Warm-up

1. A sequence is defined by $a_n = n^2 - n$. What is a_3? What is a_{10}? Find $a_{n+1} - a_n$ in terms of n. Find $a_{n+1} - 2a_n + a_{n-1}$ in terms of n.

   - $a_3 = 3^2 - 3 = 6$. $a_{10} = 90$.
   - $a_{n+1} - a_n = ((n+1)^2 - (n+1)) - (n^2 - n) = 2n$.
   - $a_{n+1} - 2a_n + a_{n-1} = (a_{n+1} - a_n) - (a_n - a_{n-1}) = 2n - 2(n-1)$ using the previous part, and this is 2. Alternatively, without using the previous part,
   $a_{n+1} - 2a_n + a_{n-1} = ((n+1)^2 - (n+1)) - 2(n^2 - n) + ((n-1)^2 - (n-1)) = 2$.

2. A sequence is defined by $a_0 = 1$ and $a_n = a_{n-1} + 3$ for $n \geqslant 1$. Find $a_0 + a_1 + \cdots + a_{10}$. Find a_{1000}.

   - This is the sum of the first 11 terms of an arithmetic sequence with first term $a = 1$ and common difference 3. So the sum is $\frac{11}{2}(2 + 10 \times 3) = 176$.
   - $a_{1000} = 3001$.

3. A sequence is defined by $a_0 = 1$ and $a_n = \frac{a_{n-1}}{3}$ for $n \geqslant 1$. Find $a_0 + a_1 + \cdots + a_{10}$. Find a_{1000}. Does the sum of all the terms of this sequence converge? If it does, what is the sum to infinity?

   - This is the sum of the first 11 terms of a geometric sequence with first term $a = 1$ and common ratio $\frac{1}{3}$. So the sum is $\frac{\left(1 - 3^{-11}\right)}{1 - 3^{-1}} = \frac{3}{2}\left(1 - 3^{-11}\right)$.
   - $a_{1000} = 3^{-1000}$.
   - The common ratio is between -1 and 1 so the sum to infinity does converge. The sum to infinity is $\frac{1}{1 - 3^{-1}} = \frac{3}{2}$.

4. A sequence is defined by $a_0 = 1$ and $a_n = 3a_{n-1} + 1$ for $n \geqslant 1$. A sequence b_n is defined by $b_n = A \times 3^n + B$ where A and B are real numbers. Find values for A and B such that $a_n = b_n$ for all $n \geqslant 0$.

   - In particular, we would need $a_0 = b_0$ so $1 = A + B$. Also, we would need $a_1 = b_1$ so $4 = 3A + B$. So $A = \frac{3}{2}$ and $B = -\frac{1}{2}$.
   - We should check that this works for all n, so let's check that if $a_{n-1} = b_{n-1}$ then $a_n = b_n$. We know that $a_n = 3a_{n-1} + 1$ and we can work out $3b_{n-1} + 1 = 3 \times \frac{3}{2}3^{n-1} - \frac{3}{2} + 1 = \frac{3}{2}3^n - \frac{1}{2}$ which is exactly b_n. So if $a_{n-1} = b_{n-1}$ then $a_n = b_n$. The sequences match for all n.

5. A sequence is defined by $a_n = An^2 + Bn + C$ where A, B, and C are real numbers. Find A, B, and C in terms of a_0, a_1, and a_2.

- At first sight, this doesn't look like enough information; we haven't been told the values of any of the terms in the sequence! The key is that we're asked to give our answer in terms of the first three terms of the sequence without solving for what those are.

- For example, if we plug in $n = 0$ then we find $a_0 = C$. So we've got an expression for C in terms of a_0.

- Now plug in $n = 1$ and $n = 2$ to get $a_1 = A + B + C$ and $a_2 = 4A + 2B + C$. We want A and B in terms of the variables a_0, a_1, a_2, and we can use the fact that $C = a_0$ to eliminate C. These are simultaneous equations for A and B with solution

$$A = \frac{1}{2}\left(a_2 - 2a_1 + a_0\right), \qquad B = \frac{1}{2}\left(-a_2 + 4a_1 - 3a_0\right)$$

- Together with $C = a_0$, that gives a solution for A, B, and C in terms of a_0, a_1, and a_2.

- Compare this with the first question where we calculated $a_{n+1} - 2a_n + a_{n-1}$ for a particular sequence of this form.

6. Simplify $2^1 + 2^2 + 2^3 + \cdots + 2^n$ for $n \geqslant 1$.

   - This is the first n terms of a geometric sequence, with sum $2\left(2^n - 1\right)/(2 - 1) = 2^{n+1} - 2$.

7. Simplify $3^4 + 3^5 + 3^6 + \cdots + 3^n$ for $n \geqslant 4$.

   - This is the first $n-3$ terms of a geometric sequence, with sum $3^4(3^{n-3}-1)/(3-1) = \frac{1}{2}\left(3^{n+1} - 3^4\right)$.

