Quantitative Hydrodynamic Limits of Stochastic Lattice Systems

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The question at hand (1)

- Old question in mathematical physics ("IPS"): continuum limit of interacting particle systems on lattice, typically regular periodic lattice $\mathbb{T}_N^d$.
- At the physical level, it corresponds to searching for a description at macroscopic scale, i.e. fluid mechanics, of the microscopic dynamics at particle level.
- Hence called hydrodynamic limit.
- At the mathematical level, it corresponds to proving the propagation in time of a **local** law of large number with a local profile solving an effective limit PDE, under an appropriate scaling of time and space.
- This means proving that a "local density" of particles converges, in an appropriate sense, towards a deterministic profile solution to a PDE.

The question at hand (2)

Many microscopic dynamics are possible, some important paradigmatic examples that we shall consider are:

- The simple exclusion process (SEP):

  Introduced by Spitzer'1970 as a simple stochastic model for transport phenomena.

  Each particle waits a random exponent mean one amount of time and then attempts a jump on neighbouring sites with given probability for each directions; however, the jump is performed only if there is no particle at the target site.

- The zero range process (ZRP):

  Also introduced by Spitzer'1970 as well (as far as I know).

  Similar to SEP but now (a) the number of particles is unrestricted at each site and (b) the jump rate depends (only) on the local particle number.

The question at hand (3)

- The Ginzburg-Landau model with Kawasaki dynamics (GLK):
  Origin Glauber'1963 extending Ising'1925 to include time evolution (non-equilibrium), and Kawasaki'1965.
  Spin lattice model: real-valued spins on each site of the lattice evolve under a single-site potential that induces a gradient flow between neighbouring sites combined with a diffusion equalizing spins between neighbouring sites.
- In dimension $d = 1$, the stochastic trajectories follow

$$\mathrm{d}x_i = \frac{N^2}{2}\left[V'(x_{i+1}) + V'(x_{i-1}) - 2V'(x_i)\right]\,\mathrm{d}t - N\left[\mathrm{d}B_{i,i-1} + \mathrm{d}B_{i,i+1}\right].$$

- Kawasaki dynamics: potential $V = V_0 + V_1$ with V_0 uniformly strictly convex and V_1 local smooth perturbation.
- Ref books: Liggett'1985, Spohn'1991, Kipnis-Landim'1999.

Existing results and questions (1)

- For the SEP and ZRP two scalings are possible:
  - hyperbolic scaling which is non-trivial when the **mean transition rate** is non-zero,
  - parabolic (diffusive) scaling when the latter is zero.
- For the GLK the parabolic scaling is built in the model.
- The hydrodynamic limit is known at a qualitative level for all these models/scalings:
  - Spitzer'1970 (SEP, ZRP, hyperbolic scaling),
  - Guo-Papanicolaou-Varadhan'1988 (ZRP, parabolic scaling),
  - Yau'1991 (ZRP, GLK, parabolic scaling, convergence in relative entropy),
  - Rezakhanlou'1991 (SEP, ZRP, hyperbolic scaling, including shocks),
  - and many variants in more recent papers. . .

Existing results and questions (2)

- However (1) none of the previous results were quantitative, (2) the error in the hydrodynamic limit is not controlled for large time in the parabolic scaling, (3) the methods used are beautiful but intricate and must be rebuilt for each model: in particular they all rely on so-called "block estimates" that take several dozens of pages in Kipnis-Landim'1999.
- An attempt at exploiting logarithmic Sobolev inequality to provide quantitative rates and more unified methods was initiated in GOVW'2009, but latter paper was still not quantitative. It was finally completely in the particular case of the Ginzburg-Landau process with Kawasaki dynamics in dimension 1, in the intricate two-parts work DMOW'2018a and DMOW'2018b.

