The Gaussian Copula and the Financial Crisis: A Recipe for Disaster or Cooking the Books?

Assignment for Module 8 (Quantitative Risk Management)

[redacted]

University of Oxford

June 2016

Contents

1 Introduction

After the dust of the 2007-2008 financial crisis had settled, many contributing causes were given. One of these was complex mathematical modelling. The financial model that came under particularly strong criticism was the Gaussian copula. This is a statistical tool that was used to price and manage the risk of Collatarised Debt Obligations (CDOs). These complex financial products were the spark to the eventual roaring fire of "the Crisis"[9] (as it shall be referred to henceforth).

One influential article that lay the blame for the Crisis squarely at the feet of the Gaussian copula was Felix Salmon's 'The Formula that Killed Wall Street'[18]. In this widely published article he describes the Gaussian copula as a "recipe for disaster". He considers it a mathematically "beautiful", but fatally flawed, model adopted for its simplicity and tractability that inflated a colossal bubble in CDOs, that kept on growing until the model "fell apart" in 2007-08, leading to trillions of dollars of losses.

In this paper, the Gaussian copula will be investigated in the context of the Crisis and its suitably as a model for CDOs discussed. It will be compared with other copulas that have been proposed as alternative, more "representative" models. An alternative hypothesis will also be investigated: that the main cause of the catastrophic losses in CDOs was not the copula choice but the gaming of credit ratings based on incorrect correlation estimations; a so-called "cooking of the books" to extend Felix Salmon's gastronomic metaphor.

The paper is structured as follows: first CDOs and their role in the Crisis will be summarised. Next, in Section 3, copulas will be introduced and how to use them to model CDOs explained. During this section the Gaussian copula and alternative copulas will be proposed and their suitability considered. Then in Section 4 these copula models will be used to actually price an idealised CDO and their performance compared. In Section 5 the alternative hypothesis above will be investigated. This includes assessing the performance of the idealised CDO under differing correlation assumptions. Finally we conclude.

2 The Financial Crisis

A complete overview of the Crisis is naturally far beyond the scope of this paper. However, it is widely accepted that the root of the crisis was Collaterized Debt Obligations (CDOs) of Asset Backed Securities (ABSs) that mainly consisted of American subprime mortgages[4]. We will summarise the pertinent aspects of the Crisis here, based on Donnelly & Embrechts[4], Mackenzie & Spears[12], and Lewis[9]

2.1 The Ingredients

Mortgage Backed Securities (MBSs) first appeared in the seventies. Bankers produce an MBS by "securitising" mortgages, i.e. pooling mortgages together and selling securities that receive parts of the total mortgage payments from the pool. This diversifies the risk and allows investors to gain exposure to mortgages without taking on idiosyncratic risk. Not long after MBSs came Collaterised Mortgage Obligations (CMOs). These were similar to MBSs but instead of just passing through payments to investors and splitting them evenly, the payments were tranched to produce securities with a range of risk profiles.

ABSs and CDOs have similar structures to MBSs and CMOs but instead of only having mortgages as underlyings, they can have any appropriate asset or debt. It is common for CDOs to contain ABSs, or even other CDOs (at least before the Crisis).

Banks were motivated to produce CDOs as they remove risky underlyings from their books and return cash. Doing this helped satisfy capital requirements while also producing fat fees.

Many institutional investors are limited to purchasing products with a high credit rating (AAA or AA) by a Credit Rating Agency (CRA) such as S&P, Moody's or Fitch. Sometimes there are not enough such products available to satisfy demand. CDOs provided a (very lucrative) solution to this pent up demand. Risky assets, rated BBB or lower for example, could be bundled together to produce large AAA rated super-senior tranches that could be sold to deep-pocketed, risk averse investors.

So why did it take over a decade and a half for the CDO market to get big? One of the significant impediments was modelling. CDOs are complex products with interconnected cashflows and, as such, are difficult to model, and therefore price, accurately. In the nineties this, along with other factors, limited the issuances.

2.2 The Recipe

The problem is correlation. If all the underlying assets of the CDO were independent the modelling would be straightforward: assess the default distribution of each individual asset separately and multiply them together to produce the joint distribution[1]. Of course, in reality the underlyings, no matter how diversified, are never fully independent (as the financial crisis so painfully proved).

Even estimating the correlation of two assets with long histories is not a simple task. This difficulty is quickly exacerbated with a CDO since there are $n(n-1)/2$ correlations between n underlyings, compounded with the fact that most obligors are idiosyncratic with limited public data (e.g. mortgages).

