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Image comparison via nonlinear elasticity

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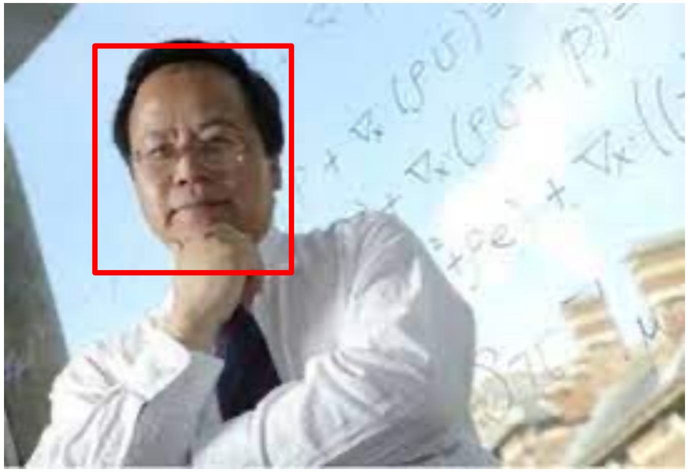
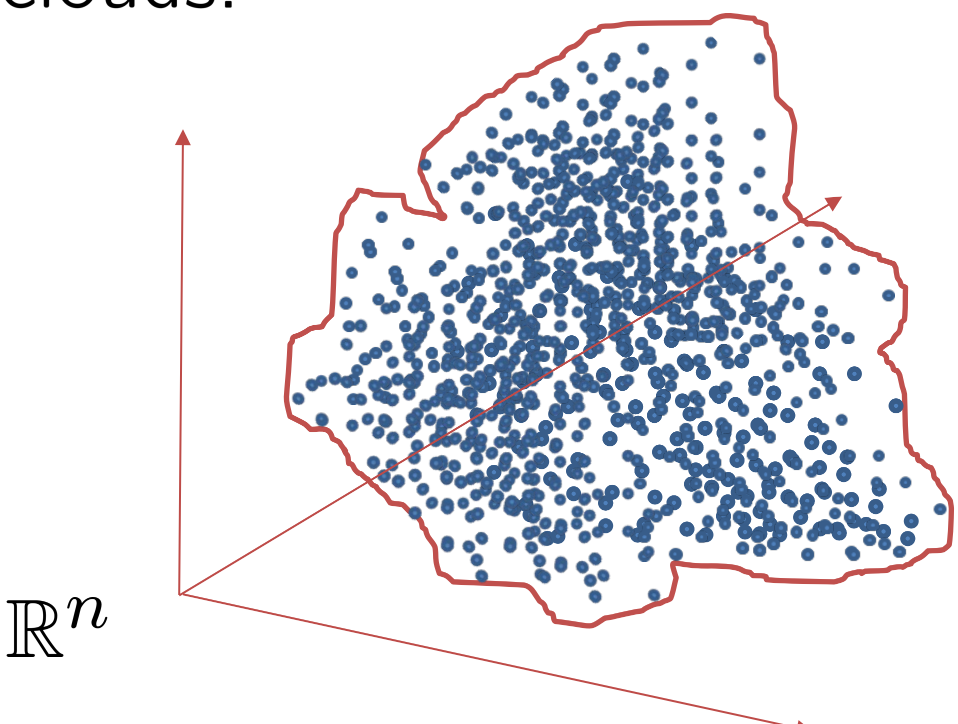
Image comparison



template

Identify an image with a pair $P = (\Omega, c)$, where $\Omega \subset \mathbb{R}^n$, is a bounded Lipschitz domain, and $c : \Omega \to \mathbb{R}^m$ is a map describing image features (grey-scale, colours ...).

Usually $n = 2$ (photograph), $n = 3$ (tomogram) or $n = 4$ (dynamic tomogram), but perhaps larger n could be relevant for continuum approximations to data clouds.



Cluster analysis

Persistent homology

Want to compare two images

$$P_1 = (\Omega_1, c_1),\ P_2 = (\Omega_2, c_2)$$

by means of a nonlinear elasticity based functional.

Other approaches use machine learning and neural nets, optimal transport and fluid flow.

Models based on linear elasticity are quite common. Nonlinear elasticity provides a conceptually clearer and more general framework, respecting rotational invariance.

Other nonlinear elasticity approaches due to Droske & Rumpf (2004), Lin, Dinov, Toga & Vese (2010), Rumpf (2009), Rumpf & Wirth (2011), Burger, Modersitski & Ruthotto (2013), Ozeré, Gout & Le Guyader (2015), Simon, Sheorey, Jacobs & Basri (2017), Debroux & Le Guyader (2018).