The contribution

- We propose in this paper an abstract (but simple) method for proving such limit quantitatively, and uniformly in time when the scaling is diffusive. We first present the abstract method and then sketch applications to the three models above.
- Our results include the previous results above, including the rate in the particular case treated in DMOW'2018a, through a conceptually unified method.
- It proves new quantitative errors for SEP, ZRP, as well as GLK in dimension $d \geq 2$, and show that they are uniform in time when the scaling is parabolic.
- The method does not need logarithmic Sobolev inequalities or the block estimates and is self-contained. The "local averaging estimates" are "pushed" onto the local Gibbs measure.

The abstract method: setting (1)

- X is the state space at a given site (number of particles, spin, etc.), which is here $\{0,1\}$ (SEP), $\mathbb{N}$ (ZRP) or $\mathbb{R}$ (GLK).
- $\mathbb{G}_N$ is a graph, in practice a lattice graph, and here the regular d-dimensional periodic lattice graph (discrete torus) $\mathbb{G}_N = \mathbb{T}_N^d$.
- The phase space of particle configurations on the graph is then $\mathsf{X}_N := \mathsf{X}^{\mathbb{G}_N}$.
- We assume that the graph $\mathbb{G}_N$ is a discrete approximation of a manifold $\mathbb{G}_\infty$; here $\mathbb{G}_\infty = \mathbb{T}^d$ is the flat torus with unit length.
- Variables in $\mathbb{G}_N$ are called microscopic and denoted by x, y, z, whereas variables in $\mathbb{G}_\infty$ are called macroscopic and denoted by u; particle configurations in X_N are denoted by η.

The abstract method: setting (2)

- The interacting particle system evolves through a stochastic process and the time-dependent probability measure describing the law of η is denoted by $\mu_t^N \in P(\mathsf{X}_N)$.
- We embed $\mathbb{G}_N = \mathbb{T}_N^d$ into $\mathbb{G}_\infty = \mathbb{T}^d$ via $\mathbb{T}_N^d \to \mathbb{T}^d$, $x \mapsto \frac{x}{N}$. So the macroscopic distance between sites of the lattice is $\frac{1}{N}$.
- Given a particle configuration $\eta \in \mathsf{X}_N$, the empirical measure is

$$\alpha_\eta^N := \sum\!\!\!\!\!\!-_{x \in \mathbb{G}_N} \eta(x) \delta_{\frac{x}{N}} \in \mathcal{M}_+(\mathbb{T}^d)$$

where $\mathcal{M}_+(\mathbb{T}^d)$ is the space of non-negative Radon measures on the torus and $\sum\!\!\!\!\!\!-$ denotes the "average sum", i.e. the sum divided by the cardinal of the set it is summed over.

The abstract method: setting (3)

- We then consider evolution systems. At the microscopic level, we consider a linear operator $\mathcal{L}_N : C_b(\mathrm{X}_N) \to C_b(\mathrm{X}_N)$ generating uniquely a Feller semigroup on $P(\mathrm{X}_N)$ so that given $\mu_0^N \in P(\mathrm{X}_N)$ the solution $\mu_t^N \in P(\mathrm{X}_N)$ satisfies

$$\forall\, \Phi \in C_b(\mathrm{X}_N), \quad \frac{\mathrm{d}}{\mathrm{d}t}\langle \mu_t^N, \Phi\rangle = \langle \mu_t^N, \mathcal{L}_N \Phi\rangle,$$

  where $\langle \cdot, \cdot \rangle$ is the standard duality bracket.
- At the macroscopic level, we consider a (possibly nonlinear) unbounded operator $\mathcal{L}_\infty : \mathcal{M}_+(\mathbb{G}_\infty) \to \mathcal{M}_+(\mathbb{G}_\infty)$ and

$$\partial_t f_t = \mathcal{L}_\infty f_t, \quad f_{t=0} = f_0.$$

- Goal: Prove that the empirical measure sampled from the law μ_t^N approximates f_t if so at $t = 0$ (with a rate!)

$$\forall\, \phi \in C_b(\mathbb{G}_\infty),\ \forall\, \epsilon > 0,\ \forall\, t \geq 0, \quad \lim_{N\to\infty} \mu_t^N \left(\left\{ \left| \langle \alpha_\eta^N, \phi\rangle - \langle f_t, \phi\rangle \right| > \epsilon \right\} \right) = 0$$

