

Polynomials – Solutions

Revision Questions

1. $p(2) = 2 \times 2^3 - 5 \times 2^2 + 7 \times 2 - 3 = 7$. So $p(2) = 7$.
2. We can rearrange to $x^2 - x - 1 = 0$ and then use the quadratic formula for $x = \frac{1}{2}(1 \pm \sqrt{5})$. If we choose the solution with the + sign then we'll get a positive number.
3. In this case, the discriminant " $b^2 - 4ac$ " is $5^2 - 4 \times 2 \times 1 = 17$.
4. The discriminant for this quadratic is $1 - 4k$. There are exactly two real solutions if this is positive, which happens when $k < \frac{1}{4}$.
5. This is not a quadratic, but if we change variable by writing $u = x^2$ then we get $u^2 - u + k = 0$. That's got two real solutions if $k < \frac{1}{4}$, one real solution if $k = \frac{1}{4}$, and no real solutions if $k > \frac{1}{4}$ (thinking about the discriminant again). But let's be careful, because that's the number of solutions there are for u , and we really want to know how many solutions there are for x .

If there are no real solutions for u then there can't be any real solutions for x . So that rules out $k > \frac{1}{4}$. If there's exactly one solution for u then we might get two real solutions for x ; they'd be $\pm\sqrt{u}$, but that only works if the solution for u is a positive number. In the case $k = \frac{1}{4}$, we've got one solution for u , and if we write down the quadratic formula then that solution is actually $\frac{1}{2}$, so we do get two real solutions for x . In the other remaining case $k < \frac{1}{4}$ there are two real solutions for u . That could give us as many as four real solutions for x . We'd get exactly two real solutions for x if and only if one of the solutions for u is positive and one is negative. Thinking about the factorisation $(u - a)(u - b)$, we can see that the constant term k in our quadratic for u would have to be negative for there to be one positive solution and one negative solution. So we would get two real solutions for x only if $k < 0$.

Putting all that together, there are two real solutions for x if $k < 0$ or if $k = \frac{1}{4}$, and for no other values of k .

6. The discriminant is $b^2 - 4$. That's positive (and the quadratic has two real solutions) if $b > 2$ or if $b < -2$. If $b = \pm 2$ then the quadratic has one solution. If $-2 < b < 2$ then the quadratic has no real solutions.
7. If I imagine multiplying out $(x + a)^2$, then I would get a term $2ax$, and I want that to match with the $4x$ term. So I'll take $a = 2$. Then if I multiply out $(x + 2)^2$, I'd get a term $+4$ at the end; that's not quite what I want, so I'll take $b = -1$ to fix the constant coefficient of this quadratic. I get $(x + 2)^2 - 1$.
8. We can write this polynomial as $-2(x - 2)^2 + 13$. The extreme value is therefore 13. This is a maximum because $-2(x - 2)^2 \leq 0$.

9. First we're asked to check that $17^3 - 13 \times 17^2 - 65 \times 17 - 51 = 0$. To make this easier, don't work out the terms individually. Instead pull out factors of 17;

$$17^3 - 13 \times 17^2 - 65 \times 17 - 51 = 17(17^2 - 13 \times 17 - 65 - 3) \text{ because } 51 = 3 \times 17.$$

$$17^2 - 13 \times 17 - 68 = 17(17 - 13 - 4) \text{ because } 68 = 4 \times 17.$$

$17 - 13 - 4 = 0$ so each line above is equal to zero. By the Factor Theorem, if $p(17) = 0$ then $(x - 17)$ is a factor of the polynomial. Doing some polynomial division, we can work out that $p(x) = (x - 17)(x^2 + 4x + 3)$. We can then write $x^2 + 4x + 3 = (x + 3)(x + 1)$ and we've factorised $p(x)$.

10. The polynomial $p(x)$ has a factor of $(x - 2)$.

11. We have $p(x) = (x - 2)q(x)$ for some polynomial $q(x)$, so $p(2) = (2 - 2)q(2) = 0$.

12. Check that $f(2) = 0$.

Now factorise $f(x) = (x - 2)(x^3 - 4x^2 + 5x - 2)$. Look for more roots; perhaps $x = 2$ is a repeated root? In fact $2^3 - 4 \times 2^2 + 5 \times 2 - 2 = 0$ so it is a repeated root.

$$f(x) = (x - 2)^2(x^2 - 2x + 1) \text{ and we can recognise that quadratic as } (x - 1)^2.$$

$$\text{So } f(x) = (x - 1)^2(x - 2)^2.$$

13. We might notice that $p(1) = 0$. Then write $p(x) = (x - 1)(x^2 - 5x + 6)$ and factorise the quadratic for $p(x) = (x - 1)(x - 2)(x - 3)$.

14. $p(3) = -9$ is not zero, so $(x - 3)$ is not a factor.

