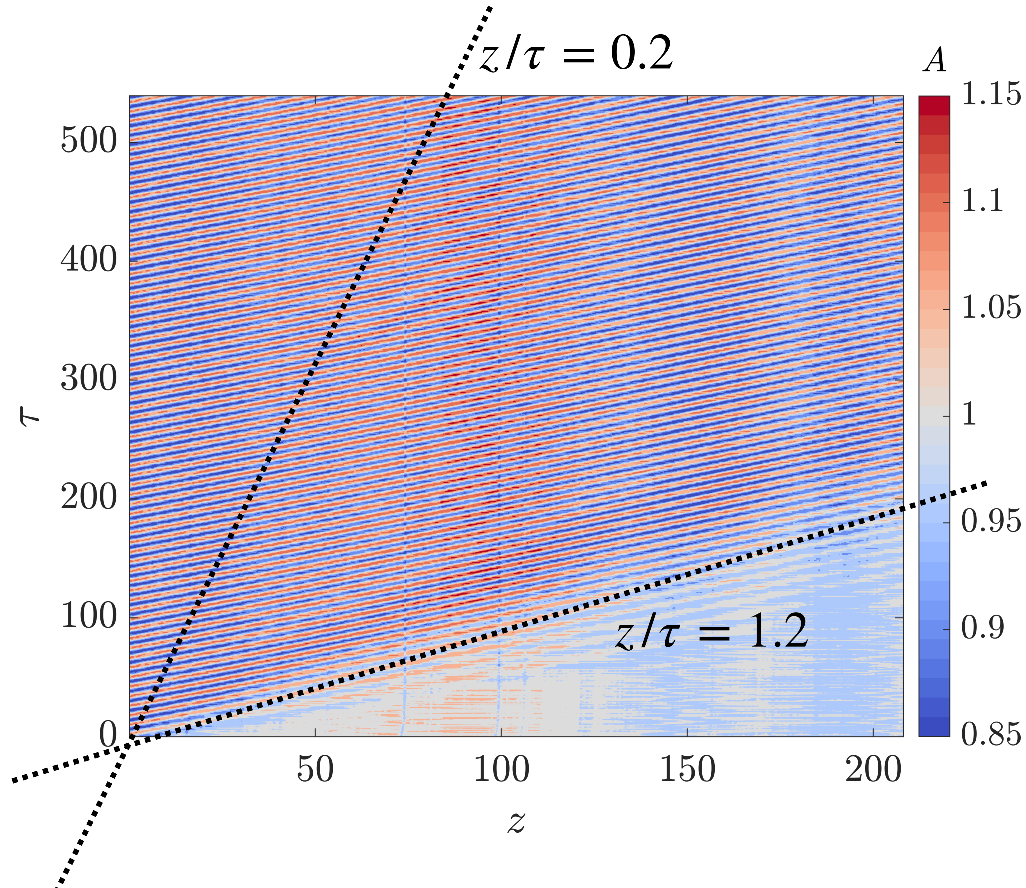
$$u(z,\tau) = \exp\left[-k_{\mathrm{Im}}\xi\tau\right]\sin\left[\left(k_{\mathrm{Re}}\xi - \omega_0\right)\tau\right] + \mathcal{O}(\tau^{-1/2})$$

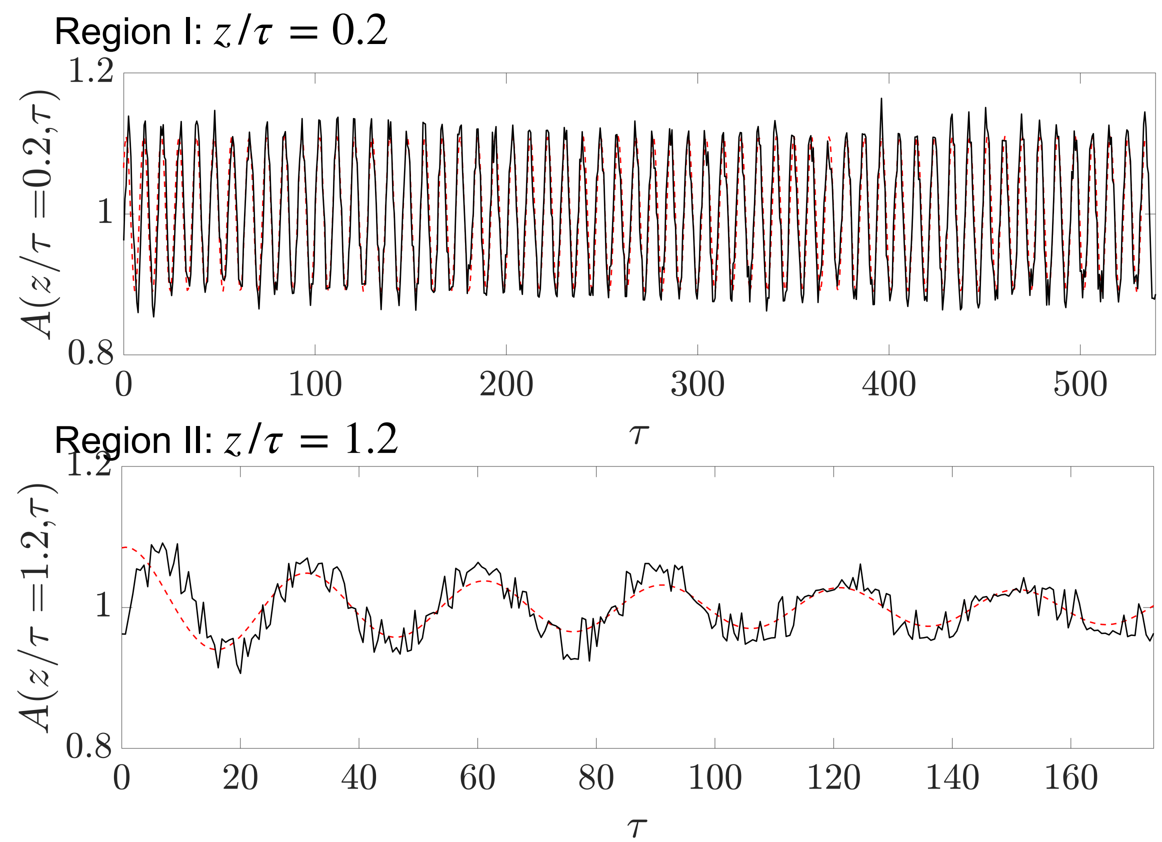
$$k_{\mathrm{Im}}(\omega_0) = \frac{\sqrt{4\omega_0^2 - 1}}{2\omega_0}, \quad k_{\mathrm{Re}}(\omega_0) = \frac{1}{2\omega_0}$$