8. When does the sum $1 + x^3 + x^6 + x^9 + x^{12} + ...$ converge? Simplify it in the case that it converges.

   - This is a geometric sequence with common ratio x^3, and the sum to infinity converges if $|x^3| < 1$, which is precisely $|x| < 1$.

   - In that case, it converges to $(1 - x^3)^{-1}$.

9. When does the sum $2 - x + \dfrac{x^2}{2} - \dfrac{x^3}{4} + \ldots$ converge? Simplify it in the case that it converges.

   - This is a geometric sequence with common ratio $-\frac{x}{2}$, and the sum to infinity converges if $\left|-\frac{x}{2}\right| < 1$, which is precisely $|x| < 2$.

   - In that case, it converges to $\dfrac{2}{1 + \frac{x}{2}} = \dfrac{4}{2 + x}$.

10. Consider the sum of the first n terms of an arithmetic sequence a_1, a_2, ..., a_n with $a_1 = a$ and $a_2 = a + d$. Explain why the sum of the ith term and the $(n + 1 - i)^{\text{th}}$ term doesn't depend on i, as long as $1 \leqslant i \leqslant n$. By considering separate cases where n is even or where n is odd, deduce that the sum of the first n terms of an arithmetic sequence is n times the average term.

   - The ith term is $a + (i - 1)d$ and the $(n + 1 - i)^{\text{th}}$ term is $a + (n - i)d$. Those sum to $2a + (n - 1)d$, which doesn't depend on i.

   - Case (i) If n is even, then we can pair up the terms like that, pairing the term a_1 to a_n and working inwards. In that case, every pair has the same sum (equal to the first term plus the last term), so the sum of all these terms is n times the average of the first and last terms.

   - Case (ii) If n is odd, then there is a left-over term in the middle when $i = (n+1)/2$. This value of this term is exactly equal to the average of the first and last terms, so overall the sum is once again the average of the first and last terms multiplied by the number of terms.

11. Consider the sum of the first n terms of an arithmetic sequence with first term a and constant difference d. Consider the special case $d = 0$. Write down the sum in this case. Now consider the case $a = 0$. In this case, write the sum in terms of the triangle numbers $T_n = 1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n + 1)$. Hence write down the sum of the first n terms of an arithmetic sequence. Check that this agrees with the formula above.

   - If $d = 0$ then every term is a, so the sum is na.

   - If $a = 0$ then the sum is $d(0 + 1 + 2 + \cdots + (n - 1))$, which is $dT_{n-1} = \frac{d}{2}(n - 1)n$.

   - The general case is the sum of the two cases above, so the sum is $na + \frac{d}{2}(n-1)n = \frac{n}{2}(2a + (n - 1)d)$

MAT questions

MAT 2016 Q1A

- The first few terms are $a_1 = 1$, $a_2 = l$, $a_3 = l^2$.

- This is a geometric sequence, but be careful because we're asked for the product of the terms and not the sum of the terms.

- We want $l_1 \times l_2 \times l_3 \times \cdots \times l_{15}$ which is $1 \times l \times l^2 \times \cdots \times l^{14}$

- Using laws of indices this is $l^{1+2+\cdots+14}$. So we actually want to add the terms of an arithmetic sequence in the exponent there!

- $1 + 2 + \cdots + 14 = 105$ using the formula for the sum of the terms of an arithmetic sequence, so the product is l^{105}.

- The answer is (d).

MAT 2016 Q1G

- The first few terms are $x_0 = 1$, $x_1 = x_0 = 1$, $x_2 = x_0 + x_1 = 2$, $x_3 = x_0 + x_1 + x_2 = 4$.

- I can spot a pattern here; for $n \geqslant 1$, it looks like $x_n = 2^{n-1}$. In fact, that's true; each term is double the one before because the sum of all previous terms includes (i) the most recent term (ii) the sum of all the terms before that, and that sum is equal to another copy of the most recent term.

- So our sequence x_n is 1, 1, 2, 4, 8, 16, 32, ...

- We're asked for the sum of one-over-each-term. That's
$$1 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots$$

- I recognise this sum (after the 1+1+ at the start), it's a geometric series.

- Overall, with the 1+1+ at the start, this adds up to 3.

- The answer is (d).

MAT 2017 Q1C

- The first few terms are
$a_1 = 2$, $a_2 = 6$, $a_3 = 3$, $a_4 = \frac{1}{2}$, $a_5 = \frac{1}{6}$, $a_6 = \frac{1}{3}$, $a_7 = 2$, $a_8 = 6$, $a_9 = 3$, $a_{10} = \frac{1}{2}$.

- This gets into a repeating pattern; $a_7 = a_1$ and also $a_8 = a_2$ and so on.

- So $a_{2017} = a_{2011} = a_{2005} = \ldots$. We should find the remainder when 2017 is divided by 6. This turns out to be 1, so $a_{2017} = a_1 = 2$.

- The answer is (d).

MAT 2016 Q5
First note that $s_1 = 2$ because the sum goes as far as $n2^n$, so if $n = 1$ then we go as far as $1 \times 2^1 = 2$. It's good to know where we're starting!