The abstract method: setting (4)

- A measure $\mu^N \in P(\mathsf{X}_N)$ is called invariant if

$$\forall \Phi \in C_b(\mathsf{X}_N), \quad \langle \mu^N, \mathcal{L}_N \Phi \rangle = 0.$$

- A local Gibbs measure on X_N associated with some $F : \mathbb{G}_N \to \mathbb{R}$ is a product measure

$$\nu_F^N(\eta) = \prod_{x \in \mathbb{G}_N} n_{F(x)}(\eta(x)) \tag{1}$$

for some $n_\lambda \in P(\mathsf{X})$ depending on one parameter only $\lambda \in \mathbb{R}$, so that ν_F^N is invariant when F constant.

- We assume that there is a function $\Theta : \mathbb{R}_+ \to \mathbb{R}$ so that

$$\mathbb{E}_{n_{\Theta(\rho)}}[\eta(x)] = \rho$$

and so that, given a smooth macroscopic profile f on $\mathbb{G}_\infty$,

$$\vartheta_f^N(\eta) := \nu^N_{\Theta(f(\frac{\cdot}{N}))}(\eta)$$

approximates the macroscopic profile f.

The abstract method: setting (5)

- The two maps $\eta \mapsto \alpha_\eta^N$ and $f \mapsto \vartheta_f^N$ allow comparisons between the microscopic and macroscopic scales:

$$
\begin{tikzcd}
P(\mathsf{X}_N) \arrow[rr, "\alpha_\eta^N"] & & P(\mathcal{M}_+(\mathbb{G}_\infty)) \arrow[dl, "\delta_f\ \text{(atomic)}"] \\
& \mathcal{M}_+(\mathbb{G}_\infty) \arrow[ul, "\vartheta_f^N"] &
\end{tikzcd}
$$

Figure: The functional setting.

The abstract method: assumptions

(H1) Microscopic stability. There is a coupling operator $\widetilde{\mathcal{L}}_N : C_b(\mathsf{X}_N^2) \to C_b(\mathsf{X}_N^2)$ and $W \in C_b(\mathsf{X}_N^2)$ s.t.

$$\widetilde{\mathcal{L}}_N\,(\cdot \otimes 1) = \mathcal{L}_N(\cdot) \otimes 1, \; \widetilde{\mathcal{L}}_N\,(1 \otimes \cdot) = 1 \otimes \mathcal{L}_N(\cdot),$$

$$\widetilde{\mathcal{L}}_N W \leq 0 \quad \text{and} \quad W(\eta,\zeta) \gtrsim \sum_{x \in \mathbb{G}_N} |\eta(x) - \zeta(x)|.$$

(H2) Macroscopic stability. There is a Banach space $\mathfrak{B} \subset \mathcal{M}_+(\mathbb{G}_\infty)$ so that if $f_0 \in \mathfrak{B}$ then there is $T \in (0,+\infty]$ and a unique macroscopic solution in $\mathfrak{B}$ with $\sup_{t\in[0,T)} \|f_t\|_{\mathfrak{B}} \lesssim \|f_0\|_{\mathfrak{B}}$. Moreover when the limit PDE has a unique constant stationary solution $f_\infty \in \mathfrak{B}$ we denote $R(t) := \|f_t - f_\infty\|_{\mathfrak{B}}$ for $t \in [0,T)$.

(H3) Consistency. Let $W_0(\zeta) := \langle \nu_\infty^N, W(\cdot,\zeta)\rangle$ (average of W in the first variable with respect to the invariant measure ν_∞^N). There is a consistency error $\mathcal{E}_1(N) > 0$ (with loss of norm):

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \vartheta_{f_t}^N, W_0 \right\rangle - \left\langle \vartheta_{f_t}^N, \mathcal{L}_N W_0 \right\rangle \leq \mathcal{E}_1(N)\, \|f_t\|_{\mathfrak{B}}\,.$$