V Exponentially decaying solution of $\mathcal{O}(\tau^{-1/2}e^{-\tau})$

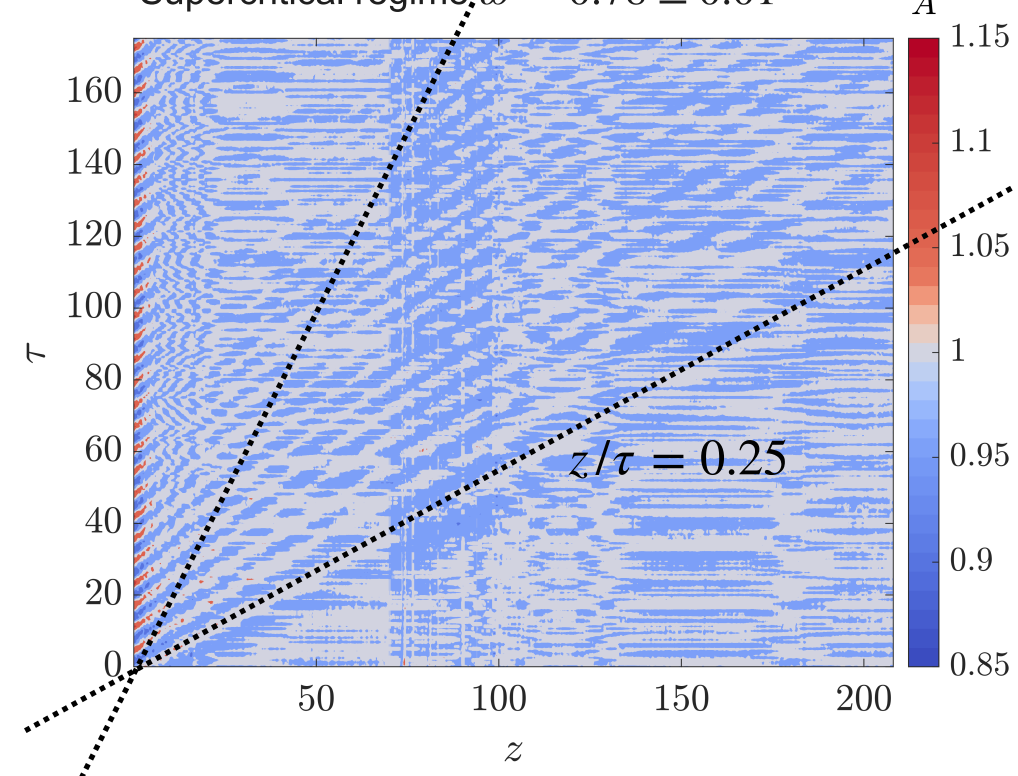
VI Damped sinusoidal wave with a $\mathcal{O}(\tau^{-1/2}e^{-\tau})$ correction

