On temperature discontinuities in the supercooled Stefan problem with noise

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18-19th July 2022
PDE CDT Alumni Event
University of Oxford

Freezing of a supercooled liquid

- 1-d model for freezing of a liquid on the positive half-line:
  - $\rightsquigarrow$ At time $t > 0$, $[0, s(t)]$ is frozen and liquid occupies $(s(t), \infty)$
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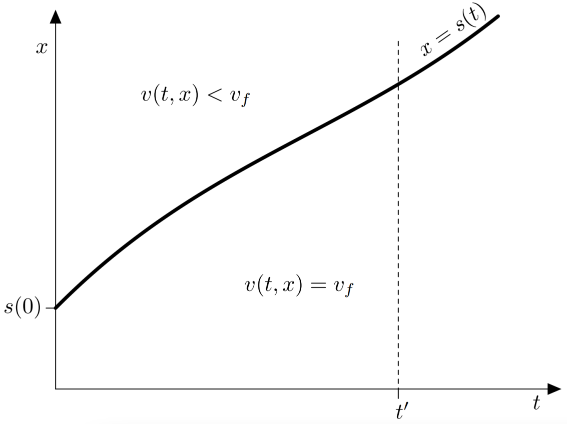
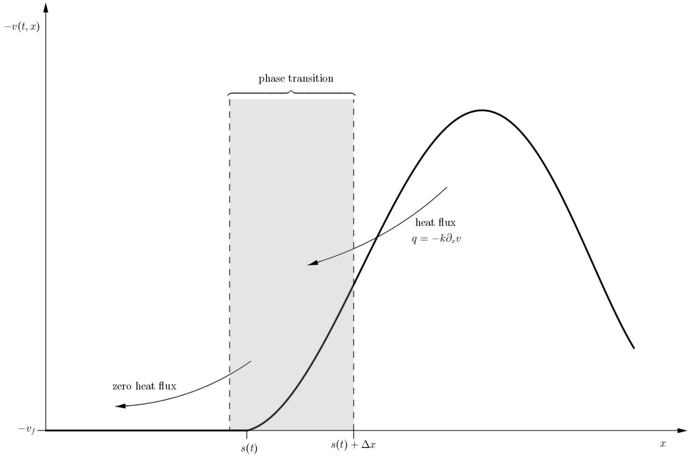
- Isothermal phase change: $v = v_f$ at the interface $(x = s(t))$, where v_f is the equilibrium freezing temperature (take $v_f = 0$)
- No temperature gradient in the frozen phase:
  - $\rightsquigarrow$ Constant equilibrium temperature v_f in $[0, s(t)]$.

The freezing front

- Illustration of the advancing *freezing front* s in the (t, x)-plane



The Stefan condition at the interface



- Flow of energy (heat flux) in the liquid is $q = -k\partial_x v(t,x)$
- At phase transition: *zero energy* escapes into the frozen phase

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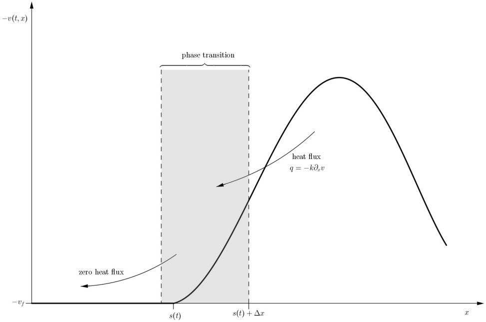

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- Change Δx in freezing front $s(t)$ over Δt amount of time

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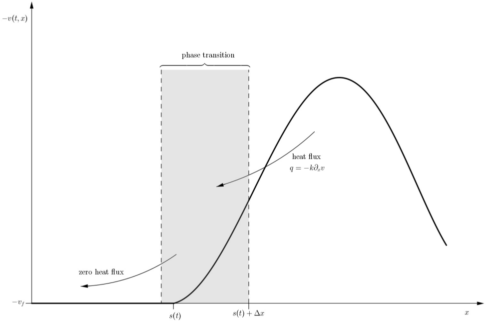

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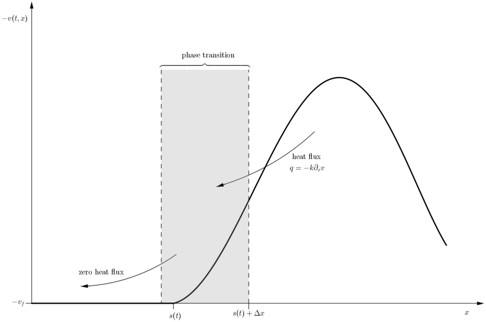