Various models for CDOs were proposed pre-millenium, such as Vasicek's analytical KMV and J.P.Morgan's Monte Carlo CreditMetrics. However, KMV's homogeneous pool was impractical and the CreditMetrics model has difficult to estimate parameters as well as being limited to a one-period framework. Then in 1999 a Chinese quant in Canada, David Li, published a paper[10] proposing a straightforward, tractable and easy to parameterise model for dependent defaults, used by actuaries to solve the broken heart syndrome[2]. This was the, now infamous, Gaussian copula. With parallels to Black-

[1] This is actually the independence copula we will discuss in Section 3.

[2] This is the well documented effect where the life expectancy of a person shortens when their spouse dies.

Scholes, Li's model also provided delta hedging ratios for single-tranche CDOs. Traders were quick to adopt it.

Armed with this new model, banks could issue CDOs at a much faster pace[3] and hedge them. Importantly for its popularity, it allowed traders to book day one P&L boosting their bonuses. Also, crucially, the CRAs adopted the Gaussian copula to rate CDO tranches[4].

2.3 Growth in Popularity

In the early 2000s, the adoption of the Gaussian copula to model CDOs coincided with low interest rates leading to high demand for yield on safe assets. It was possible to create senior CDO tranches that could offer higher yields than alternative products while having the rating of treasuries, thus they were in great demand.

The growth in the CDO market was not completely smooth. In 2005 there was a so-called "correlation crisis". Much like the volatility skew in options, CDO tranches also exhibited a "correlation skew". The market prices of a CDO's tranches did not all match the same correlation parameter. In May 2005 General Motors and Ford were both downgraded to junk status despite benign economic conditions. This idiosyncratic risk caused equity tranche protection to increase and mezzanine protection to fall, sharply steepening the correlation skew. This crisis only lasted a few days and was barely reported on. The CDO market continued to grow unabated.

At the beginning of 2000, when Li's article was published, the total issuance of CDOs was worth less than $70 billion. By 2006 and 2007, at the height of the CDO bubble, global issuance had grown to over $500 billion per year.

2.4 Hygiene standards decline

Soon the number of available risky assets to collaterize was running low so banks looked for more and more risky assets. This led to a boom in the subprime mortgage market. As CDOs snapped up subprime ABSs the more experienced and thorough ABS investors were side-lined. But the bankers in the "assembly line" producing CDOs were not as careful as the traditional investors. They were focused on achieving the best CRA ratings, by gaming the CRAs' models, and were not concerned with the accuracy of these rating since the risk would be sold on. The standard of subprime mortgage underwriting started to fall.

The chain between original loan and eventual holder had become so long and complicated it was nigh on impossible to actually understand what was going on[5]. This led to investors in CDOs and, more alarmingly, CDO managers often relying purely on

[3] In only a matter of days according to Mackenzie[12]

[4] Moody's adopted it in August of 2004. Only a week later S&P changed its methodology as well[7]. It appears there is high correlation between the actions of CRAs.

[5] In The Big Short[9], when Michael Burry says he's looking at the mortgages underlying MBSs and actually reading the contracts the reply from his investor is "No one reads them. Only the lawyers who put them together read them".

the CRA's ratings, obtained using the Gaussian copula model, to judge tranche risk. In Mackenzie [12] it's remarked that in an interview with a senior figure at a firm that managed ABS CDOs Mackenzie was shocked to find out the firm had no model of its own. This meant that if the CRAs were wrong, as they turned out to be, huge swathes of the market were also wrong.

Then in mid-2007, subprime mortgages started to default en masse. This was due to a combination of a change in bankruptcy laws as well as adjustable teaser-rates ending. Despite mortgages being widely modelled as consisting of an economic factor and idiosyncratic factor, large numbers defaulted even though the economy was doing relatively well.

This is when Salmon[18] says the Gaussian copula model started to show cracks "that eventually became gaping chasms". It was no longer possible to match market prices using the model, no matter what correlation parameter was used.

2.5 Disaster

In the late 2000s, default rates reached above 14% in America, a number many investors had claimed was "impossible"[9]. Huge numbers of mezzanine tranches of CDOs, crammed with subprime mortgages, defaulted, despite being rated low risk by CRAs. 80% of ABS CDOs issued in 2007 had a default event. Then all the synthetic CDOs based on the same underlyings defaulted as well. Banks lost billions. Citigroup lost $34 billion on ABS CDOs, Merrill Lynch $26 billion, UBS $22 billion and AIG lost $33 billion (receiving the equivalent of 1% of US GDP to bail it out)[12]. This lead to the largest financial crash since the Great Depression, earning the Financial Crisis its capital letters.

There were many concordant problems in finance that led to the Crisis but the one this paper is concerned with is that clearly CDO risks were horrifically underestimated[6]. With this we turn to how to price CDOs.