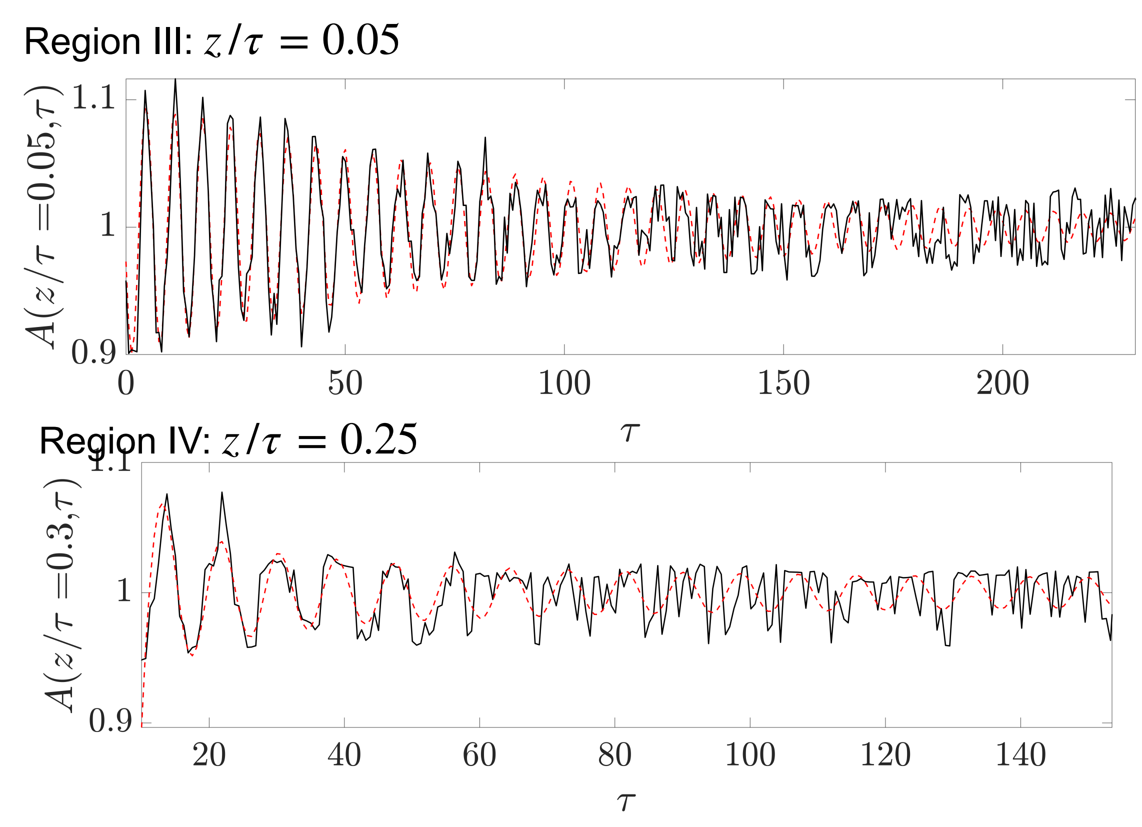
Comparison with Experiment





- Supercritical regime $\omega = 0.78 \pm 0.01$





Generalizations

radiation condition

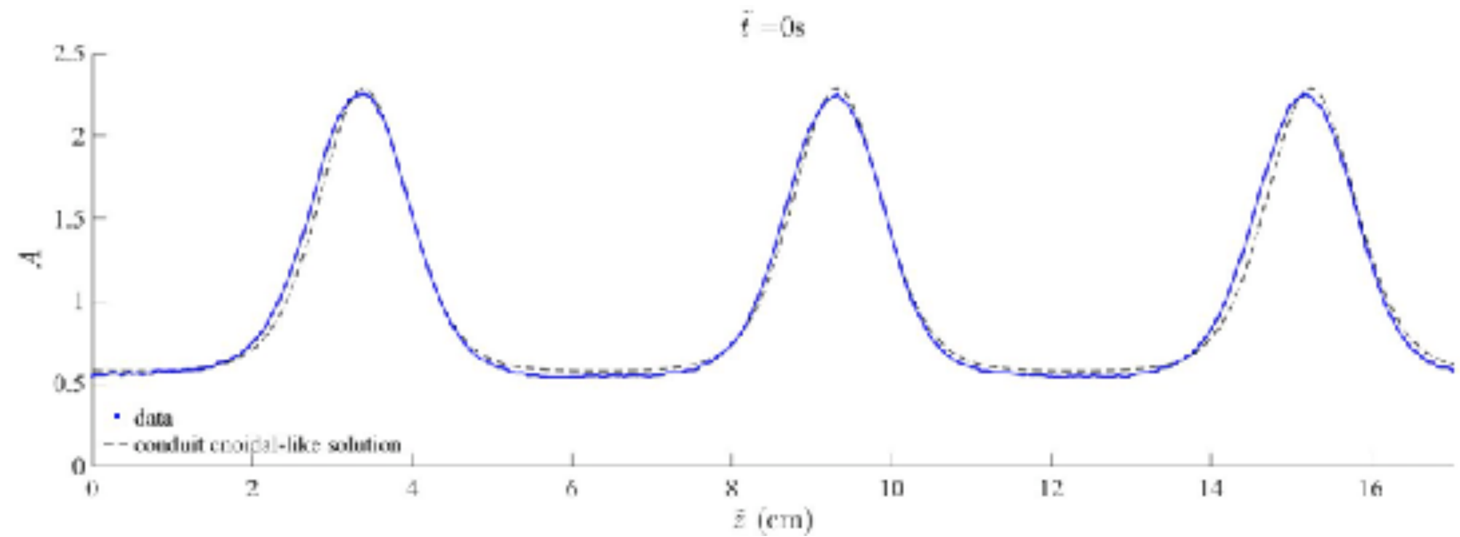
$\Longleftrightarrow$

unique D2N map

[Fokas, van der Weele *SAPM* 2021]