Define a 'distance' $d(P_1, P_2) \geq 0$ between P_1, P_2 by

$$d(P_1, P_2) = \inf I_{P_1,P_2}(y),$$

where the inf is over invertible maps $y : \Omega_1 \to \mathbb{R}^n$ such that $y(\Omega_1) = \Omega_2$, and

$$I_{P_1,P_2}(y) = \int_{\Omega_1} \psi(c_1(x), c_2(y(x)), Dy(x))\, dx,$$

where $\psi : \mathbb{R}^s \times \mathbb{R}^s \times M_+^{n\times n} \to [0, \infty)$, and

$$\begin{aligned} M_+^{n\times n} &= GL^+(n, \mathbb{R}) \\ &= \{\text{real } n \times n \text{ matrices } A \text{ with } \det A > 0\}. \end{aligned}$$

Note that we do not specify y on $\partial\Omega_1$, only that $y(\Omega_1) = \Omega_2$. Thus we allow 'sliding at the boundary'. This is not typically done in the computer vision literature, but is considered in the context of elasticity by Iwaniec & Onninen (2009).

$$I_{P_1,P_2}(y) = \int_{\Omega_1} \psi(c_1(x), c_2(y(x)), Dy(x))\, dx$$

$y : \Omega_1 \to \Omega_2$ invertible with $y(\Omega_1) = \Omega_2$.

Properties of ψ.

(i) *Invariance under rotation and translation*

For two images $P = (\Omega, c)$ and $P' = (\Omega', c')$ write $P \sim P'$ if P, P' are related by a rigid translation and rotation, i.e.

$$\Omega' = E\Omega,\ c'(Ex) = c(x)$$

for some proper rigid transformation $Ex = a + Rx$, $a \in \mathbb{R}^n$, $R \in SO(n)$.

If $P_1 \sim P_1'$, $P_2 \sim P_2'$, with corresponding rigid transformations $E_1 x = a_1 + R_1 x, E_2 x = a_2 + R_2 x$, we require that

$$I_{P_1,P_2}(y) = I_{P_1',P_2'}(E_2 \circ y \circ E_1^{-1}),$$

or, equivalently,

$$\int_{\Omega_1} \psi(c_1(x), c_2(y(x)), R_2 Dy(x) R_1^T)\, dx = \int_{\Omega_1} \psi(c_1(x), c_2(y(x)), Dy(x))\, dx.$$

This holds for all P_1, P_2 and invertible $y : \Omega_1 \to \Omega_2$ with $y(\Omega_1) = \Omega_2$ iff $\psi(c_1, c_2, \cdot)$ is *isotropic*, i.e.

$$\psi(c_1, c_2, QAR) = \psi(c_1, c_2, A)$$

for all $c_1, c_2 \in \mathbb{R}^n, A \in M_+^{n\times n}, R, Q \in SO(n)$.

A standard result of nonlinear elasticity gives that $\psi(c_1, c_2, \cdot)$ is isotropic iff

$$\psi(c_1, c_2, A) = H(c_1, c_2, v_1(A), ..., v_n(A))$$

with H symmetric with respect to permutations of the last n arguments, where the $v_i(A)$ are the singular values of A (that is, the eigenvalues of $\sqrt{A^T A}$).

Furthermore we require that

$$\psi(c_1, c_2, A) = 0 \text{ iff } c_1 = c_2 \text{ and } A \in SO(n).$$

Then $I_{P_1,P_2}(y) = 0$ for some $y : \Omega_1 \to \Omega_2$ with $y(\Omega_1) = \Omega_2$ iff $P_1 \sim P_2$. Indeed if $I_{P_1,P_2}(y) = 0$ then $Dy(x) \in SO(n)$ a.e. and so under some regularity on y (which we will assume) $Dy(x) = R$ a.e. for some $R \in SO(n)$, and thus $y(x) = a + Rx$ and $c_1(x) = c_2(a + Rx)$, so that $P_1 \sim P_2$.

(ii) *Symmetry with respect to interchanging images*

We require that $I_{P_1,P_2}(y) = I_{P_2,P_1}(y^{-1})$.

