Mean field limits and singular kernels
Some recent advances
including
Derivation of the Vlasov-Fokker-Planck eqs

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Goal of the talk:

Well adapted dynamical weights to get result with low regularity

Topics:

- Compressible Navier-Stokes equations:
  Global existence à la Leray
- Mean Field limits (with singular kernels)
  - First order systems
  - Second order systems

A) Compressible Navier-Stokes equations:

$$\partial_t \rho + \mathrm{div}(\rho u) = 0$$

with velocity field u such that

$$u \in L^2_t W^{1,2}_x$$

and

$$\mathrm{div} u \approx \mathcal{A} p(\rho) \in L^q_{t,x} \text{ with } p > 2.$$

where

- $\mathcal{A}$ zero order non-local operator
- $s \mapsto p(s)$ a given function.

Compressible framework namely

$$\mathrm{div} u \text{ not necessarly } L^\infty.$$

A) Compressible Navier-Stokes equations;

Goal: Quantitative estimates (Stability)
$\implies$ To get global existence of weak solutions.

Tool:

$$\|\rho\|_{p,\theta} = \sup_{h \le 1/2} |\log h|^{-\theta} \int_{\Omega^2} \frac{|\rho(t,x) - \rho(t,y)|^p}{(h + |x-y|)^d} dx\, dy$$

Using that for $s > 0$, $0 < \theta < 1$ and $p \in [1, +\infty)$ we have

$$W^{s,p} \subset W^p_{\log,\theta} \subset L^p$$

which are compact.

Déf: $W^p_{\log,\theta} = \{u \in L^p : \|u\|_{p,\theta} < +\infty\}$.

1) Let us look at the propagation of the information

$$\int_\Omega \frac{|\rho_n(t,x) - \rho_n(t,y)|}{(h + |x-y|)^d}(w_n(t,x) + w_n(t,y))\,dxdy$$

with w_n solution of

$$\partial_t w_n + u_n \cdot \nabla w_n + \lambda P_n w_n = 0,$$

where $w_n|_{t=0} = 1$ and with P_n a positive penalization associated to (ρ_n, u_n) to be chosen and λ a large enough parameter to be chosen.

2) We must show some properties on the weigths w_n i.e.

$$0 \le w_n \le c_n \le 1, \qquad \int_\Omega \rho_n |\log w_N|^q < +\infty$$

with $q > 0$ to hope to get rid of the weights at the end and use the compact embedding given in the previous slide.

B) Mean Field limits

1) First order:
Liouville of forward Kolmogorov equation (general kernels)

$$\partial_t \rho_N + \sum_{i=1}^{N} \mathrm{div}_{x_i}\Big(\rho_N \frac{1}{N} \sum_{j\neq i}^{N} K(x_i - X_j)\Big) = \sum_{i=1}^{N} \sigma_N \Delta_{x_i} \rho_N$$

2) Second order:
Linear advection-diffusion equation (repulsive)

$$\partial_t f_N + \sum_{i=1}^{N} v_i \cdot \nabla_{x_i} f_N + \sum_{i=1}^{N} \frac{1}{N} \sum_{j\neq j}^{N} K(x_i - x_j) \cdot \nabla_{v_i} f_N = \frac{\sigma^2}{2} \Delta_{v_i} f_N$$

with

$$K = \nabla V.$$

$\mathrm{div}\, K$ not necessarly L^∞.

Goal: Quantitative estimates

$\Longrightarrow$ To get mean field limit justification

Tools:

Appropriate weights: modification of pure Gaussian

- 1) Modulated free energy (First order system).
- 2) L^p estimate (Second order system).

References

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Mean Field limits and particle systems:

- D.B., P.–E. J. and Z.W: C.R. Acad Sciences Maths, vol. 357, Issue 9, 708-720 (2019)
- D.B., P.–E.J and Z.W. Repulsive kernels: https://slsedp.centre-mersenne.org/journals/SLSEDP/
- D.B., P.–E. J. and Z.W: Attractive kernels: See https://arxiv.org/pdf/2011.08022.pdf
- D.B., P.–E. J. and J.S.: A new approach to the mean-field limit of Vlasov-Fokker-Planck equations. See https://arxiv.org/pdf/2203.15747.pdf

and references cited therein.