3 CDO Pricing Models

For this paper we will focus on Copula models that define the default time dependence structure. This is within the "bottom up approach for pricing CDOs, wherein "tranche premiums depend upon the individual credit risk of names in the underlying portfolio and the dependence structure between default times"[3]. This is unlike the "top down" approach which starts by modeling the aggregate loss process and from this derive the behaviour of the individual underlyings.

The copula approach was the industry standard in the run up to the Crisis and continues to be, although in a modified form. The early CDO pricing models such as Vasicek's KMV and the CreditMetrics model actually indirectly use a Gaussian copula.

Before Li's "recipe" was published, banks could model the individual marginal default probability of each underlying and they could model a large homogeneous pool but they couldn't combine the knowledge from both. The genius in Li's idea was to use copulas to

[6]Mackenzie[11] reports that the realised incidence of default for CDOs issued in 2005-2007 was hundreds of times higher than that predicted by one of the CRAs' models.

create a bridge between the marginal default distributions and the joint default distribution. This bridge would "couple" together the constituent elements, thus specifying the dependence structure. This allows the dependence structure to be modelled separately from the marginal distributions.

3.1 Introduction to Copulas

A copula (from the Latin for "link") is a mapping from univariate marginals to their multivariate distribution. As Li defines it[10]: for n uniform random variables $U_i, ..., U_n$, the joint distribution function C, defined as

$$C(u_1, ..., u_n, \rho) = \mathbb{P}(U_1 \leq u_1, ..., U_n \leq u_n)$$

can be called a copula function. Then, for n given univariate marginal distributions $F_1(x_1), ..., F_n(x_n)$, the copula function

$$C(F_1(x_1), ..., F_n(x_n)) = F(x_1, ..., x_n)$$

results in a multivariate distribution function with univariate marginal distributions as specified. The simplest copula is the independence copula:

$$C(u_1, ..., u_n) = \prod_{i=0}^{n} u_i$$

Correspondingly the copula for perfectly correlated variables is:

$$C(u_1, ..., u_n) = \min_i(u_i)$$

One of the most popular copulas, and the one proposed by Li, is the Gaussian copula:

$$C_{n,R}(u_1, ..., u_n) = \Phi_{n,R}\left(\Phi^{-1}(u_1), ..., \Phi^{-1}(u_n)\right)$$

where $\Phi_{n,R}$ denotes the joint cumulative distribution function of a multivariate Gaussian distribution with mean vector zero and $n \times n$ correlation matrix R. Φ^{-1} denotes the inverse cumulative distribution function of a standard univariate normal distribution.

Other copulas will be discussed shortly but before that: how does one use a copula to model a CDO?

3.2 Modelling a CDO

A CDO consists of n underlyings of notionals K_i, $i = 1, 2, ..., n$, with corresponding recovery rates of δ_i and exposure at default E_i. When the i^{th} underlying obligor defaults there is a loss of $K_i E_i (1 - \delta_i)$. We denote the cumulative fractional loss of the underlying pool at time t by $L(t)$.

For a CDO with a standard waterfall structure, each tranche has a simple payoff function that is dependent on $L(t)$. Let K_1 denote the attachment point of the tranche.

If $L(t) < K_1$ then the tranche suffers no losses. Let K_2 denote the detachment point. If $L(t) \geq K_2$ then the tranche is completely wiped out with 100% loss. Thus the width of the tranche is $K_2 - K_1$. Therefore the fractional loss of the tranche at time t is

$$L(t, K_1, K_2) = \frac{\max(L(t) - K_1, 0) - \max(L(t) - K_2, 0)}{K_2 - K_1}$$

For cash CDOs, the payoffs come from the actual cashflows of the underlyings. For synthetic CDOs however, since they are based on CDSs, tranches receive premiums (the premium leg) for assuming the risk of loss on the tranche (the protection leg) which must be covered in the case of tranche default. The value of these are the risk neutral discounted expected cashflows (ignoring charges and margins etc.).

Therefore the present value of the premium leg at time zero is:

$$\text{Premium leg PV} = S(K_1, K_2) \sum_{i=1}^{N_T} \Delta(t_{i-1}, t_i) Z(t_i) \mathbb{E}\left[1 - \frac{L(t_{i-1}, K_1, K_2) + L(t_i, K_1, K_2)}{2}\right]$$

where $S(K_1, K_2)$ is the spread or coupon paid (usually quarterly) on the surviving notional of the tranche, N_T is number of coupon payments within T, $\Delta(t_{i-1}, t_i)$ is the year fraction between t_{i-1} and t_i, and $Z(t)$ is the price of a zero-coupon bond maturing at t[6]. This only applies when K_2 is less than the maximum loss, as otherwise the tranche protection payer would continue to pay premiums even after all the underlyings have defaulted. Therefore when K_2 is is greater than the maximum loss we instead use

$$L\left(t, K_1, \min\left(K_2, 1 - L(t)\frac{\delta}{1-\delta}\right)\right)$$

in the expectation of the premium leg PV. This means we reduce the total notional by the recovered amount upon each default. Thus premiums are not paid on the riskless recovered notional.