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  - ⇝ Heat flux of $q\Delta t = -k\partial_x v\Delta t$ and energy release of $\rho\ell\Delta x$
  - ⇝ Fourier's law enforces energy balance: $0 - k\partial_x v\Delta t = \rho\ell\Delta x$

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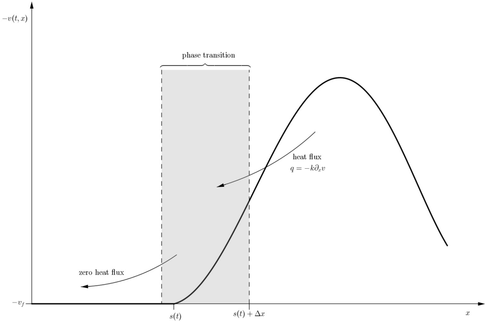

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  - $\rightsquigarrow$ Fourier's law enforces energy balance: $0 - k\partial_x v\Delta t = \rho\ell\Delta x$
  - $\rightsquigarrow$ Conclude that $\partial_x v(t, s(t)+) = -\lambda\dot{s}(t)$ with $\lambda = \frac{\rho\ell}{k}$

The supercooled Stefan problem

Free boundary problem

$$\begin{cases} \partial_t v(t,x) = \kappa \partial_{xx} v(t,x) \ \text{ for } \ x \in (s(t), \infty) \\ v(t,x) = v_f \ \text{ for } \ x \in [0, s(t)) \\ v(t,s(t)) = v_f, \quad \partial_x v(t,s(t)) = -\lambda \dot{s}(t) \end{cases}$$

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  - ⇝ Internal energy change of ΔQ from temperature change $\Delta\bar{v}$

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- Agrees with what can be obtained from integration by parts:
  - ⇝ $\lambda(s(t) - s(r)) = \frac{1}{\kappa}\left(\int_0^\infty v(t,x)dx - \int_0^\infty v(r,x)dx\right)$

Supercooled Stefan problem with noise

Weak formulation of the problem

$$\begin{cases} \int_0^\infty v(t,x)\phi(x-s(t))\mathrm{d}x - \int_0^\infty v_0(x)\phi(x-s(0))\mathrm{d}x = \\ \int_0^\infty \kappa v(r,x)\partial_{xx}\phi(x-s(r))\mathrm{d}x\mathrm{d}r - \int_0^t \int_0^\infty v(r,x)\partial_x\phi(x-s(r))\mathrm{d}s(r) \\ \quad - \int_0^t \int_0^\infty b v(r,x)\partial_x\phi(x-s(r))\mathrm{d}x\mathrm{d}W_r \end{cases}$$

$$\text{and} \quad \lambda(s(t)-s(0)) = \frac{1}{\kappa}\Big(\int_0^\infty v(t,x)\mathrm{d}x - \int_0^\infty v_0(x)\mathrm{d}x\Big)$$

- Smooth test functions $\phi \in C^\infty(\mathbb{R})$ with $\phi(0) = 0$
- Initial condition $v_0 \in L^\infty \cap L^1$ supported in $(s(0), \infty)$
- Solution $v(t,\cdot) \in L^\infty \cap L^1$ $\forall t \geq 0$ with $t \mapsto v(t,\cdot)$ càdlàg

Allowing a temperature discontinuity: $v(t-,x) \curvearrowright v_f$

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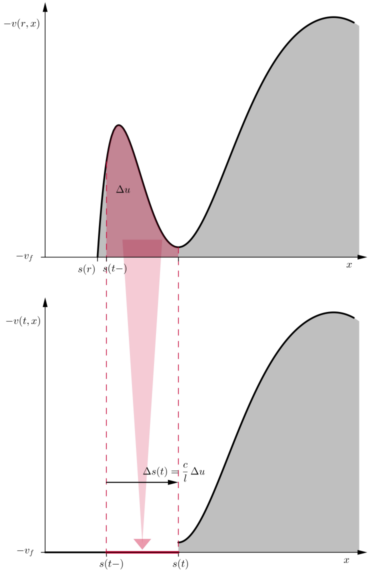
  - $\rightsquigarrow$ *Freezing front:* jumps from $s(t-)$ to $s(t) := s(t-) + \Delta s$
  - $\rightsquigarrow$ *Away from the interface:* at all positions $x \in (s(t), \infty)$, the temperature remains unchanged $v(t, x) = v(t-, x)$