That is

$$\begin{aligned}\int_{\Omega_1} \psi(c_1(x), c_2(y(x)), Dy(x))\, dx \\ &= \int_{\Omega_2} \psi(c_2(y), c_1(x(y)), Dx(y))\, dy \\ &= \int_{\Omega_1} \psi(c_2(y(x)), c_1(x), Dy(x)^{-1}) \det Dy(x)\, dx.\end{aligned}$$

Taking c_1, c_2 constant and $y(x) = Ax$ this holds iff

$$\psi(c_1, c_2, A) = \psi(c_2, c_1, A^{-1}) \det A.$$

Examples

Let

$$\psi(c_1, c_2, A) = \Psi(A) + f(c_1, c_2, \det A),$$

where

(i) $\Psi \geq 0$ is isotropic, $\Psi(A) = \det A \cdot \Psi(A^{-1})$, $\Psi^{-1}(0) = SO(n)$,

(ii) $f \geq 0$, $f(c_1, c_2, \delta) = \delta f(c_2, c_1, \delta^{-1})$, $f(c_1, c_2, 1) = 0$ iff $c_1 = c_2$.

In particular we can take

$$f(c_1, c_2, \delta) = (1 + \delta)|c_1 - c_2|^2,$$

or

$$f(c_1, c_2, \delta) = |c_1 - c_2\delta|^2 + \delta^{-1}|c_1\delta - c_2|^2,$$

which are both convex in δ.

Existence of minimizers

We suppose that $\bar{\Omega}_1$ and $\bar{\Omega}_2$ are diffeomorphic. Let $p > n$ and

$$\mathcal{A} = \{y \in W^{1,p}(\Omega_1, \mathbb{R}^n) : y : \Omega_1 \to \Omega_2 \text{ a homeomorphism}, \\ y^{-1} \in W^{1,p}(\Omega_2, \mathbb{R}^n)\}.$$

Hypotheses on ψ.

(H1) $\psi : \mathbb{R}^s \times \mathbb{R}^s \times M_+^{n\times n} \to [0, \infty)$ is continuous,

(H2) $\psi(c, d, A) \geq C(|A|^p + \det A \cdot |A^{-1}|^p) + h(\det A)$ for all $c, d \in \mathbb{R}^s, A \in M_+^{n\times n}$, where $h = h(\delta)$ is bounded below and $\lim_{\delta\to 0+} h(\delta) = \infty$,

(H3) $\psi(c, d, \cdot)$ is polyconvex for each $c, d \in \mathbb{R}^s$, i.e. there is a function $g : \mathbb{R}^s \times \mathbb{R}^s \times \mathbb{R}^{\sigma(n)} \times (0, \infty) \to \mathbb{R}$ with $g(c, d, \cdot)$ convex, such that

$g(c, d, \mathbf{J}_{n-1}(A), \det A) = \psi(c, d, A)$ for all $A \in M_+^{n\times n}$,

where $\mathbf{J}_{n-1}(A)$ is the list of all minors of A of order $\leq n-1$ and $\sigma(n)$ is the number of such minors,

(H4) $c_1 \in L^\infty(\Omega_1, \mathbb{R}^s)$, $c_2 \in L^\infty(\Omega_2, \mathbb{R}^s)$.

Theorem Under the hypotheses (H1)-(H4) there exists an absolute minimizer in $\mathcal{A}$ of

$$I_{P_1,P_2}(y) = \int_{\Omega_1} \psi(c_1(x), c_2(y(x)), Dy(x))\, dx.$$

Proof. This follows the usual pattern for proving existence of minimizers in nonlinear elasticity for a polyconvex stored-energy function. However there are some extra issues.

We use the change of variables formula of Marcus and Mizel (1973), which gives that

$$\int_{y(E)} \varphi(z)\, dz = \int_E \varphi(y(x)) \det Dy(x)\, dx$$

for all $\varphi \in L^1(\Omega_2)$, $y \in \mathcal{A}$, and measurable $E \subset \Omega_1$, whenever one side is meaningful. In particular

$$|y(E)| = \int_E \det Dy(x)\, dx.$$

It follows from this that $c_2(y(\cdot))$ is measurable and independent a.e. of the representative of c_2, so that the integral is well-defined.

Since $\bar{\Omega}_1, \bar{\Omega}_2$ are diffeomorphic $\mathcal{A}$ is nonempty. Let $y^{(j)}$ be a minimizing sequence for I_{P_1,P_2} in $\mathcal{A}$ with corresponding inverses $\xi^{(j)}(y)$. From (H2) we can suppose that

$$y^{(j)} \rightharpoonup y \text{ in } W^{1,p}(\Omega_1, \mathbb{R}^n),\ \xi^{(j)} \rightharpoonup \xi \text{ in } W^{1,p}(\Omega_2, \mathbb{R}^n).$$

Passing to the limit in

$$y^{(j)}(\xi^{(j)}(z)) = z, \ \ \xi^{(j)}(y^{(j)}(x)) = x$$

we see using the compact embeddings of $W^{1,p}(\Omega_i, \mathbb{R}^n)$ in $C^0(\bar{\Omega}_i, \mathbb{R}^n)$ that $y \in \mathcal{A}$ with inverse ξ.