The protection leg just pays out the trance losses at the time of loss:

$$\text{Protection leg PV} = \int_0^T Z(s)\mathbb{E}\left[dL(s, K_1, K_2)\right]$$

where $dL(t, K_1, K_2)$ is the amount lost over a small period dt[6].

Therefore the fair price of tranche protection can be found by setting the spread $S(K_1, K_2)$ such that both legs are equal. The values of both legs are functions of the expected fractional tranche loss which is equivalent to the probability of tranche default.

To find this one can use any model that gives the the distribution of defaults (e.g. Vasicek's single factor model), loss fraction distribution (e.g. the Large Homogeneous Portfolio approximation), or can be used for Monte Carlo simulations of defaults (e.g. copula models).

3.3 Using a copula to model a CDO

Copulas can be used to model a CDO by modelling defaults of the underlyings. Li's proposal[10] was a semi-dynamic model. In this type of model the term structure of each obligor's survival probability is given. Each underlying defaults at time τ_i.[7] We denote the marginal probability of default and survival of each underlying by:

$$F_i(t_i) = \mathbb{P}(\tau_i \leq t_i), \quad S_i(t_i) = \mathbb{P}(\tau_i > t_i)$$

where $\mathbb{P}$ is a suitable pricing probability measure. The joint distribution function and survival function are similarly denoted:

$$F(t_1, ..., t_n) = \mathbb{P}(\tau_1 \leq t_1, ..., \tau_n \leq t_n), \quad S(t_1, ..., t_n) = \mathbb{P}(\tau_1 > t_1, ..., \tau_n > t_n)$$

The chosen copula, in Li's case a Gaussian copula, is used to generate $U_1, ..., U_n$, which have a uniform marginal distribution. These are used the default times are found via[8]

$$\tau_i = F_i^{-1}(U_i)$$

From this the loss distribution can be obtained either from analytic or semi-analytic solutions or via Monte Carlo simulation, depending on complexity and assumptions.

The marginal survival and default probability distributions can be estimated using an intensity-based model, calibrated to data. If the intensity, λ, is assumed to be constant (as it will be for the simulations later) it can easily be backed out from the spread, S, of CDSs on the obligor:

$$\lambda = \frac{S}{1-\delta}$$

From this the risk-neutral probability of survival and default up to time t is:

$$S(t) = \mathbb{P}(\tau > t) = \exp(-\lambda t), \quad F(t) = \mathbb{P}(\tau \leq t) = 1 - \exp(-\lambda t)$$

Therefore, given U from the copula we can solve for τ:

$$\tau = -\frac{1}{\lambda} \ln(1-U) \tag{1}$$

With the default times of each underlying it is straightforward to infer the pool losses within the life of the CDO contract. From this the tranche payoffs can be calculated as in Section 3.2.

[7] It must be noted that could occur at any future time, which may be beyond the lifetime of the CDO.

[8] For symmetric copulas such as Gaussian or Student-t either the survival or default probability can be used.

3.4 Suitable Copulas

There is an infinite number of copulas that can be chosen. However not all of these are suitable for CDO modelling. For example, the copulas for independent and perfectly correlated variables are naturally unsuitable for practical applications.

Due to its familiarity and tractability, the Gaussian copula was a natural choice for Li to use as an example[9]. There is also a mathematical justification for using it from information theory, as it is an entropy maximising distribution, see [2]. However, with strong parallels to Brownian motion and Black-Scholes, in real financial data, heavier tails than in a normal distribution are often observed[14]. Multivariate normal distributions of credit risk, such as produced by the Gaussian copula, fail to capture default clustering; in times of crisis if one obligor defaults then it is likely that other obligors also default within a short period of time[4].

The tail behaviour of a copula can be examined via its tail dependence. The Gaussian copula is tail independent. For credit risk, this means that olbigor defaults become independent as their size of default increases[4]. This is unrealistic.

The natural alternative would therefore be the Student-t copula as it is similar to the Gaussian but it has positive tail dependence[6]. However it introduces another parameter to estimate, the degrees of freedom ν. This makes it useful for risk management since as ν increases the Student-t becomes closer to the Gaussian. Therefore one can start with a high ν and lower it to observe the effect of fattening tails[2]. But it also makes it difficult to calibrate to data.

The Student-t copula is symmetric which means it has both upper and tail dependence. However, there is not an equivalent of default clustering at the other end of the spectrum. We expect that large negative co-movements are much more likely in real-life than large positive co-movements. Thus it is natural to consider copulas with only lower tail dependence.