Interacting particles - The questions

Consider N particles, identical and interacting two by two through the kernel K. For $X_i(t) \in \Pi^d$ the position of the i-th particle,

$$d\,X_i = \frac{1}{N} \sum_{j \neq i} K(X_i - X_j)\,dt + \sqrt{2\,\sigma}\,dW_i,$$

with the mean field scaling and for N independent Brownian motions W_i^t where

$$K = -\nabla V \qquad \text{Gradient flow}$$

The most classical case: Poisson law with $d = 2$ (Patlak-Keller-Segel)

$$V(x) = \lambda \log |x| + \underline{\textit{perturbation}},$$

Main question: Behavior of the system as $N \to \infty$.

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For simplicity in the talk: σ is fixed but $\sigma = \sigma_N$ is also of interest.

Motivations:

- Attractive Kernel :
  The Patlak-Keller-Segel.
- Repulsive Kernel :
  More general kernel than Riesz potential, Coulomb potential.

Here Focus on attractive PKS Kernel
and comment on repulsive cases too.

Focus the joint law $\rho_N(t, x_1, \cdots, x_N)$ of the process $(X_1, \cdot, X_N)$ which solves the Liouville or forward Kolmogorov equation

$$\partial_t \rho_N + \sum_{i=1}^N \mathrm{div}_{x_i}(\rho_N \frac{1}{N} \sum_{j \neq i}^N K(x_i - x_j)) = \sigma \sum_{i=1}^N \Delta_{x_i} \rho_N$$

$$\rho_N|_{t=0} = \rho_N^0 \text{ such that } \int_{\Pi^{dN}} \rho_N^0 = 1.$$

Some of the questions to answer

- Well posedness for a finite N.
  Difficulty: Singularity of the attractive force kernel.
  See Cattiaux-Pédèches, Fournier-Jourdain if $\lambda < \sigma$.
- Stability estimates for finite N.
  Difficulty: Still the singularity of the force kernel and exponential growth in N of the estimates.
- The Mean Field limit: Use the large number of particles to try to find and justify that a continuum equation provides a good approximation to the system. Here the tentative limit is

$$\begin{cases} \partial_t \bar\rho + \mathrm{div}\,(\bar\rho\, u) = \sigma\,\Delta\,\bar\rho, \\ u = -\nabla V \star_x \bar\rho, \end{cases}$$

where $\bar\rho$ is the 1-particle distribution, $\bar\rho \geq 0$, $\int \bar\rho = 1$.

Our guiding example: The Patlak-Keller-Segel system

For $V = \lambda \log|x|$ on $\mathbb{R}^d$ with $d = 2$, then one finds for the limit

$$\begin{cases} \partial_t \bar\rho + \mathrm{div}\,(\bar\rho\, u) = \sigma\,\Delta\,\bar\rho, \\ u = \nabla\Phi, \quad -\Delta\Phi = 2\,\pi\,\lambda\,\bar\rho. \end{cases}$$

Well known basic model of chemotaxis (but not very accurate).
Solutions may not exist for all times as the singular attractive interactions can lead to concentration: From Dolbeault-Perthame

- Global existence of classical solution if $\lambda \leq 4\,\sigma$ (or $\lambda \leq 2\,d\,\sigma$).
- Always blow-up if $\lambda > 4\,\sigma$.

Based on the free energy of the system

$$\int \bar\rho \log\bar\rho\,dx + \frac{\lambda}{2}\int \log|x-y|\,\bar\rho(x)\,\bar\rho(y)\,dx\,dy.$$

In Blanchet-Dolbeault-Perthame :
Existence of global weak satisfying free energy control with subcritical mass.

Some of the Existing Literature

The mean field limit has been proved for:

- The Lipschitz case is still important case to further understand the framework (see for example Golse, Hauray, Mischler, Mouhot, Ricci...).
- Deterministic singular cases are much better understood (2d incompressible Euler in Goodman, Hou and Lowengrub, Schochet, general result by Hauray) for instance in fluid mechanics.
- For 2d Navier-Stokes, if $K = \nabla^{\perp} V$, qualitative convergence by Osada, Fournier-Hauray-Mischler.
- For the Patlak-Keller-Segel system, various attempts by Cattiaux-Pédèches, Godinh-Quininao, Haskovec-Schmeiser... Recently Fournier-Jourdain proved limit for $\lambda < \sigma$, with no quantitative estimates.