$$\left(1 - A_{-2}\partial_{xx}\right) u_t = iA_0 u + A_1 u_x + iA_2 u_{xx} + A_3 u_{xxx}$$

$$A_2 \geq 0, \quad A_3 \leq 0$$



$$A_t + 2AA_z - AA_{zzt} + A_t A_{zz} = 0, \quad A(0,t) = \phi(-\omega t)$$

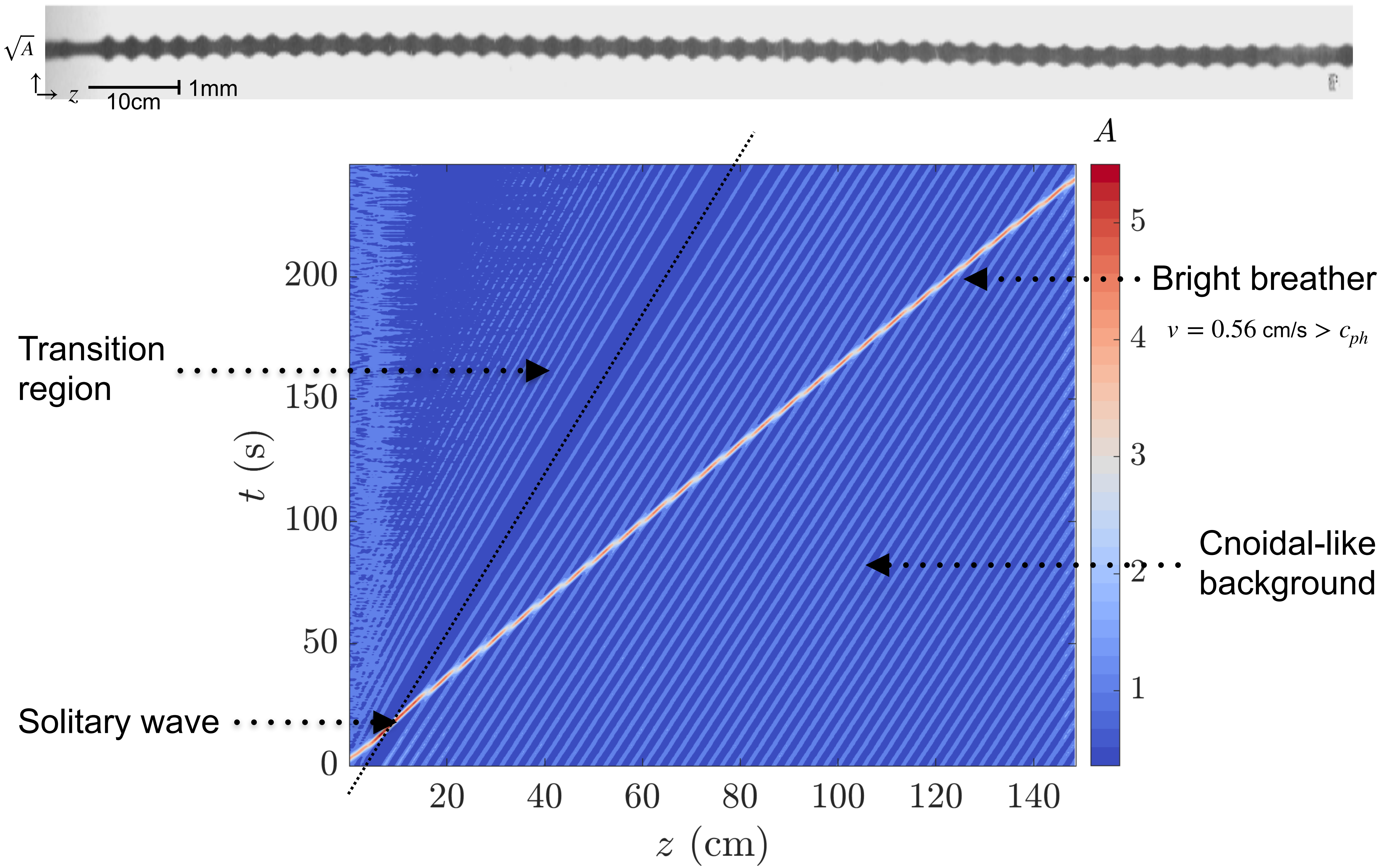
$$A(z,t) = \phi(kz - \omega t) \;\Rightarrow\; \left(\phi'\right)^2 = -\frac{2}{k^2}\phi - \frac{2}{\omega k}\phi^2 \ln\phi + A + B\phi^2$$