The Clayton copula is a common proposal (see [2], [3], and [15]) since it possesses lower tail dependence. It is an Archimedean copula, which is a family of copulas that take the form

$$C(u_1, ..., u_n) = \psi^{-1}(\psi(u_1) + ... + \psi(u_n))$$

$\psi(x)$ is the so-called "generator" of the copula. In the case of the Clayton copula the generator is

$$\psi_\theta(x) = x^{-\theta} - 1, \quad \theta > 0$$

The parameter θ controls the dependence with dependence increasing as θ increases.

Other copulas have also been proposed such as the Marshall-Olkin copula and double-t copulas, see [3]. However this paper will focus on the three copulas above as these are the most common modelling choices discussed pre-crisis. Therefore they are the ones to investigate if a different copula choice could have averted the Crisis.

[9] In his seminal paper[10] he does also discuss other copulas, such as the Frank copula, and does not comment on the Gaussian copula's validity as a modelling choice for financial applications.

Rating at issuance	AAA	AA+	AA	AA-	A+	A	A-	BBB+	BBB
5-yr default probability (%)	0.05	0.19	0.26	0.36	0.56	0.62	0.92	1.20	1.89
Rating at issuance	BBB-	BB+	BB	BB-	B+	B	B-	CCC+	CCC
5-yr default probability (%)	3.63	5.74	8.11	12.50	17.09	21.36	27.08	33.64	37.64

Table 1: Part of Fitch's CDO default matrix.

Tranche (Rating)	**Attachment Points**
Senior (AAA)	42% - 100%
Mezzanine 1 (AA-)	34% - 42%
Mezzanine 2 (A-)	28% - 34%
Mezzanine 3 (BBB-)	20% - 28%
Mezzanine 4 (BB-)	11% - 20%
Equity (NA)	0% - 11%

Table 2: Example CDO tranches, chosen to be as large as possible for their rating.

4 Comparing Copulas

In order to judge the effect a different choice of CDO pricing model may have had in averting the Crisis, we look at an artificial, stylized ABS CDO. This CDO consists of 100 homogeneous, underlying ABSs, each of notional 1 and constant spread of 100bps[10]. The underlyings are modelled with a simple intensity model, as described in Section 3.3. For simplicity we will assume the exposure at default is 1 and that default rates are constant. We will also set the risk free rate to zero as it has little effect on results. The CDO contract is for five years, a typical contract length. The code used for the simulations is provided in the appendix along with all the simulation results.

The CDO tranches were constructed by following the example of banks and credit rating agencies pre-crisis. That is by starting with the pre-crisis industry standard modelling choices of Gaussian copula, recovery rate of 40%, and $\rho = 30\%$[12]. Then, using simulations finding the largest tranches that will be given each rating by a credit rating agency according to calculated default probability. The rating requirements used were according to Fitch's CDO matrix[17], listed in Table 1. The resulting tranches are listed in Table 2.

The behaviour of this CDO using alternative copula models was investigated. These alternative copulas, as recommended in Section 3.4, were the Clayton copula and the Student-t copula with 5, 10 and 15 degrees of freedom[11]. To simulate default times with the Gaussian and Student-t copula the inbuilt MATLAB function `copularnd` was used to generate variables with uniform marginal distribution. For the Clayton copula the

[10]This spread is in the ballpark of actual MBS spreads around 2007 according to Investopedia[8].

[11]It was found that it was not possible to back out implied correlations for Student-t copulas with lower degrees of freedom, as performed later, so they were not considered

algorithm from Bluhm and Overbeck[2] was used:

1. Generate i.i.d random variables $U_1, ..., U_n \sim U([0,1])$.
2. Generate random variable $X \sim \Gamma(1/\theta, 1)$ independent of all the other variables.
3. Using these variables calculate for each i, $v = -\ln(U_i)/X$.
4. Convert this to a uniformly distributed variable by $\psi_\theta^{-1}(v) = (1+v)^{-1/\theta}$

The default times were then found using equation (1) from Section 3.3.

In order to compare this CDO under these different copulas, we need to use equivalent parameters. Linear correlation is unsuitable for comparing non-linear copulas. Therefore, following the example of Bluhm and Overbeck[2], we used the copula parameters with the same Kendall's tau. This was found using MATLAB's inbuilt functions `copulastat` and `copulaparam`. For a correlation of 30%, the Gaussian and Student-t copulas have a tau of 0.194. The θ that gives the same tau for the Clayton copula is 0.4813.

The example CDO was then reevaluated using these alternative copulas instead of the Gaussian copula. The resulting probabilities of default for each tranche under each copula are plotted on figure 1. The change of copula affects the ratings that would be given by Fitch if they were to evaluate this CDO, as displayed in Table 4.