Two new approaches

Two key results appeared recently

- In Jabin-Wang, new estimates relative entropy of joint law

$$\frac{1}{N}\int_{\Pi^{dN}} \rho_N(t,X^N)\log\left(\frac{\rho_N(t,X^N)}{\bar\rho_N(t,X^N)}\right)dX^N$$

where $\bar\rho_N = \bar\rho^{\otimes N} = \Pi_{i=1}^N \bar\rho(t,x_i)$ and ρ_N the joint law of the process $(X_1,\cdots,X_N)$ which satisfies the Liouville equation

$$\partial_t\rho_N + \sum_{i=1}^N \mathrm{div}_{x_i}\left(\rho_N \frac{1}{N}\sum_{j\neq i}^N K(x_i-x_j)\right) = \sigma\sum_{i=1}^N \Delta_{x_i}\rho_N.$$

They give optimal rates of convergence in $\frac{1}{\sqrt{N}}$ provided that K, $\mathrm{div}\, K \in W^{-1,\infty}$.

$\longrightarrow$ very well – 2d Navier-Stokes because $\mathrm{div} K = 0$.

$\longrightarrow$ very poorly – gradient flows: Log-Lipschtiz Kernels.

See L. St Raymond, Bourbaki 70ème année, 2017–2018, no 1143.

Remark/idea: $\operatorname{div} K$ in relative entropy propagation is a bad term!

Similar quantity $\operatorname{div} u$ appears for compressible Navier-Stokes eqs. D.B., P.–E. Jabin (published *Ann. Math.* 2018) introduced weights satisfying PDE related to the unknowns in the quantity encoding the low regularity to cancel bad terms and prove some quantitative estimates on the density leading to compactness.

Modify the relative entropy here to cancel $\operatorname{div} K$ and conclude ?

This was the starting point question with P.–E. Jabin and Z. Wang.

- Duerinckx-Serfaty and Serfaty (for $\sigma = 0$):
  Modulated potential energy

$$\frac{1}{2}\int_{\Pi^{2d}\cap\{x\neq y\}} V(x-y)(\mu_n(dx)-\bar{\rho}(x)dx)(\mu_N(dy)-\bar{\rho}(y)dy)$$

(with $\mu_N = \left(\sum_{i=1}^{N}\delta(x-X_i(t))\right)/N$ the empirical measure) allows to deal with repulsive singular Riesz potential of the type

$$V = \frac{C}{|x|^{\alpha}} \text{for } C > 0 \text{ and } \alpha < d$$

$\longrightarrow$ Works beyond Poisson kernel.

$\longrightarrow$ Does not work

for stochastic systems, attractive potentials.

$\longrightarrow$ Use the explicit formula of the kernel
allowing to reformulate the energy
in terms of potential or extension representation
(for the fractional laplacian) by Caffarelli-Silvestre

We can write a result with a singular kernel combinaison of regular, attractive and repulsive kernels: See Note CRAS.

For simplicity we separate in the sequel and this explains the two papers mentioned in the introduction.

Our new repulsive result

Repulsive kernels hypothesis: Consider even potentials

$$V(-x) = V(x) \text{ where } V \in L^p(\Pi^d) \text{ for some } p > 1$$

and the point-wise controls for all $x \in \Pi^d$: $\exists k$ and $C > 0$ such that

$$|\nabla V(x)| \leq \frac{C}{|x|^k}, \qquad |\nabla^2 V(x)| \leq \frac{C}{|x|^{k'}} \text{ for some } k, k' > 1/2$$

and

$$|\nabla V(x)| \leq C \frac{V(x)}{|x|}.$$

We also assume that

$$\lim_{|x| \to 0} V(x) = +\infty, \qquad V(x) \leq C V(y) \text{ for all } |y| \leq 2|x|.$$

We also assume the following properties on the Fourier transform

$$\hat{V}(\xi) \geq 0 \text{ for all } \xi \in \mathbb{R}^d$$

and

$$\nabla_\xi \hat{V}(\xi) \leq \frac{C}{1+|\xi|}(\hat{V}(\xi)+f(\sigma)\frac{1}{1+|\xi|^{d-\alpha}}) \text{ with } 0 < \alpha < d \text{ for all } \xi \in \mathbb{R}^d$$

where $f(\sigma) = 0$ if $\sigma \to 0$ and $f(\sigma) = 1$ if σ is fixed.