15. Yes, the polynomial could have a repeated root. For example, $p(x) = 2(x - 1)^2(x - 2)$

16. • $y = 2x^6 + x^3 + 1$. Choosing $u = x^3$ gives $y = 2u^2 + u + 1$.

• $y = x + \sqrt{2x}$. Choosing $u = \sqrt{x}$ gives $y = u^2 + \sqrt{2}u$.

• $y = 3e^{-3x} + 6e^{-6x}$. Choosing $u = e^{-3x}$ gives $y = 3u + 6u^2$.

• $y = \frac{1 + x}{(1 - x)^2}$. We can rearrange this to $y = \frac{(x - 1) + 2}{(1 - x)^2} = \frac{-1}{1 - x} + \frac{2}{(1 - x)^2}$.

Choosing $u = \frac{1}{1 - x}$ gives $y = -u + 2u^2$.

17. $q(x)$ could be $17(x - 2)(x + 3)(x - 1)$ or $39(x - 2)^2(x + 3)^2(x - 1)^2$ or $-(x - 3)(x - 2)(x - 1)x(x + 1)(x + 2)(x + 3)$. We aren't told if these are repeated roots or not, or whether there are any other roots, or what the leading coefficient is.

18. $v(1) = 3 + a + b$ and that must be zero. Try polynomial division;

$$v(x) = (x - 1)(x^2 + 3x + (a + 3)),$$

provided that $3 + a + b = 0$. Now we want $x = 1$ to be root of that quadratic, so we need $1 + 3 + a + 3 = 0$. Solve these equations for $a = -7$ and $b = 4$.

MAT Questions

MAT 2016 Q1F

- $(x + 1)$ is a factor of this polynomial if and only if -1 is a root of the polynomial, which would mean that

$$(3 + (-1)^2)^n - (-1 + 3)^n(-1 - 1)^n = 0.$$

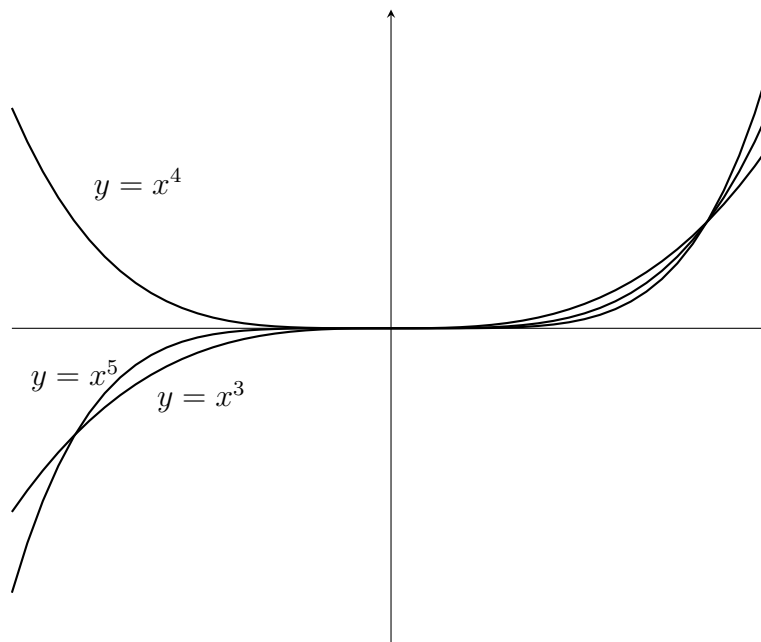
- This simplifies to

$$4^n - 2^n(-2)^n = 0.$$

- If n is even then $(-2)^n$ is the same as 2^n and the equality is true. If n is odd then we have $4^n + 4^n$ on the left-hand side, which is not zero for any positive integer n .
- So the answer is (b).

MAT 2015 Q1I

- Here's a sketch.



- The three curves all pass through $(0, 0)$ and $(1, 1)$. The curves $y = x^3$ and $y = x^5$ both pass through $(-1, -1)$.
- There's one region above all three curves, one region below all three curves, and a total of seven regions between the curves.
- The answer is (d).

MAT 2007 Q2

- (i) Plugging in $n = 3$, we have $f_3(x) = ((2 + (-2)^3))x^2 + (3 + 3)x + 3^2 = -6x^2 + 6x + 9$.

Completing the square, we can write this as $-6\left(x - \frac{1}{2}\right)^2 + \frac{21}{2}$

The polynomial is usually a quadratic (unless the leading coefficient $2 + (-2)^n$ happens to be zero), in which case it has a maximum if and only if the leading coefficient is negative (if it's a "sad" quadratic), which happens if n is odd. Watch out for the special case though; $2 + (-2)^n$ is zero if $n = 1$, in which case the polynomial is a linear function without a maximum.