The results reveal that the choice of copula does have a significant effect on tranche pricing and ratings. As the tail dependence increases, the riskiness of the equity tranche decreases whereas it increases for all of the more senior tranches. This is intuitive as increasing tail dependence increases the probability that lots of underlyings fail at the same time, pushing the expected losses further up the tranches and away from the equity tranche. The choice of copula significantly affects the rating of the tranches, particularly for the more senior tranches. In the most extreme case causing the Mezzanine 1 tranche to be downgraded down six ratings! Both Mezzanine 2 and 3 are downgraded from investment grade to junk.

What instead of inputting chosen parameters we instead inferred them from market prices? What effect does the copula choice make then? To investigate this the dependence parameter that gave the same equity tranche premiums was found for each copula. As shown by Burtschell et al.[3], this is mathematically valid as equity tranche premiums are monotonic with respect to the copulas parameter. It also practically makes sense since one can think of it as backing out base correlations. The resulting parameters are listed in Table 3.

Using these parameters the CDO was again reevaluated for each copula. The results are given in figure 2 and table 5. This time we see copula choice makes much less difference to tranche default probabilities. It has no effect at all on ratings. These results are in accordance with those of Burtschell et al.[3] who also found that Gaussian, Student-t and Clayton copulas produce similar spreads for CDO tranches.

This implies that if one wishes to price a synthetic CDO tranche using base correlations implied from the market, these three copulas will produce similar results.

Copula	Parameter		
Gaussian	30%	55%	80%
Student t - 15	25%	52%	79%
Student t - 10	23%	50%	78%
Student t - 5	15%	45%	75%
Clayton	0.20	0.46	1.00

Table 3: Copula parameters that match equity tranche premiums.

5 Correlation

We have seen that choice of copula does make a difference when pricing a CDO from the ground up. But what about the other vital modelling choice when using copula models, the input parameter? Pre-crisis papers criticising CDO modelling practices, such as Adelson[1], often only discuss correlation, with no mention of copulas. As the saying goes, "garbage in, garbage out", so it is worth investigating the hypothesis that the mispricing of CDOs was not due to the choice of copula but instead to the estimation of correlation.

This is an especially compelling argument since the huge losses were in CDOs of subprime ABSs. Whereas CDOs of corporate debt did suffer losses in the Crisis, they were not catastrophic[12]. This implies that the fatal flaw may have been correlation estimation.

Since ABSs were a relatively new security and they defaulted rarely (pre-crisis) there was little econometric data available[11]. Thus credit rating agencies relied largely on judgement-based ABS correlation estimates. When S&P introduced their Gaussian copula CDO rating system in 2001, *CDO Evaluator*, ABSs from the same sector (e.g. based on subprime mortgages) were estimated to have the same correlation (30%) as was used for corporations in the same industry[12]. This was actually seen as conservative at the time. The average correlation between the losses on pairs of pools of American residential mortgages between 1995 and 2002 was reported as only 0.06[11].

However, this was during a prolonged bull market in the U.S. housing market, marked with low default rates. Additionally the U.S.A. had not had a widespread crash in the housing market since the Great Depression, thus the risk in mortgages had mostly been idiosyncratic rather than systemic. This allows the diversification in an ABS to be a "free lunch". But as Mackenzie remarks in his paper[11], "the lunch was frequently being eaten twice". By this he means that ABSs diversified away idiosyncratic (particularly geographic) risk and then CDOs diversified away ABS risk due to (assumed) modest correlation between them.

This double luncheoning was allowed by the credit rating agencies as, due to the copula approach, the ABSs and the CDO were normally rated separately by different teams. This separation of labour meant that holistic assessments were not made. This proved fatal as ABSs started being packed with sub-prime mortgages. ABS risk became more and more systemic as these mortgages were all exposed to the same factors: a fall in house prices, a rise in interest rates, or contractual (e.g. teaser rates expiring). Therefore the correlation between ABSs increased and the ratings given to CDOs for their diversification were no longer justifiable.

Mackenzie reported that in 2006, CDO specialists at a CRA actually performed a complete drilldown assessment of an ABS CDO as an intellectual curiosity[11]. Using only Vasicek's model to pools of mortgages they calculated the correlation between each pair of ABS tranches. They discovered the correlation to be in the region of 80%, much higher than the "conservative" 30% used for ratings. Unfortunately for organisational reasons this did not affect the CRAs methodology[12].

So how would our ABS CDO example perform under more conservative (and potentially more accurate) correlation assumptions? The CDO was reevaluated for correlations of 30%, 42.5%, 55%, 67.5% and 80%. The results are displayed in Table 6 and Figure 3. From this we can see that increasing the correlation affects the tranche ratings and probability of default significantly. With a correlation of 80%, as recommended by the anecdote above, the probability of default for the senior tranche is over eighty times higher and it has dropped nine ratings. Naturally by increasing dependence we see similar behaviour as we saw by increasing the tail dependence by changing copula, with the equity tranche becoming safer and the senior less so.