Our new repulsive results

Theorem

Assume $K = -\nabla V$ with V satisfying the above conditions. Consider $\bar\rho$ a smooth enough solution with $\inf \bar\rho > 0$. There exists $C > 0$ and $\theta > 0$ (or $\eta(N)$ with $\eta(N) \to 0$ as $N \to +\infty$) s.t. for $\bar\rho_N = \Pi_{i=1}^N \bar\rho(t, x_i)$, and for the joint law ρ_N on Π^{dN} of any entropy solution to the SDE system, for σ fixed

$$H_N(t) + |\mathcal{K}_N(t)| \leq e^{C_{\bar\rho}\,\|K\|\,t} \left(H_N(t=0) + |\mathcal{K}_N(t=0)| + \frac{C}{N^\theta} \right),$$

Hence if $H_N^0 + |\mathcal{K}_N^0| \leq C\,N^{-\theta}$, for any fixed marginal $\rho_{N,k}$

$$\|\rho_{N,k} - \Pi_{i=1}^k \bar\rho(t, x_i)\|_{L^1(\Pi^{k\,d})} \leq C_{T,\bar\rho,k}\,N^{-\theta}.$$

where
$\rho_{N,k}(t, x_1, \cdots, x_k) = \int_{\Pi^{(N-k)d}} \rho_N(t, x_1, \cdots, x_N)\,dx_{k+1} \cdots dx_N$ *observable or marginal of the system at a fixed rank k.*

Our new repulsive results

Theorem

Assume $K = -\nabla V$ with V satisfying the above conditions. Consider $\bar\rho$ a smooth enough solution with $\inf \bar\rho > 0$. There exists $C > 0$ and $\theta > 0$ (or $\eta(N)$ with $\eta(N) \to 0$ as $N \to +\infty$) s.t. for $\bar\rho_N = \Pi_{i=1}^N \bar\rho(t, x_i)$, and for the joint law ρ_N on Π^{dN} of any entropy solution to the SDE system, for $\sigma = \sigma_N \to 0$,

$$\sigma_N \mathcal{K}_N(t) \leq e^{C_{\bar\rho}\, \|K\|\, t} \left(\sigma_N \mathcal{K}_N(t=0) + \sigma_N \mathcal{H}_N((t=0) + \eta(N)\right),$$

Hence if $\sigma_N \mathcal{K}_N(t=0) + \sigma_N \mathcal{H}_N((t=0) \leq \eta(N)$, for any fixed marginal $\rho_{N,k}$

$$W_1(\rho_{N,k},\ \Pi_{i=1}^k \bar\rho(t, x_i)) \leq C_{T,\bar\rho,k}\, \eta(N).$$

Our new attractive result

Attractive kernel hypothesis:

We introduce the following hypothesis

$$V(-x) = V(x), \qquad V \in L^p(\Pi^d) \cap \mathcal{C}^2(\Pi^d \setminus \{0\}) \text{ for some } p > 1$$

$$V(x) \geq \lambda \log |x| + C \text{ for some } 0 \leq \lambda < 2d\sigma$$

$$|\nabla V(x)| \leq \frac{C}{|x|}$$

Theorem

Assume $K = -\nabla V$. Consider $\bar\rho \in L^\infty(0,T; W^{2,\infty}(\Pi^d))$ solves the limit system with $\inf \bar\rho > 0$. Assume finally that $\lambda < 2d\sigma$. There exists $C > 0$ and $\theta > 0$ such that $\bar\rho_N = \Pi_{i=1}^N \bar\rho(t, x_i)$ and for the joint law ρ_N on Π^{dN} of any entropy solution to the SDE system then

$$\mathcal{H}_N(t) + |\mathcal{K}_N(t)| \le e^{C_{\bar\rho}\|K\| t}\Big(\mathcal{H}_N(t=0) + |\mathcal{K}_N(t=0)| + \frac{C}{N^\theta}\Big).$$

Hence if $\mathcal{H}_N(t=0) + |\mathcal{K}_N(t=0)| \le CN^{-\theta}$, then for any fixed marginal $\rho_{N,k}$

$$\|\rho_{N,k} - \Pi_{i=1}^k \bar\rho(t, x_i)\|_{L^1(\Pi^{kd})} \le C_{T,\bar\rho,k} N^{-\theta}.$$

The Liouville equation

The Liouville eq. describes the evolution of the law $\rho_N(t, x_1, \ldots, x_N)$ of the distribution of the particles

$$\partial_t \rho_N + L_N \, \rho_N = 0,$$

where

$$L_N \, \rho_N = \sum_{i=1}^{N} \frac{1}{N} \sum_{j \neq i} K(x_i - x_j) \cdot \nabla_{x_i} \rho_N - \sigma \sum_i \Delta_{x_i} \rho_N.$$

The Liouville equation encompasses all the relevant statistical information about the dynamics.