The abstract result

Theorem. Let $\phi \in C_c^\infty(\mathbb{G}_\infty)$, $\mu_0^N \in P(\mathsf{X}_N)$, $f_0 \in \mathfrak{B}$, $\pi_0^N \in P(\mathsf{X}_N^2)$ a coupling between μ_0^N and $\nu_{f_0}^N$, and let us denote the initial sampling and coupling error

$$\mathcal{E}_2(N) := \left\langle \mu_0^N, \left| \left(\alpha_\eta^N - f_0, \phi \right) \right| \right\rangle + \left\langle \pi_0^N, W \right\rangle$$

where $(\cdot,\cdot)$ is the duality bracket $(\mathcal{M}_+(\mathbb{G}_\infty), C_b(\mathbb{G}_\infty))$. Then, assuming **(H1)**-**(H2)**-**(H3)**, for all $t \in [0, T)$,

$$\left\langle \mu_t^N, \left| \left(\alpha_\eta^N - f_t, \phi \right) \right| \right\rangle \lesssim \mathcal{E}_1(N) \left(\int_0^t R(s)\, \mathrm{d}s \right) + \mathcal{E}_2(N) + N^{-\frac{d}{d+2}}$$

where f_t solves the limit PDE in $\mathfrak{B}$.

Remarks. (1) When f_0 smooth enough, one can construct an initial particle distribution μ_0^N for which the initial assumptions hold, namely $\mu_0^N := \vartheta_{f_0}^N = \nu^N_{\Theta(f_0(\frac{\cdot}{N}))}$.
(2) The convergence above implies the convergence in probability of the random variable $\langle \alpha_\eta^N, \phi \rangle$ to the deterministic object $\langle f_t, \phi \rangle$.

Concrete results: the SEP (1)

- The state space at each site is $\mathsf{X} = \{0, 1\}$.
- Given the choice of a transition function $p \in P(\mathbb{Z}^d)$, the base generator (before scaling) writes for $\Phi \in C_b(\mathsf{X}_N)$ and $\eta \in \mathsf{X}_N$:

$$\mathcal{L}_N \Phi(\eta) = \sum_{y \in \mathbb{T}_N^d} p(y - x)\eta(x)[1 - \eta(y)]\left[\Phi(\eta^{x,y}) - \Phi(\eta)\right]$$

  where $\eta^{x,y}(z)$ is $\eta(x) - 1$ if $z = x$, $\eta(y) + 1$ if $z = y$, and $\eta(z)$ otherwise.
- The local Gibbs measure ν_λ^N is constructed from

$$n_\lambda(\eta(x)) := \lambda^{\eta(x)}(1 - \lambda)^{1-\eta(x)}$$

  and $\Theta(z) = z$ which yields

$$\vartheta_f^N(\eta) := \prod_{x \in \mathbb{T}_N^d} f(x)^{\eta(x)}\left[1 - f(x)\right]^{1-\eta(x)}.$$

- The mean transition rate is

$$\gamma := \sum_{x \in \mathbb{Z}^d} x p(x) \in \mathbb{R}^d.$$

Concrete results: the SEP (2)

- When $\gamma \neq 0$, the first non-zero asymptotic dynamics as $N \to \infty$ is given by the hyperbolic scaling $N\mathcal{L}_N$.
- The corresponding expected limit equation is the nonlinear transport equation $\partial_t f = \gamma \cdot \nabla[f(1-f)]$.
- Due to the nonlinearity, even for smooth initial data, the solution can develop shocks.
- Our abstract theorem applies but only as long as the limit solution f_t is regular (C^2).
- The macroscopic stability is then guaranteed by the assumed limit regularity.
- The consistency estimate is a tedious (but not difficult) calculation based on the regularity of f_t.
- The microscopic stability will be discussed later.