These results imply that it is vital to estimate the magnitude of correlation correctly, or at the very least use high dependence when stress testing CDOs.

6 Conclusion

There are flaws with the copula approach to modelling CDOs[13], such as the correlation skew and the separate calibration of marginal distributions and copula parameter. But one of the largest flaws is that there is no systematic method to choose which copula to use. As shown in Section 4, this choice is influential particularly for Credit Rating Agencies.

However, in the Crisis the probability of default predicted by the CRAs were wrong by a factor of a hundred[12]. Changing copula did not produce the magnitude of change in default probability to match this. Thus even if CRAs had adopted an alternative copula, the Crisis would most likely not have been averted.

It resulted that dependence, in the form of correlation choice, impacted the probability of default for tranches even more than copula choice. The changes in probability of default here were more of the Crisis triggering magnitude.

In both cases the gaming of tranche sizes proved to make the ratings very unstable. That is, other than for the Mezzanine 4 tranche, curiously, which remained quite stable

[12] Mackenzie also mentions that in 2006 Goldman Sachs also started modelling ABS CDOs with a drilldown approach, which may have influenced their crucial decision to liquidate or hedge its mortgage-related positions[11].

[13] See [14] for a good summary.

for all calculations. It would be interesting to investigate whether it is possible instead to game tranche sizes for stability to reduce model risk for a synthetic CDO tranche.

These results support the arguments of Donnelly & Embrechts[4] and Mackenzie & Spears[12], that Li and the Gaussian copula were not to blame for the Crisis. As an interviewee in Mackenzie & Spears[12] said, "the base corr guys [users of Gaussian copula base correlation models] are still standing...I am not sure that there was anything in terms of an Armageddon for the models". Instead it appears that the gaming of the model beyond its original assumptions, the outsourcing of CDO risk management to credit rating agencies, and the failure to perform holistic risk assessment seem far more to blame.

The Gaussian copula continues to be used in practice, although it is often now extended to compensate for some of its deficiencies. This is done, for example, by incorporating stochastic correlation[5], time-varying copulas[13], or dynamic recovery rates[3].

The simulation results in this paper show that it is more important to focus on parameter estimation than copula choice. This leads to the observation that when it comes to mathematical financial modelling: in order to avoid a disaster, the cooking is more important than the recipe.

References

[1] Mark H. Adelson. CDO and ABS underperformance: A correlation story. *The Journal of Fixed Income*, 13(3):53–63, 2003.

[2] Christian Bluhm and Ludger Overbeck. *Structured credit portfolio analysis, baskets and CDOs*. CRC Press, 2006.

[3] Xavier Burtschell, Jon Gregory, and Jean-Paul Laurent. A comparative analysis of CDO pricing models. *The Journal of Derivatives*, 16(4):9–37, 2009.

[4] Catherine Donnelly and Paul Embrechts. The devil is in the tails: Actuarial mathematics and the subprime mortgage crisis. *ASTIN Bulletin: The Journal of the International Actuarial Association*, 40:1–33, May 2010.

[5] Gregory and J. P. Laurent. Beyond the gaussian copula: stochastic and local correlation. Technical report, Working Paper, BNP Paribas, 2005.

[6] Daniel Jones. Credit risk lecture notes. *MSc Mathematical Finance Lectures, Module 8*, April 2016.

[7] Sam Jones. The formula that felled wall street. *The Financial Times*, 22(04), 2009.

[8] Daniel Kurt. How mortgage backed securities (MBS) ETFs work. `http://www.investopedia.com/articles/investing/030215/how-mortgage-backed-securities-mbs-etfs-work.asp`, March 2015. Accessed 30/05/2016.

[9] Michael Lewis. *The Big Short: Inside the Doomsday Machine.* WW Norton & Company, 2015.

[10] David X. Li. On default correlation: A copula function approach. *Social Science Research Network Working Paper Series*, December 1999.

[11] Donald MacKenzie. The credit crisis as a problem in the sociology of knowledge 1. *American Journal of Sociology*, 116(6):1778–1841, May 2011.

[12] Donald MacKenzie and Taylor Spears. 'the formula that killed wall street': The gaussian copula and modelling practices in investment banking. *Social Studies of Science*, 44(3):393–417, June 2014.

[13] Hans Manner and Olga Reznikova. A survey on Time-Varying copulas: Specification, simulations, and application. *Econometric Reviews*, 31(6):654–687, November 2012.