Why is the entropy critical

Due to the additive nature of the entropy

$$\frac{1}{k}\int_{\Pi^{k\,d}} \rho_{N,k}\,\log \rho_{N,k} \leq \frac{1}{N}\int_{\Pi^{N\,d}} \rho_N\,\log \rho_N,$$

and similarly

$$\frac{1}{k}\int_{\Pi^{k\,d}} \rho_{N,k}\,\log \frac{\rho_{N,k}}{\bar\rho^{\otimes k}} \leq \frac{1}{N}\int_{\Pi^{N\,d}} \rho_N\,\log \frac{\rho_N}{\bar\rho_N}.$$

$\Longrightarrow$ Controls $\|\rho_{N,k}-\bar\rho^{\otimes k}\|_{L^1}$ by Csiszár-Kullback-Pinsker ineq.

The Gibbs equilibria

Denote by G_N the Gibbs equilibrium of the system, and by $G_{\bar\rho_N}$ the corresponding distribution where the exact field is replaced by the mean field limit according to the law $\bar\rho$,

$$G_N(t, X^N) = \exp\Big(-\frac{1}{2N\sigma}\sum_{i\neq j} V(x_i - x_j)\Big),$$

$$G_{\bar\rho_N}(t, X^N) = \exp\Big(-\frac{1}{\sigma}\sum_{i=1}^{N} V\star\bar\rho(x_i) + \frac{N}{2\sigma}\int_{\Pi^d} V\star\bar\rho\,\bar\rho\Big),$$

$$G_{\bar\rho}(t, x) = \exp\Big(-\frac{1}{\sigma} V\star\bar\rho(x) + \frac{1}{2\,\sigma}\int_{\Pi^d} V\star\bar\rho\,\bar\rho\Big).$$

Our method uses the modified relative entropy

$$E_N\Big(\frac{\rho_N}{G_N} \,\Big|\, \frac{\bar\rho_N}{G_{\bar\rho_N}}\Big) = \frac{1}{N}\int_{\Pi^{dN}} \rho_N(t, X^N)\log\Big(\frac{\rho_N(t, X^N)}{G_N(X^N)}\,\frac{G_{\bar\rho_N}(t, X^N)}{\bar\rho_N(t, X^N)}\Big)\,dX^N.$$

A modified free energy

One may also write

$$E_N\left(\frac{\rho_N}{G_N} \mid \frac{\bar\rho_N}{G_{\bar\rho_N}}\right) = \mathcal{H}_N(\rho_N|\bar\rho_N) + \mathcal{K}_N(G_N|G_{\bar\rho_N}),$$

where

$$\mathcal{H}_N(\rho_N|\bar\rho_N) = \frac{1}{N}\int_{\Pi^{dN}} \rho_N(t,X^N)\log\left(\frac{\rho_N(t,X^N)}{\bar\rho_N(t,X^N)}\right) dX^N$$

is exactly the relative entropy introduced in Jabin-Wang and

$$\mathcal{K}_N(G_N|G_{\bar\rho_N}) = -\frac{1}{N}\int_{\Pi^{dN}} \rho_N(t,X^N)\log\left(\frac{G_N(t,X^N)}{G_{\bar\rho_N}(t,X^N)}\right) dX^N$$

is expectation of modulated potential energy in Serfaty, Duerinckx multiplied by $1/\sigma$.

$\longrightarrow$ E_N is a modulated free energy for the system.