$\phi(\theta) =$ periodic traveling wave solution, cnoidal-like waves

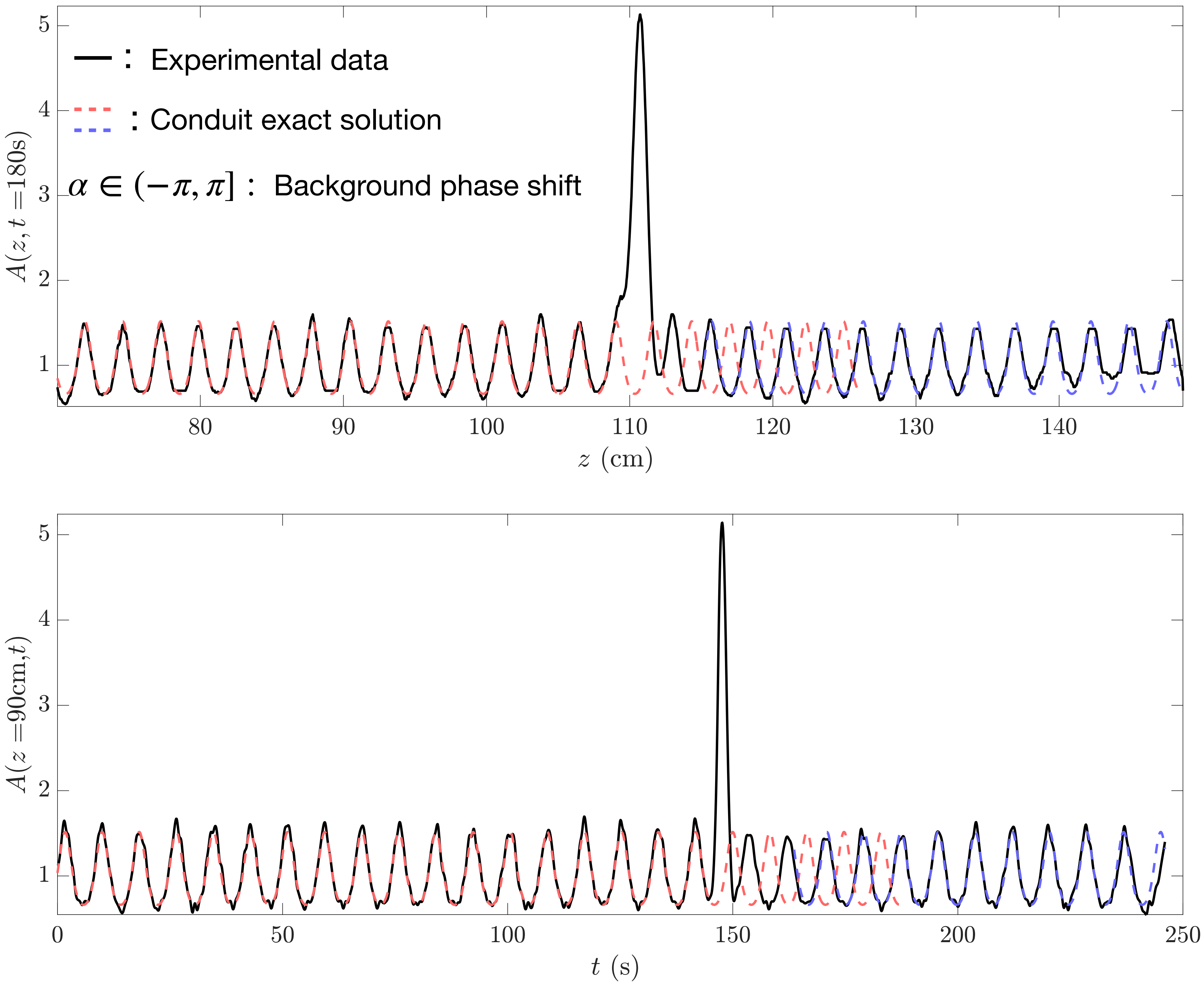
Soliton + Cnoidal Wave = Breather

Mao, Chandramouli, Hoefer *in preparation* 2022

Breather = Solitary Wave + Cnoidal-like Wave



Bright Breather



Cnoidal-like bkgrd:

$\bar{A} = 1$,

$a = 0.85$,

$k = 2.39$ rad/cm,

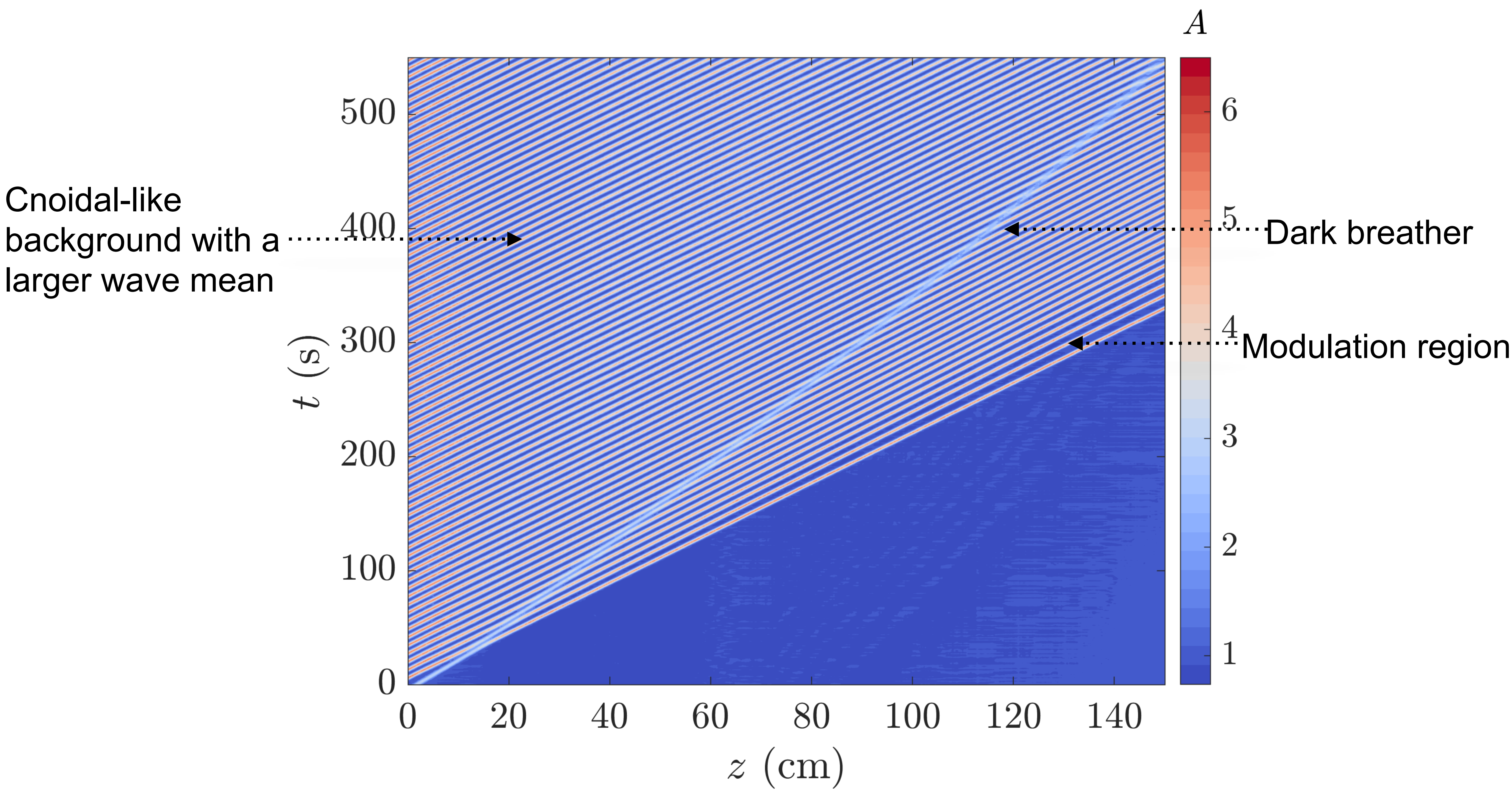
$\omega = 0.80$ rad/s.

Phase shifts:

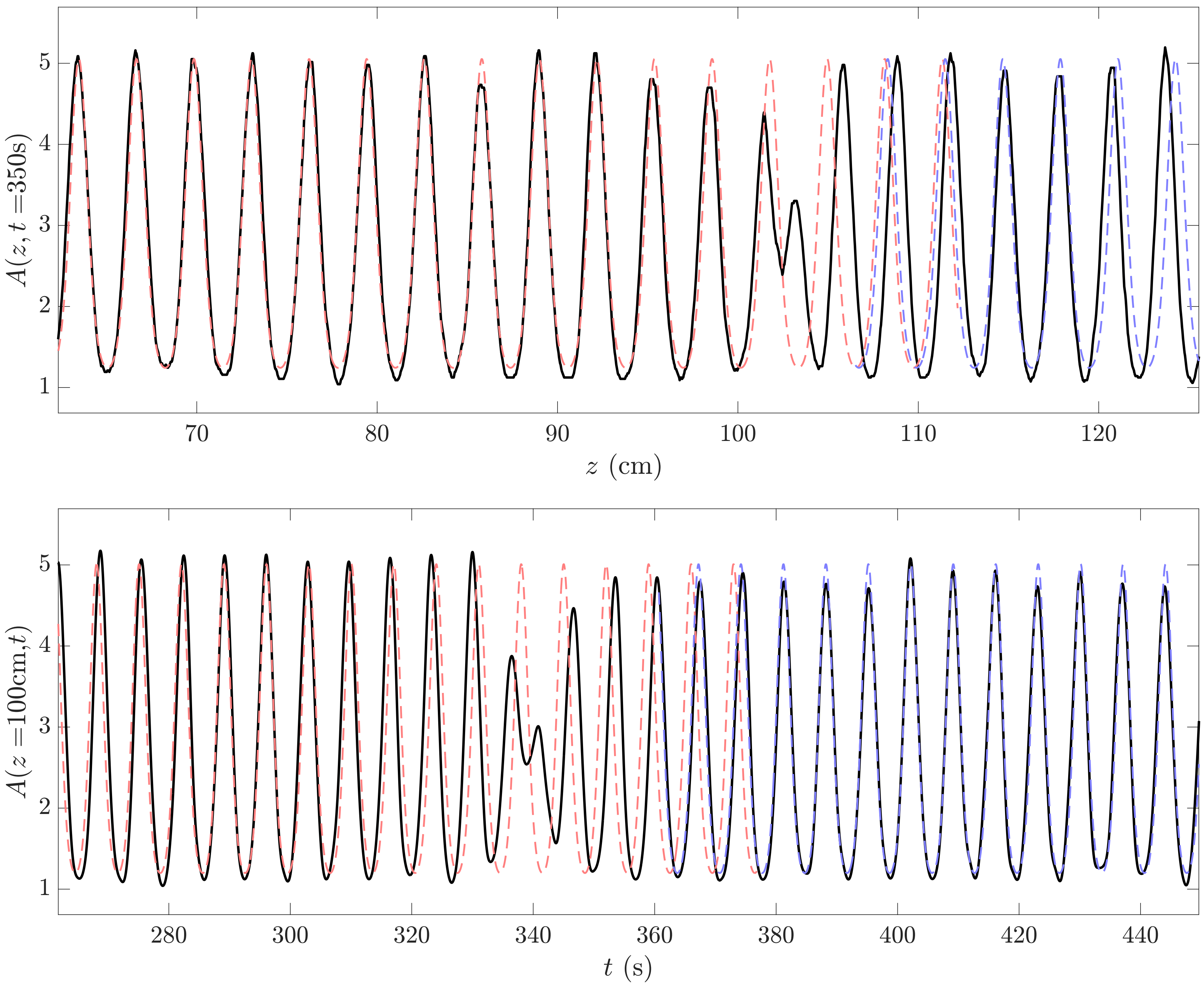
$\alpha_z = 0.95\pi$,

$\alpha_t = 0.93\pi$.

Dark Breather



Dark Breather Phase Shift



Cnoidal background:

$\bar{A} = 2.63,$

$a = 3.81,$

$k = 1.93$ rad/cm,

$\omega = 0.93$ rad/s.

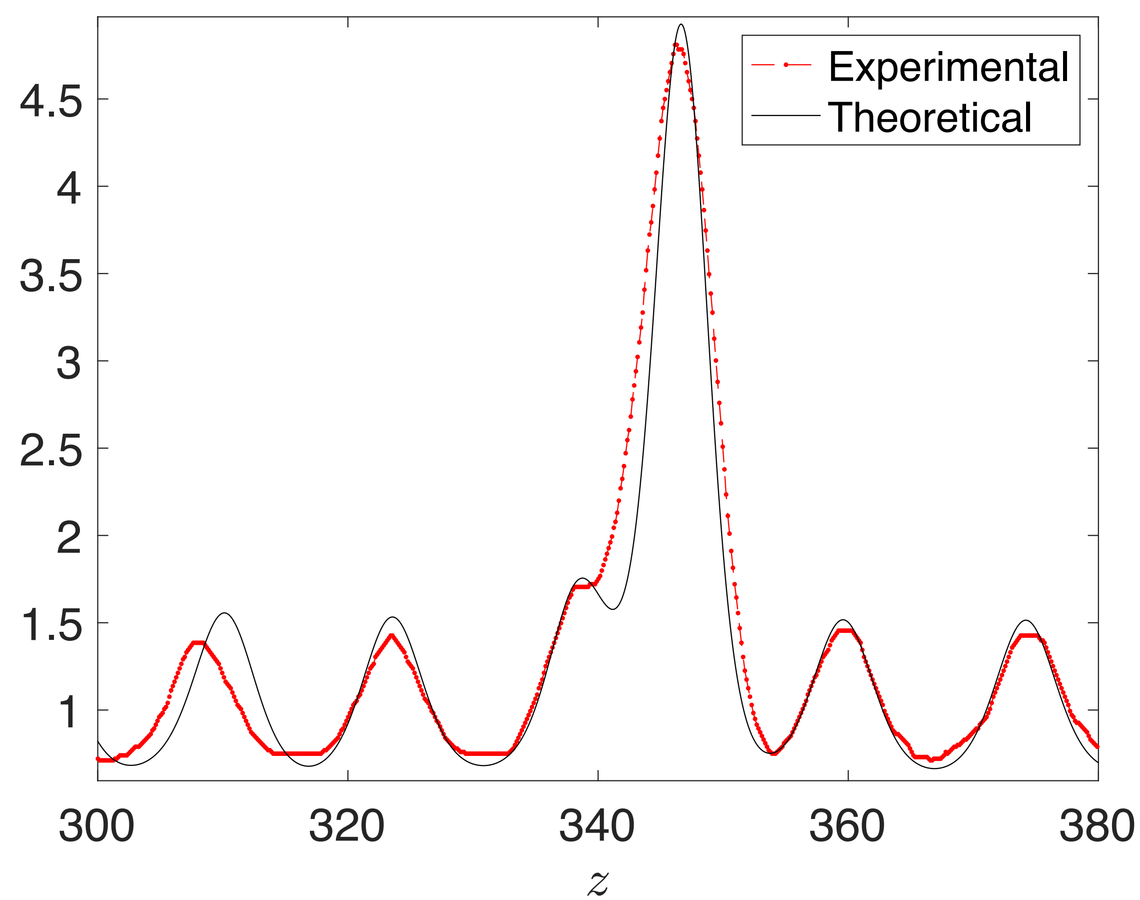
Phase shifts:

$\alpha_z = -0.10\pi,$

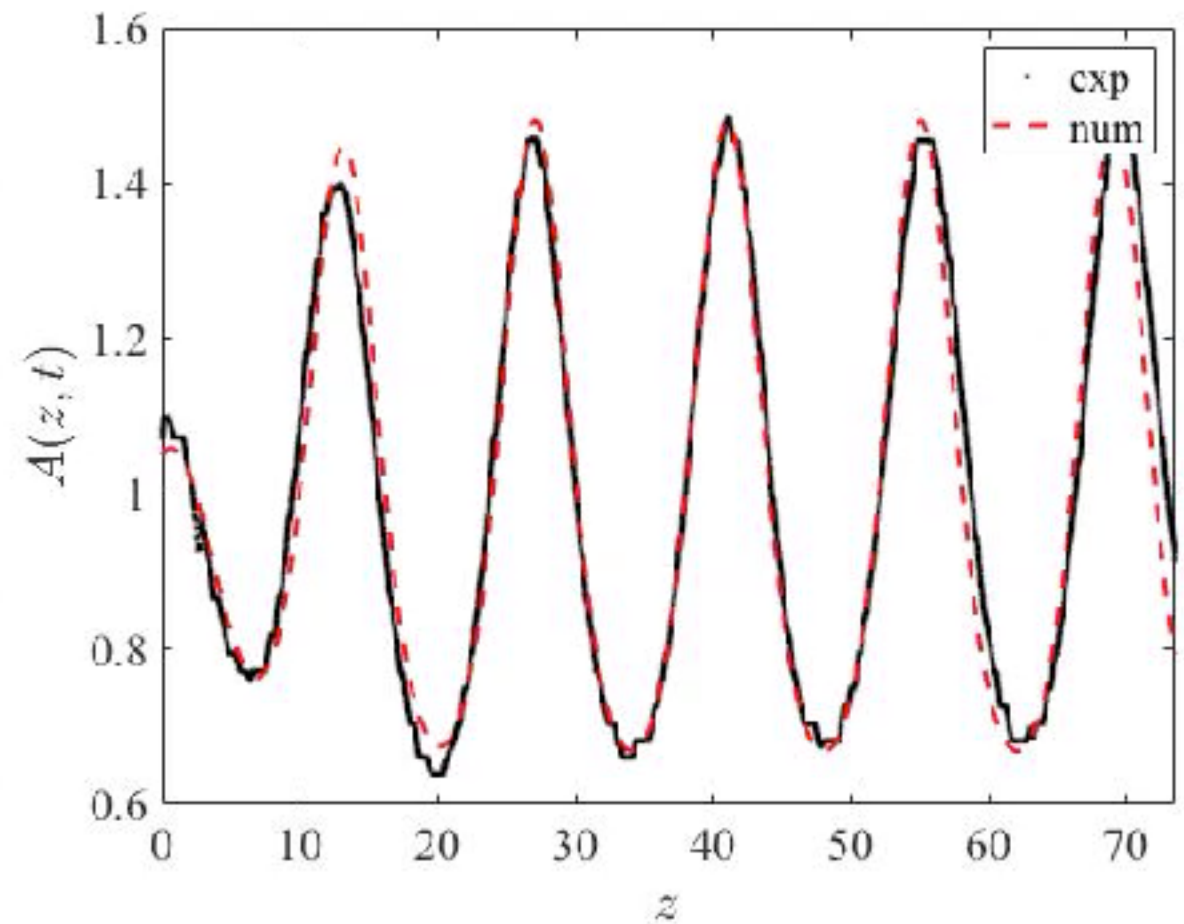
$\alpha_t = -0.35\pi.$

Breather solutions of model equation

$$A_t + 2AA_z - AA_{zzt} + A_tA_{zz} = 0$$



stable bright breather solution



stable dark breather solution

Conclusions

- Viscous core-annular flows: versatile system for dispersive hydrodynamics
- Can test sophisticated mathematics in an accessible system
- Multiscale modeling: microscopic (Navier-Stokes), mesoscopic (conduit eq.), macroscopic (Whitham eqs.), and megascopic (kinetic eq.) scales
- Excellent agreement between theory and experiment

Thank you for your interest