- (ii) $f_1(x) = 4x + 1$.

$$f_1(f_1(x)) = 4(4x + 1) + 1 = 4^2x + 4 + 1 = 16x + 5.$$

$$f_1(f_1(f_1(x))) = 4(4^2x + 4 + 1) + 1 = 4^3x + 16 + 4 + 1 = 64x + 21$$

In general, $f_1(f_1(\dots f_1(x) \dots))$ with f_1 applied k times is equal to $4^kx + 4^{k-1} + 4^{k-2} + \dots + 4 + 1$.

The constant term is a geometric series, so we can simplify to

$$4^kx + \frac{4^k - 1}{4 - 1} = 4^kx + \frac{4^k - 1}{3}.$$

- (iii) $f_2(x) = 6x^2 + 5x + 4$ is a quadratic. Each time we repeatedly square, the degree gets multiplied by 2. So the degree of $f_2(f_2(\dots f_2(x) \dots))$ with f_2 applied k times is 2^k .

Extension

- $f_n(x)$ with $n > 2$ is still a quadratic, so the degree of $f_n(f_n(\dots f_n(x) \dots))$ with f_n applied k times is 2^k just like in the last part of the question.
- Let's look at what happens for $f(x) = ax^2$ for real non-zero a (only the highest power really matters for this question). $f(f(x)) = a(ax^2)^2 = a^3x^4$ and $f(f(f(x))) = a(a^3x^4)^2 = a^7x^8$, so it looks like the coefficient of x^{2^k} is a^{2^k-1} . For the quadratic we're talking about here, the coefficient ends up being $(2 + (-2)^n)^{2^k-1}$.
- For n odd and greater than 3, g is a quadratic with a maximum value. $n = 3$ is special; $g_3(x) = 9$. That has a maximum value of 9 (which is happens to take for all x).
- For $n \neq 3$ the degree is 2^k again. For $n = 3$ the degree is zero because the outcome after all those function applications is still just the value 9.

MAT 2011 Q2

- (i) Multiply both sides by x to get $x^4 = 2x^2 + x$.

Then multiply both sides by x again for $x^5 = 2x^3 + x^2$. Now use the fact that $x^3 = 2x + 1$ to write $x^5 = 2(2x + 1) + x^2 = 2 + 4x + x^2$.

- (ii) In general we can multiply by x and use the initial fact about x^3 to remove any x^3 term we get. In general, it looks like this;

$$x^{k+1} = (x^k) x = (A_k + B_k x + C_k x^2) x = A_k x + B_k x^2 + C_k x^3 = A_k x + B_k x^2 + C_k (2x + 1).$$

Now remember that $x^{k+1} = A_{k+1} + B_{k+1}x + C_{k+1}x^2$. We can match this up with the expression on the right by taking $A_{k+1} = C_k$ and $B_{k+1} = A_k + 2C_k$ and $C_{k+1} = B_k$.

- (iii) By the previous part,

$$A_{k+1} + C_{k+1} - B_{k+1} = C_k + B_k - (A_k + 2C_k).$$

That simplifies to $B_k - A_k - C_k$. So $D_{k+1} = -D_k$.

We can also note that $D_4 = 1$, so $D_5 = -1$ and $D_6 = 1$ and so on; we have $D_k = (-1)^k$. Then use the definition of D_k and rearrange $A_k + C_k - B_k = (-1)^k$ by adding B_k to both sides.

- (iv) We're asked to show that $A_{k+1} + C_{k+1} + A_{k+2} + C_{k+2} = A_{k+3} + C_{k+3}$. The way to approach this which is clearest to me is to use the previous part to replace the A_{k+3} and C_{k+3} for things with subscript $k + 2$, then replace everything that has a subscript $k + 2$ for things with subscript $k + 1$, and then hope that everything balances.

I have $A_{k+1} + C_{k+1} + (A_{k+2} + C_{k+2}) = A_{k+1} + C_{k+1} + (C_{k+1} + B_{k+1})$ on the left.

I have $A_{k+3} + (C_{k+3}) = C_{k+2} + (B_{k+2}) = B_{k+1} + (A_{k+1} + 2C_{k+1})$ on the right.

[The brackets here are just to make it clearer which terms I'm replacing with which.]

These are equal, so the fact is true.

Alternatively, replace the $A_k + C_k$ terms with $B_k + (-1)^k$ and go from there.

Extension

- Given $x^2 = x + 1$, we could multiply both sides by x for $x^3 = x^2 + x$, then replace the x^2 for $x + 1$ to get $x^3 = 2x + 1$. The roots of that quadratic are $x = \frac{1 \pm \sqrt{5}}{2}$. The other solution to $x^3 = 2x + 1$ is $x = -1$.

The quantity D_k is just the value of the quadratic at $x = -1$, and so we have

$$A_k + B_k(-1) + C_k(-1)^2 = (-1)^k.$$