Concrete results: the SEP (3)

- When $\gamma = 0$, the first non-zero asymptotic dynamics as $N \to \infty$ is the given by the parabolic scaling $N^2 \mathcal{L}_N$.
- The corresponding limit equation is the diffusion equation

$$\partial_t f = \sum_{i,j=1}^{d} a_{ij} \partial_{ij}^2 f \quad \text{with} \quad a_{ij} := \sum_{x \in \mathbb{Z}^d} p(x) x_i x_j .$$

- The microscopic stability is as above, the macroscopic stability is trivial and the consistency is another tedious calculation.
- The conclusion is that if the quantity

$$E_N(t) := \int_{X_N} \left| \sum_{x \in \mathbb{T}_N^d} \eta(x) \phi\left(\frac{x}{N}\right) - \int_{\mathbb{T}^d} f_t(u) \phi(u) \, \mathrm{d}u \right| \mathrm{d}\mu_t^N(\eta)$$

satisfies $E_N(0) \lesssim N^{-\frac{d}{d+2}}$, then $\sup_{t \geq 0} E_N(t) \lesssim N^{-\frac{d}{d+2}}$.

Concrete results: the ZRP (1)

- The state space at each site is $X = \mathbb{N}$.
- Given the choice of a transition function $p \in P(\mathbb{T}_N^d)$ and a jump rate function $g : \mathbb{N} \to \mathbb{R}_+$, the base generator writes for $\Phi \in C_b(X_N)$ and $\eta \in X_N$ (with $\eta^{x,y}$ defined as before)

$$\mathcal{L}_N \Phi(\eta) = \sum_{y \in \mathbb{T}_N^d} p(y-x) g(\eta(x)) \left[\Phi(\eta^{x,y}) - \Phi(\eta) \right].$$

- The local Gibbs measure is constructed from

$$n_\lambda(\eta(x)) := \frac{\lambda^{\eta(x)}}{g(\eta(x))! Z(\lambda)} \quad \text{with} \quad Z(\lambda) := \sum_{n=0}^{+\infty} \frac{\lambda^n}{g(n)!}$$

  and the notation $g(n)! := g(n)g(n-1)\cdots g(1)$.
- $\Theta := \sigma$ is the inverse function of $R(\lambda) = \lambda \partial_\lambda \ln Z(\lambda)$, and $\vartheta_f^N = \nu^N_{\sigma(f(\frac{\cdot}{N}))}$. Note the relation $\mathbb{E}_{\nu^N_{\Theta(\alpha)}}[g(\eta)] = \sigma(\alpha)$.
- The mean transition rate is defined as before.

Concrete results: the ZRP (2)

- When $\gamma \neq 0$, the first non-zero asymptotic dynamics as $N \to \infty$ is given by the hyperbolic scaling $N\mathcal{L}_N$.
- The corresponding expected limit equation is the nonlinear transport equation $\partial_t f = \gamma \cdot \nabla[\sigma(f)]$.
- When $\gamma = 0$, the first non-zero asymptotic dynamics as $N \to \infty$ is the given by the parabolic scaling $N^2\mathcal{L}_N$.
- The corresponding limit equation is then the nonlinear diffusion equation

$$\partial_t f = \sum_{i,j=1}^{d} a_{ij}\partial^2_{ij}[\sigma(f)] \quad \text{with} \quad a_{ij} := \sum_{x\in\mathbb{Z}^d} p(x)x_i x_j.$$

Concrete results: the ZRP (3)

- We prove these scaling limits under the following assumptions on the jump rate g:
  - (ZR1) **Non-degeneracy**: $g(0) = 0$ and $g(n) > 0$ for all $n > 0$.
  - (ZR2) **Lipschitz-property**: $\sup_{n \geq 0} |g(n+1) - g(n)| \leq g^* < +\infty$.
  - (ZR3) **Spectral gap**: $\exists n_0 \in \mathbb{N}^*, \delta > 0$ s.t. $\min_{n \geq j + n_0} g(n) - g(j) \geq \delta$.
  - (ZR4) **Attractivity**: g non-decreasing.
- The conclusion is then similar to the SEP result.
- The macroscopic stability is either assumed again in the hyperbolic scaling (before shocks) or proved for the nonlinear diffusion: the assumptions on g imply that σ' is uniformly positive and bounded (De Giorgi and Schauder theories).
- The consistency estimate is yet another tedious calculations based on the regularity of f_t.
- The microscopic stability will be discussed later.