The time evolution of E_N

The modulated free energy E_N has the right algebraic structure with for any V even that

$$E_N\left(\frac{\rho_N}{G_N}\,\Big|\,\frac{\bar\rho_N}{G_{\bar\rho_N}}\right)(t) \le E_N\left(\frac{\rho_N}{G_N}\,\Big|\,\frac{\bar\rho_N}{G_{\bar\rho_N}}\right)(0)$$
$$-\frac{\sigma}{N}\int_0^t\int_{\Pi^{d\,N}} d\rho_N\left|\nabla\log\frac{\rho_N}{\bar\rho_N}-\nabla\log\frac{G_N}{G_{\bar\rho_N}}\right|^2$$
$$-\frac{1}{2}\int_0^t\int_{\Pi^{dN}}\int_{\Pi^{2\,d}\cap\{x\neq y\}}\nabla V(x-y)\cdot\left(\nabla\log\frac{\bar\rho}{G_{\bar\rho}}(x)-\nabla\log\frac{\bar\rho}{G_{\bar\rho}}(y)\right)$$
$$(d\mu_N-d\bar\rho)^{\otimes 2}d\rho_N,$$

where $\mu_N=\frac{1}{N}\sum_{i=1}^N\delta(x-x_i)$ is the empirical measure.

The main points of the proof

The previous simple expression leaves two main points in the proof

- Bound the right-hand side in terms of E_N.
- Show that E_N is almost positive or more specifically that for some constant C

$$E_N\left(\frac{\rho_N}{G_N}\,\Big|\,\frac{\bar\rho_N}{G_{\bar\rho_N}}\right)(t)\geq\frac{1}{C}\,\mathcal{H}_N\left(\rho_N\,|\,\bar\rho_N\right)(t)-\frac{C}{N^\theta}.$$

Remark. Combining the relative entropy with a modulated energy: Very successfully used for various limit in kinetic theory.
– Quasi neutral limit (see D.H Kwan, M. Puel, L. Saint-Raymond),
– Vlasov-Maxwell-Boltzmann to incompressible viscous EMHD (see D. Arsenio, L. Saint-Raymond)

Main points of the proof - Attractive case

Upper bound: One uses $|\nabla V(x)| \leq C/|x|$ to ensure an L^∞ bound on

$$\nabla V(x-y) \cdot \left(\nabla \log \frac{\bar\rho}{G_{\bar\rho}}(x) - \nabla \log \frac{\bar\rho}{G_{\bar\rho}}(y) \right)$$

and use Large deviation theorem by Jabin–Wang (Inventiones 2018).

Lower Bound: We prove that there exists $\delta < 1$ and $\eta > 0$ depending on $\|V\|_{L^p}$ and λ st. for any smooth function χ with $\chi(x) = 1$ if $x < 1/2$ and $\mathrm{supp}\chi \in [0,1]$ then

$$-\frac{1}{2\sigma} \int_{\Pi^{dN}} \int_{\Pi^2 \cap \{x \neq y\}} V(x-y)\chi(|x-y|/\eta)(d\mu_N - d\bar\rho)^{\otimes 2} \rho_N dX^N$$
$$\leq \delta \mathcal{H}(\rho_N | \bar\rho_N) + \frac{C}{N^{1/(2d+1)}}(\|\log \bar\rho\|_{W^{1,\infty}} + \eta^{-1}) + \frac{C}{N^{1/(2(2d+1))}}.$$

Main points of the proof - Repulsive case

This uses an important regularization lemma. There exists smooth approximation V_ε of V and a function $\eta(\varepsilon)$ with $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that

$$\hat{V}_\varepsilon \geq 0$$

$$\|V_\varepsilon - V\|_{L^1(\Pi^d)} \leq \eta(\varepsilon), \qquad V_\varepsilon(x) \leq V(x) + \varepsilon \text{ for all } x.$$

$$\|1_{|x|\geq\delta}(V - V_\varepsilon)\|_{L^1} \leq C\frac{\varepsilon}{\delta^k}, \qquad \|1_{|x|\geq\delta}(\nabla V - \nabla V_\varepsilon)\|_{L^1} \leq C\frac{\varepsilon}{\delta^{k'}}$$

Lower bound. Show by truncation and regularization that there exists a function $\eta(N)$ with $\eta(N) \to 0$ as $N \to +\infty$, on a that

$$\int_{\Pi^{2d}\cap\{x\neq y\}} V(x-y)(d\mu_N - d\bar{\rho})^{\otimes 2} \geq -\eta(N).$$

Upper bound. In Duerinckx, Serfaty proofs, the modulated energy is rewritten in terms of a potential h^μ satisfying an elliptic equation. In the Coulomb case, the potential is explicit and in the Riesz case, the introduction on an extra space variable allows to transform the non local operator in a local operator (Caffarelli-Silvestre paper).

Here we do not use such transformation.

$\Longrightarrow$ We come back to Fourier transform of the quantity and use the properties on $\hat{V}$ to get an appropriate controlable upper-bound

Remark: Use the regularization Lemma again.