From the weak continuity of minors we have that

$$(\mathbf{J}(Dy^{(j)}), \det Dy^{(j)}) \rightharpoonup (\mathbf{J}(Dy), \det Dy) \text{ in } L^1(\Omega_1; \mathbb{R}^{\sigma(n)+1}),$$

so that we need to prove that

$$\begin{aligned} I_{P_1,P_2}(y) &= \int_{\Omega_1} g(c_1, c_2(y), \mathbf{J}(Dy), \det Dy)\, dx \\ &\leq \liminf_{j\to\infty} \int_{\Omega_1} g(c_1, c_2(y^{(j)}), \mathbf{J}(Dy^{(j)}), \det Dy^{(j)})\, dx \\ &= \liminf_{j\to\infty} I_{P_1,P_2}(y^{(j)}). \end{aligned}$$

For this, by the convexity of $g(c, d, \cdot)$ and a standard lower semicontinuity result, it is enough to show that for a subsequence $c_2(y^{(j)}(x)) \to c_2(y(x))$ a.e.. As pointed out by Rumpf (2009) this is not so obvious for c_2 discontinuous, and he gives conditions on c_2 under which this holds. However an argument using the change of variables formula and weak continuity of the determinant shows that it holds for any $c_2 \in L^\infty$, completing the proof. □

Properties of minimizers. Are minimizers weak solutions of the Euler-Lagrange equation? Are they smooth? Do they satisfy $0 < \mu \leq \det Dy(x) \leq M < \infty$ a.e.? Can the Lavrentiev phenomenon occur?

These are all open questions, as they are for nonlinear elasticity.

Parenthetical comment on existence hypotheses

In Ball (1976) an apparently stronger polyconvexity hypothesis was made, namely that for all $A \in M_+^{n\times n}$

$$\psi(A) = g(\mathbf{J}_{n-1}(A), \det A) \qquad (*)$$

where $g : \mathbb{R}^{\sigma(n)} \times \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ is continuous convex and $g(H, \delta) = \infty$ for $\delta \leq 0$.

However in B/Murat (1984) this condition was apparently weakened to those just given (dropping the dependence on c_1, c_2), namely there is a convex function $g : \mathbb{R}^{\sigma(n)} \times (0, \infty) \to \mathbb{R}$ such that $(*)$ holds, and

$$\psi(A) \to \infty \text{ as } \det A \to 0+.$$

But is this condition weaker? Namely are there convex functions $g : \mathbf{J}_{n-1} \times [0, \infty) \to [0, \infty]$ with $g(H, 0) \neq \infty$ for some H and

$$\psi(A) = g(\mathbf{J}_{n-1}(A), \det A) \to \infty \text{ as } \det A \to 0+?$$

Such g can be constructed using the following result:

Proposition. For any proper lower semicontinuous convex function $\Phi : \mathbb{R}^\sigma \to (-\infty, \infty]$ there is a lsc convex extension $\varphi : [0, \infty) \times \mathbb{R}^\sigma \to (-\infty, \infty]$ such that
(i) $\varphi = \varphi(x, y)$ is finite and smooth for $x > 0$
(ii) $\lim_{x \to 0+} \varphi(x, y) = \varphi(0, y) = \Phi(y)$ for each $y \in \mathbb{R}^\sigma$.

Example. $\Phi(y) = \begin{cases} 0, & y = 0 \\ \infty, & y \neq 0 \end{cases}$, for which a smooth convex extension is $\varphi(x, y) = \frac{|y|^2}{x}$.

Magnification and linear transformations

Suppose $P_1 = (\Omega_1, c_1)$ and $P_2 = (\Omega_2, c_2)$ are linearly related, i.e. for some $M \in M_+^{n\times n}$ we have

$$\Omega_2 = M\Omega_1, \;\; c_2(Mx) = c_1(x).$$

Can we choose ψ such that the unique minimizer y of I_{P_1,P_2} is $y(x) = Mx$?

For simplicity consider ψ of the form

$$\psi(c_1, c_2, A) = \Psi(A) + (1 + \det A)|c_1 - c_2|^2$$

Thus we require that for all invertible y with $y(\Omega_1) = M\Omega_1$

$$\int_{\Omega_1} \left(\Psi(Dy(x)) + (1 + \det Dy(x))|c_1(x) - c_2(y(x))|^2\right) dx$$
$$\geq \int_{\Omega_1} \Psi(M)\, dx$$

with equality iff $y(x) = Mx$.