(i) We're told to set n to some small numbers. Just like in one of the warm-up problems, this gives us simultaneous equations for A and B and C. In this question, we have $s_1 = 2$ and $s_2 = 2 + 8 = 10$ and $s_3 = 2 + 8 + 24 = 34$. We also have $f(1) = 2(A + B) + C$, $f(2) = 4(2A + B) + C$, and $f(3) = 8(3A + B) + C$. So

$$2A + 2B + C = 2$$
$$8A + 4B + C = 10$$
$$24A + 8B + C = 34$$

(ii) Now we need to solve those equations; for example we could subtract the first equation from the second, and subtract the second equation from the third, and simplify, to get $3A + B = 4$ and $4A + B = 6$. The difference between these equations gives $A = 2$, then substitute this in to find $B = -2$ and $C = 2$.

(iii) Now we're asked to "show that" something, so we'll need to be careful to explain our answer. If $s_k = f(k)$ then $2 + 8 + 24 + \cdots + k2^k = (2k - 2)2^k + 2$. We'd like to know about s_{k+1}, which is just $s_k + (k + 1)2^{k+1}$ because it's the same sum with one extra term at the end. We need to show that this sum is equal to $f(k + 1)$.

Plugging in $f(k)$ for s_k and multiplying out all the brackets, I get $s_{k+1} = (2k-2)2^k + 2 + (k+1)2^{k+1} = k2^{k+1} - 2^{k+1} + 2 + k2^{k+1} + 2^{k+1}$. Collecting like terms, this is $(2k)2^{k+1} + 2$. I'll also work out $f(k + 1) = (2(k + 1) - 2)\,2^{k+1} + 2 = (2k)2^{k+1} + 2$, which matches exactly. So if $s_k = f(k)$ then $s_{k+1} = f(k + 1)$.

(iv) I'm looking for a sum like $2 + 8 + 24 + \ldots$ so that I can re-use the work we did for s_n. If I ignore the n's in t_n, I can see terms like $2(-1)$ and $4(-2)$ and $8(-3)$ which is the sum I'm looking for. So I'm going to try multiplying out the brackets;

$$t_n = n + 2n - 2 + 4n - 8 + 8n - 24 + \cdots + 2^{n-1}1.$$

Thinking carefully about what I'm doing to each bracket there, I can also write $2^{n-1}1$ as $2^{n-1}(n - (n - 1))$ and expand that bracket too! So

$$t_n = n + 2n - 2 + 4n - 8 + 8n - 24 + \cdots + 2^{n-1}n - 2^{n-1}(n - 1).$$

This is a geometric sequence $n + 2n + 4n + 8n + \cdots + 2^{n-1}n$ with s_{n-1} subtracted off. I've got expressions for both of those, so my answer is

$$t_n = (2^n - 1)n - (2(n - 1) - 2)2^{n-1} - 2 = 2^{n+1} - n - 2$$

The sum u_n is just t_n divided by 2^n with the terms in the opposite order. So $u_n = 2 - n2^{-n} - 2^{1-n}$.

(v) We've got a formula for s_k, and we want

$$\sum_{k=1}^{n} s_k = \sum_{k=1}^{n} \left((2k-2)2^k + 2\right) = 2\sum_{k=1}^{n} \left(k2^k\right) - 2\sum_{k=1}^{n} \left(2^k\right) + \sum_{k=0}^{n} 2$$

The last sum is simple, it's just n copies of 2 added together, so it's $2n$. The middle sum is the sum of the terms of a geometric sequence. The first sum is exactly the definition of s_n, which we've got a formula for! So altogether we've got

$$\sum_{k=1}^{n} s_k = 2s_k - 2(2^{n+1} - 1) + 2n$$

$$= 2\left((2n-2)2^n + 2\right) - 4(2^n - 1) + 2n$$
$$= n2^{n+2} - 2^{n+3} + 8 + 2n$$

Reflection

- Spotting a pattern was invaluable in two of the multiple-choice MAT questions. I can only spot a pattern if I work out a few of the terms. How many terms should I work out? I suppose I might keep working out terms until I can see what's going on, or until the sums get too hard.

- Are there other ways to work out t_n? Perhaps we could do something like the previous part of the question; guess a formula and choose constants to make it work.

- Are there other ways to work out $\sum_{k=1}^{n} s_k$? Perhaps we could expand each s_k in terms of the original definition and re-group the terms somehow?

- Lots of these questions ask us to switch between x_k and x_n and x_{n+1} and x_{n-1}. Are you happy making that substitution? Given the formula $s_n = (2n-2)2^n + 2$, can you work out s_{n-3} and s_{2n}? (We didn't need either of those to do the question, but you should be able to work them out without too much trouble!)

- MAT questions might involve more tricky sequences that aren't geometric sequences or arithmetic sequences, like the sequence in the long MAT question above. Even so, we might be able to use what we know about geometric sequences or arithmetic sequences along the way.