D. B., P.–E. Jabin, J. Soler. A new approach to the mean-field limit of Vlasov-Fokker-Plank equations. See https://arxiv.org/pdf/2203.15747.pdf.

Second order particle system:

$$\frac{d}{dt}X_i(t) = V_i(t), \quad X_i(t=0) = X_i^0,$$
$$dV_i(t) = \frac{1}{N}\sum_{j\neq i} K(X_i - X_j)\,dt + \sigma\,dW_i, \quad V_i(t=0) = V_i^0, \tag{1}$$

where the W_i are N independent Wiener process, and where K is a pairwise interaction kernel deriving from a potential

$$K = -\nabla\phi$$

for a positive, even potential ϕ. For simplicity we take the positions X_i on the torus Π^d, while the velocities lie in $\mathbb{R}^d$.

Goal: Derive the mean-field limit for (1) which is a kinetic, Vlasov-Fokker-Planck equation posed on the limiting 1-particle density $f(t,x,v)$

$$\partial_t f + v \cdot \nabla_x f + (K \star_x \rho) \cdot \nabla_v f = \frac{\sigma^2}{2} \Delta_v f \quad \text{with} \quad \rho = \int_{\mathbb{R}^d} f dv.$$

The mean-field of the Vlasov-Poisson equation has remained a long-standing open problem due to the difficulty in general of handling singular kernels for second-order model.

Known results:
In one dimension: Singular kernels.

Higher space dimensions: Regularized and cutted-off Kernels ; convergence results of discrete approximation for multidimensional Vlasov-Poisson systems with mollified mass and charge density.

The marginals $f_{k,N}$ associated to the N-particle joint law f_n satisfies the BBGKY system

$$\partial_t f_{k,N} + \sum_{i=1}^{k} v_i \cdot \nabla_{x_i} f_{k,N} + \sum_{i\leq k} \frac{1}{N} \sum_{j\leq k} K(x_i - x_j) \cdot \nabla_{v_i} f_{k,N}$$
$$+ \frac{N-k}{N} \sum_{i\leq k} \nabla_{v_i} \cdot \int_{\Pi^d \times \mathbb{R}^d} f_{k+1,N}\, K(x_i - x_{k+1})\, dx_{k+1}\, dv_{k+1}$$
$$= \frac{\sigma^2}{2} \sum_{i\leq k} \Delta_{v_i} f_{k,N}.$$

where

$$f_{k,N}(t, x_1, v_1, \ldots, x_k, v_k) =$$
$$\int_{\Pi^{d(N-k)} \times \mathbb{R}^{d(N-k)}} f_N(t, x_1, v_1 \ldots, x_N, v_N)\, dx_{k+1}\, dv_{k+1} \ldots dx_N\, dv_N.$$

Goal: The limit system is the tensorized Vlasov hierarchy

Theorem. Assume f is a smooth solution to VPFP. Assume K in L^p with $p > 1$ and $K = -\nabla\phi$ with $\phi \geq 0$. Consider initial data such that

$$\int |f_{N,k}^0|^p \exp(e_k/\lambda) \leq F^k$$

for all k independent of N. Then $f_{N,k}$ weakly converge to $f^{\otimes k}$ in L^q_{loc} space for any $q > 2$ such that $1/q + 1/p \leq 1$ on $[0, T^*]$ with T^* independent of N.

Main idea:

Derive uniform L^q_w with appropriate dynamical weights w.

Let $d = 2$ and consider the Poisson kernel $K = -\nabla\phi$ with associated potential $\phi(x) = -\ln|x|$. Then the convergence leading to the Vlasov-Poisson-Fokker-Planck system holds.

The dynamical weight $w = \exp(\lambda(t)e_k)$ is linked to a reduced energy e_k namely an energy of reduced k particles

$$e_k(x_1, v_1, \cdots, x_k, v_k) = \sum_{i\le k}(1 + |v_i|^2) + \frac{1}{N}\sum_{i,j\le k}\phi(x_i - x_j)$$

observing that

$$L_k e_k = 0$$

where

$$L_k = \sum_{i\le k} v_i \cdot \nabla_{x_i} + \frac{1}{N}\sum_{i\le k}\sum_{j\le k} K(x_i - x_j)\cdot\nabla_{v_i}$$

and $\lambda(t) = 1/(\Lambda(1+t))$.

Thank you !!