In particular, if this holds for all c_1 we have that

$$\fint_{\Omega_1} \Psi(Dy)\,dx \geq \Psi(M)$$

for y invertible with $y(\Omega_1) = M\Omega_1$, a stronger version of *quasiconvexity at* M, in which the usual requirement that $y(x) = Mx$ for $x \in \partial\Omega_1$ is weakened.

We show that we can satisfy this condition if $M = \lambda\mathbf{1}$, $\lambda > 0$ (or more generally if $M = \lambda R$, $R \in SO(n)$), so that P_2 is a magnification of P_1. For simplicity we give the construction for $n = 2$, and let

$$\Psi(A) = v_1^\alpha + v_2^\alpha + v_1 v_2 \left(v_1^{-\alpha} + v_2^{-\alpha}\right) + h(v_1 v_2),$$

where $v_i = v_i(A)$ are the singular values of A, $\alpha > 2$, and $h = h(\delta) = \delta h(\delta^{-1})$ is C^1, convex and bounded below, with $h'(1) = -2$ and $\lim_{\delta\to 0+} h(\delta) = \infty$.

Then Ψ is isotropic, $\Psi(A) = \det A \cdot \Psi(A^{-1})$, $\Psi \geq 0$, $\Psi^{-1}(0) = SO(2)$, and ψ satisfies (H1)-(H4).

Let y be invertible with $y(\Omega_1) = \lambda\Omega_1$. By the AM≥GM inequality we have that, since $\det Dy = v_1 v_2$,

$$\begin{aligned}
\int_{\Omega_1} \Psi(Dy)\,dx &\geq \int_{\Omega_1} \Big(2(\det Dy)^{\frac{\alpha}{2}} + 2(\det Dy)^{1-\frac{\alpha}{2}} \\
&\qquad\qquad + h(\det Dy)\Big)\,dx \\
&= \int_{\Omega_1} H(\det Dy(x))\,dx \\
&\geq |\Omega_1| H\left(\frac{1}{|\Omega_1|}\int_{\Omega_1} \det Dy(x)\,dx\right) \\
&= |\Omega_1| H(\lambda^2) \\
&= |\Omega_1| \Psi(\lambda \mathbf{1}),
\end{aligned}$$

as required.

Note that we have equality only when $v_1 = v_2 = \lambda$, i.e. $Dy(x) = \lambda R(x)$ for $R(x) \in SO(2)$, which implies that $R(x) = R$ is constant and $a + \lambda R\Omega_1 = \lambda\Omega_1$, which for generic Ω_1 implies $a = 0$ and $R = \mathbf{1}$, hence $y(x) = \lambda x$.

What about general M?

Theorem

$$\fint_{\Omega_1} \Psi(Dy)\,dx \geq \Psi(M)$$

for all invertible y with $y(\Omega_1) = M\Omega_1$ and for every Ω_1 and $M \in M_+^{n\times n}$ iff

$$\Psi(A) = H(\det A)$$

for some convex H.

Sketch of proof. If $y = Mx$ is a minimizer, then we can construct a variation that slips at the boundary, so that the tangential component at the boundary of the 'Cauchy stress' is zero, i.e.

$$D\Psi(M)M^T = p(M)\mathbf{1},$$

from which it follows that Ψ corresponds to an elastic fluid, i.e. $\Psi(M) = H(\det M)$. But then $H(\det M)$ is quasiconvex, and so H is convex.

Conversely, if H is convex then

$$\begin{aligned}\int_{\Omega_1} H(\det Dy(x))\,dx &\geq |\Omega_1| H\left(\frac{1}{|\Omega_1|}\int_{\Omega_1} \det Dy(x)\,dx\right) \\ &= |\Omega_1| H(\det M).\end{aligned}$$

Comparing parts of images

Regard $P_1 = (\Omega_1, c_1)$ as a template image and $P_2 = (\Omega_2, c_2)$ as the target.

Possibilities

(i) Minimize I_{P_1,P_2} with the constraint $y(\Omega_1) = \Omega_2$ replaced by $y(\Omega_1) \subset \Omega_2$.

(ii) Minimize both over subdomains $\tilde{\Omega} = a + \lambda R\Omega_1 \subset \Omega_2$, where $a \in \mathbb{R}^n$, $R \in SO(n)$, $\alpha \leq \lambda \leq \beta$, and maps $y : \Omega_1 \to \tilde{\Omega}$.

Thanks for listening