Concrete results: the GLK (1)

- The state space at each site is $\mathsf{X} = \mathbb{R}$.
- Given the choice of a single-site potential $V \in C^2(\mathbb{R})$, the base generator writes for $\Phi \in C_b(\mathsf{X}_N)$ and $\eta \in \mathsf{X}_N$:

$$\begin{aligned}\mathcal{L}_N\Phi(\eta) =& \frac{1}{2}\sum_{x\sim y\in\mathbb{T}_N^d}\left(\frac{\partial}{\partial\eta(x)}-\frac{\partial}{\partial\eta(y)}\right)^2 \\ &-\frac{1}{2}\sum_{x\sim y\in\mathbb{T}_N^d}\left[V'(\eta(x))-V'(\eta(y))\right]\left(\frac{\partial}{\partial\eta(x)}-\frac{\partial}{\partial\eta(y)}\right)\end{aligned}$$

  where $x \sim y$ means that x and y are neighbours.
- The local Gibbs measure is constructed from

$$n_\lambda(\eta(x)) := \frac{e^{\lambda\eta(x)}}{Z(\lambda)} \quad \text{with} \quad Z(\lambda) := \int_{\mathbb{R}} e^{\lambda z - V(z)}\,\mathrm{d}z$$

- $\Theta := \sigma$ is the inverse function of $\partial_\lambda \ln Z(\lambda)$, and $\vartheta_f^N = \nu^N_{\sigma(f(\frac{\cdot}{N}))}$.

Concrete results: the GLK (2)

- The hyperbolic scaling is empty and the parabolic scaling $N^2\mathcal{L}_N$ leads to the nonlinear diffusion equation $\partial_t f = \Delta[\sigma(f)]$.
- We assume that $V(u) = V_0(u) + V_1(u)$ and there exist $C, \lambda > 0$ so that

$$V_0''(u) \geq \lambda \text{ and } \|V_1\|_{L^\infty(\mathbb{T}^d)} \leq C, \ \|V_1'\|_{L^\infty(\mathbb{T}^d)} \leq C.$$

  This assumption is similar to those in previous works GOVW'2009, Fathi'2013, DMOW'2018. It includes some double-well potentials that are strictly convex at infinity.
- Same results as for SEP and ZRP (in their parabolic versions).
- Macroscopic stability proved by studying regularity of the limit nonlinear diffusion equation (De Giorgi and Schauder): assumptions imply that σ' is uniformly positive and bounded.
- The consistency is a tedious calculation once more.
- The microscopic stability is the core and will be discussed later.

Sketch of the abstract proof (1)

The proof proceeds in three steps. Denote by η^ℓ for $0 < \ell < N$, the local ℓ-averaged configuration $\eta^\ell(x) := \sum_{|y-x|\le\ell} \eta(y)$.

Step 1. Quantitative Local Law of Large Numbers

(1) This is proved **only** for the local Gibbs measure, which is where we "gain" as compared with block estimates (we separate local averaging error from correlation error).

(2) Given a certain nonlinear moment relation at each site, we prove a similar approximate relation in a box at intermediate scale ℓ.

Proposition: Let $f \in C^1$ and θ on X and β on $\mathbb{R}$ so that

$$\forall\, r \ge 0, \quad \beta(r) = \mathbb{E}_{n_{\Theta(r)}}(\theta(\eta(x))) \quad \text{then}$$

$$\sum_{x \in \mathbb{T}_N^d} \mathbb{E}_{\nu_\infty^N \otimes \vartheta_f^N} \left[W(\eta(x), \zeta(x)) \left| (\theta \circ \zeta)^\ell(x) - \beta\left[\zeta^\ell(x)\right] \right| \right] \lesssim \frac{1}{\ell^{d/2}} + \frac{\ell}{N}.$$

Sketch of the abstract proof (2)

Step 2. Sampling rate.