Temperature discontinuity: $v(t-, x) \curvearrowright v_f$



Allowing the temperature to jump: $v(t,x) \curvearrowright v_f$

- As a mathematical abstraction, one could think of *instantly freezing* a portion of the liquid
- Allow temperatures to *jump instantly* to the *freezing point* v_f *close to the interface* (at some time t):
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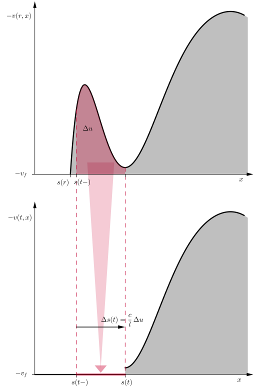
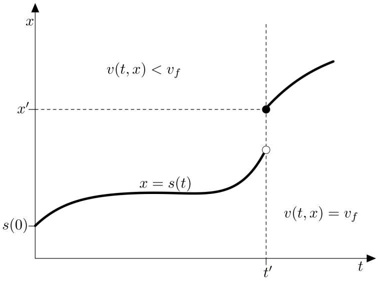
- Which corresponds to an *instant change of heat* of size

$$\Delta\bar{v} = -\int_{s(t-)}^{s(t-)+\frac{c}{\ell}\Delta\bar{v}} v(t-,x)dx$$

Allowing the temperature to jump: $v(t,x) \curvearrowright v_f$

- Illustration of the freezing front with a discontinuity
- Recall we need $\Delta s(t) = \frac{1}{\kappa}\Delta\bar{v} = -\int_{s(t-)}^{s(t-)+\frac{c}{\ell}\Delta\bar{v}} v(t-,x)dx$





A possible story for temperature discontinuity

- *Exogenous shock:* temperatures jump to v_f ($v(t-,x) \curvearrowright v_f$) in a small Δx-neighbourhood of $s(t-)$ with freezing front moving to $s(t-) + \Delta x$

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  - $\rightsquigarrow$ Let this cause a further movement of the freezing front by $\frac{\Delta Q}{\rho \ell} = \frac{c\rho \Delta \bar{v}^{(0)}}{\rho \ell} = \frac{c}{\ell} \Delta \bar{v}^{(0)}$

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  - $\rightsquigarrow$ Further heat change $\Delta\bar{v}^{(1)} := -\int_{s(t-)+\Delta x}^{s(t-)+\Delta x+\frac{c}{\ell}\Delta\bar{v}^{(0)}} v(t-,y)\mathrm{d}y$

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$$\sum_{n=0}^{\infty} \Delta \bar{v}^{(n)} = \int_{s(t-)}^{s(t-)+\Delta x + \frac{c}{\ell}\sum_{n=0}^{\infty} \Delta \bar{v}^{(n)}} v(t-,y)\mathrm{d}y$$

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- This way, we indeed end up with energy balance

$$\Delta \bar{v} = \int_{s(t-)}^{s(t-)+\frac{c}{\ell}\Delta \bar{v}} v(t-,y)\mathrm{d}y$$

Probabilistic representation

- Recall $v(t,x)$ lives on $(t,x) \in [0,\infty) \times [s(t),\infty)$
- Can be represented as $-v(t,x)\mathrm{d}x = \mathbb{P}(X_t \in \mathrm{d}x, t < \tau \mid W)$

Conditional McKean–Vlasov problem

$$\begin{cases} X_t = X_0 + 2\sqrt{\kappa}\beta B_t + bW_t \\ \tau = \inf\{t \geq 0 : X_t \leq s(t)\} \\ s(t) = s(0) + \frac{1}{\lambda\kappa}\mathbb{P}(\tau \leq t \mid W) \end{cases}$$

- Here $\beta = \sqrt{1 - (\frac{b}{2\sqrt{\kappa}})^2}$ and recall also that $\lambda\kappa = \frac{\ell}{c}$
- As before, the freezing front $t \mapsto s(t)$ must satisfy

$$\lambda(s(t) - s(0)) = \frac{1}{\kappa}\Big(\int_0^\infty v(t,x)\mathrm{d}x - \int_0^\infty v_0(x)\mathrm{d}x\Big)$$

Continuity in time versus heat discontinuities

- For *any* given *weak solution* v, consider the **first time** the **temperature** undergoes a **discontinuity** in time:

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Theorem: global continuity

If $v_0(x) < \lambda\kappa = \frac{\ell}{c}$ for all $x \in (s(0), \infty)$, then

$$\mathbb{P}(\tau < \infty) = 0.$$

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- Show that, on this event, we also have

$$\nu_\varsigma(I) \geq \nu_0(c+I) - \varepsilon \quad \text{and} \quad \int_I x\nu_\varsigma(dx) \leq \int_{c+I}(x-c)\nu_0(dx) + \varepsilon$$

Thank you!