Given $f \in C^1$, it is easy to obtain by optimising the intermediate scale $\ell = N^{\frac{2}{d+2}}$ in step 1 the following sampling rate

$$\mathbb{E}_{\vartheta_f^N} \left[\left| \sum_x \eta(x) \phi\left(\frac{x}{N}\right) - \int_{\mathbb{T}^d} f(u)\phi(u) \, \mathrm{d}u \right| \right] \lesssim N^{-\frac{d}{d+2}}.$$

Step 3. The complete error estimate.

Given a coupling π_0^N between μ_0^N and $\vartheta_{f_0}^N$ at initial time, we define π_t^N solution to, for all $\Phi \in C_b(\mathsf{X}_N^2)$

$$\partial_t \langle \pi_t^N, \Phi \rangle = \langle \pi_t^N, \widetilde{\mathcal{L}}_N \Phi \rangle + \langle \partial_t \vartheta_{f_t}^N, \Phi_2 \rangle - \langle \vartheta_{f_t}^N, \mathcal{L}_N \Phi_2 \rangle$$

where $\Phi_2(\zeta) := \int_{\mathsf{X}_N} \Phi(\eta, \zeta) \, \mathrm{d}\nu_\infty^N(\eta)$.

Then observe that the first marginal solves the equation for μ_t^N, and the second marginal solves an equation to which $\vartheta_{f_t}^N$ is solution.

Sketch of the abstract proof (3)

Thus π_t^N is a coupling between μ_t^N and $\vartheta_{f_t}^N$ at each $t \in [0, T)$, and:

$$
\begin{aligned}
&\mathbb{E}_{\mu_t^N}\left[\left|\sum_x \eta(x)\phi\left(\frac{x}{N}\right) - \int_{\mathbb{T}^d} f_t(u)\phi(u)\,\mathrm{d}u\right|\right] \\
&= \int_{\eta,\zeta}\left|\sum_x \eta(x)\phi\left(\frac{x}{N}\right) - \int_{\mathbb{T}^d} f_t(u)\phi(u)\,\mathrm{d}u\right| \mathrm{d}\pi_t^N \\
&\leq \int_{\eta,\zeta}\left|\sum_x (\eta(x)-\zeta(x))\phi\left(\frac{x}{N}\right)\right| \mathrm{d}\pi_t^N \\
&\qquad\qquad + \int_{\zeta}\left|\sum_x \zeta(x)\phi\left(\frac{x}{N}\right) - \int_{\mathbb{T}^d} f_t(u)\phi(u)\,\mathrm{d}u\right| \mathrm{d}\vartheta_{f_t}^N \\
&\lesssim \int_{\eta,\zeta} W(\eta,\zeta)\,\mathrm{d}\pi_t^N + N^{-\frac{d}{d+2}} \\
&\lesssim \int_{\eta,\zeta} W(\eta,\zeta)\,\mathrm{d}\pi_0^N + \left(\int_0^t R\right)\mathcal{E}_1(N) + N^{-\frac{d}{d+2}}
\end{aligned}
$$

Microscopic stability: the coupling operators (1)

For the SEP (with $c(x,y) := b(\eta(x),\eta(y)) \wedge b(\zeta(x),\zeta(y))$)

$$\widetilde{\mathcal{L}}_N\Phi(\eta,\zeta) := \sum_{x,y} p(y-x)c(x,y)\Big[\Phi(\eta^{xy},\zeta^{xy}) - \Phi(\eta,\zeta)\Big]$$

$$+ \sum_{x,y} p(y-x)\Big(b(\eta(x),\eta(y)) - c(x,y)\Big)\Big[\Phi(\eta^{xy},\zeta) - \Phi(\eta,\zeta)\Big]$$

$$+ \sum_{x,y} p(y-x)\Big(b(\zeta(x),\zeta(y)) - c(x,y)\Big)\Big[\Phi(\eta,\zeta^{xy}) - \Phi(\eta,\zeta)\Big].$$

For the ZRP (with $c(x) := g(\eta(x)) \wedge g(\zeta(x))$)

$$\widetilde{\mathcal{L}}_N\Phi(\eta,\zeta) := \sum_{x,y} p(y-x)c(x)\Big[\Phi(\eta^{xy},\zeta^{xy}) - \Phi(\eta,\zeta)\Big]$$

$$+ \sum_{x,y} p(y-x)\Big(g(\eta(x)) - c(x)\Big)\Big[\Phi(\eta^{xy},\zeta) - \Phi(\eta,\zeta)\Big]$$

$$+ \sum_{x,y} p(y-x)\Big(g(\zeta(x)) - c(x)\Big)\Big[\Phi(\eta,\zeta^{xy}) - \Phi(\eta,\zeta)\Big].$$

Microscopic stability: the coupling operators (2)

- The two coupling operators for the SEP and ZRP go back to Liggett'1985 and are widely used in optimal transport.
- They correspond to choosing the coupling so that jumps happen as simultaneously as possible, thus keeping the cost of transporting from one measure to the other minimum.
- Reminiscent also of Tanaka'1973-1978 in the case of binary collisions for the Kac random walk.
- In the hyperbolic limit, the type of Wasserstein estimates we use correspond to Kruzhhov'1970 estimates for scalar conservation laws, and in the parabolic limit, they correspond to the counterpart Carrillo'1999 for nonlinear diffusions.
- The latter theories are based on the "doubling of variable argument", which has conceptual connexion with coupling argument, as shown for instance in Bolley-Brenier-Loeper'2005.

Microscopic stability: the coupling operators (3)

For the GLK we propose the following coupling operator.

Denote $\mathfrak{D}_\eta^{x,y} := \frac{\partial}{\partial\eta(x)} - \frac{\partial}{\partial\eta(y)}$ and $\mathfrak{D}_\zeta^{x,y} := \frac{\partial}{\partial\zeta(x)} - \frac{\partial}{\partial\zeta(y)}$.

Define $(\mathfrak{D}_\eta^{x,y})^* = -\mathfrak{D}_\eta^{x,y} + [V'(\eta(x)) - V'(\eta(y))]$ (adjoint wrt ν_∞^N) so that

$$\mathcal{L}_N = -\frac{1}{2}\sum_{x\sim y}(\mathfrak{D}_\eta^{x,y})^*\mathfrak{D}_\eta^{x,y}$$

Finally we define:

$$\begin{aligned}
\widetilde{\mathcal{L}}_N\Phi(\eta,\zeta) := &-\frac{1}{2}\sum_{x\sim y}\left[\left(\mathfrak{D}_\eta^{x,y}\right)^*\left(\mathfrak{D}_\eta^{x,y}\right)\otimes 1\right]\Phi(\eta,\zeta)\\
&-\frac{1}{2}\sum_{x\sim y}\left[1\otimes\left(\mathfrak{D}_\zeta^{x,y}\right)^*\left(\mathfrak{D}_\zeta^{x,y}\right)\right]\Phi(\eta,\zeta)\\
&+(2+K)\sum_{x\sim y}\left(\mathfrak{D}_\eta^{x,y}\right)\otimes\left(\mathfrak{D}_\zeta^{x,y}\right)\Phi(\eta,\zeta)
\end{aligned}$$

Microscopic stability: the coupling operators (4)

Observe first that it is a coupling operator:
(1) if $\Phi = \Phi(\eta)$, then $\widetilde{\mathcal{L}}_N \Phi(\eta, \zeta) = \mathcal{L}_N \Phi(\eta)$
(2) if $\Phi = \Phi(\zeta)$, then $\widetilde{\mathcal{L}}_N \Phi(\eta, \zeta) = \mathcal{L}_N \Phi(\zeta)$.

Then given the weight $W(\eta, \zeta) := \sum_x |\eta(x) - \zeta(x)|^2$ we compute

$$\widetilde{\mathcal{L}_N} \left(\sum_x |\eta(x) - \zeta(x)|^2 \right)$$
$$= \frac{1}{2} \sum_x \Big[-\lambda |\eta(x) - \zeta(x)|^2 + C|\eta(x) - \zeta(x)| - 4K \Big]$$

where $\lambda > 0$ is the convexity constant of V_0 (bound from below on V_0'') and C is the bound from above on V_1'.
This can be made non-positive for a suitable choice of K.