[14] Thomas Mikosch. Copulas: Tales and facts. *Extremes*, 9(1):3–20, March 2006.

[15] Dominic O'Kane. *Modelling single-name and multi-name credit derivatives*, volume 573. John Wiley & Sons, 2011.

[16] Jörn Rank. *Copulas: From theory to application in finance.* Risk books, 2007.

[17] Fitch Ratings. Global rating criteria for collateralised debt obligations, September 2004.

[18] Felix Salmon. The formula that killed wall street. *Significance*, 9(1):16–20, 2012.

Appendix A Simulation results

The probabilities of default and Fitch ratings for tranches of the stylised CDO obtained via simulations run for:

- Different copulas and the same Kendall's tau
- Different copulas and the same equity tranche premium
- The Gaussian copulas and different correlation parameter value

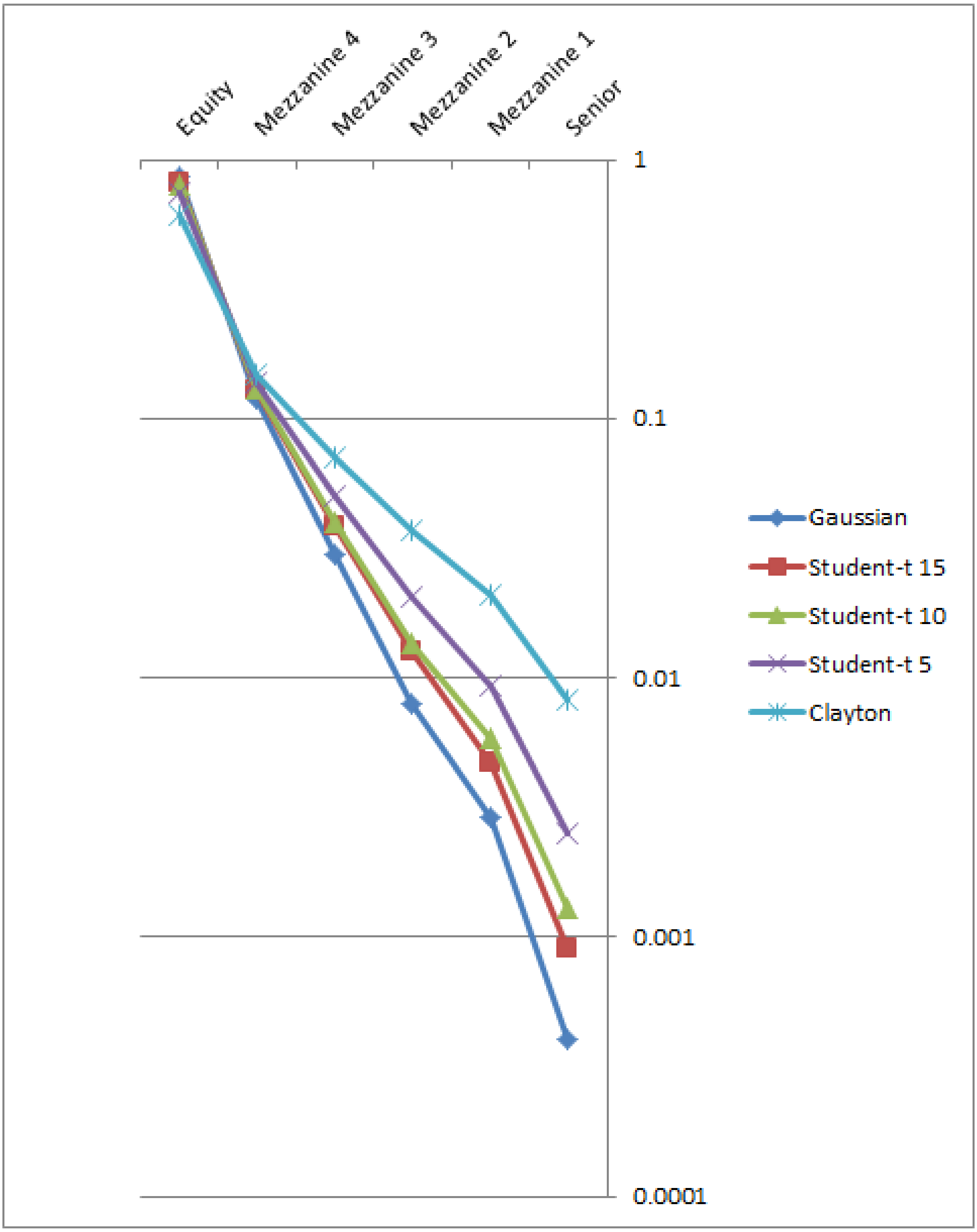

Figure 1: Probability of default for each tranche of the example CDO, as calculated by different copulas with the same Kendall's tau.

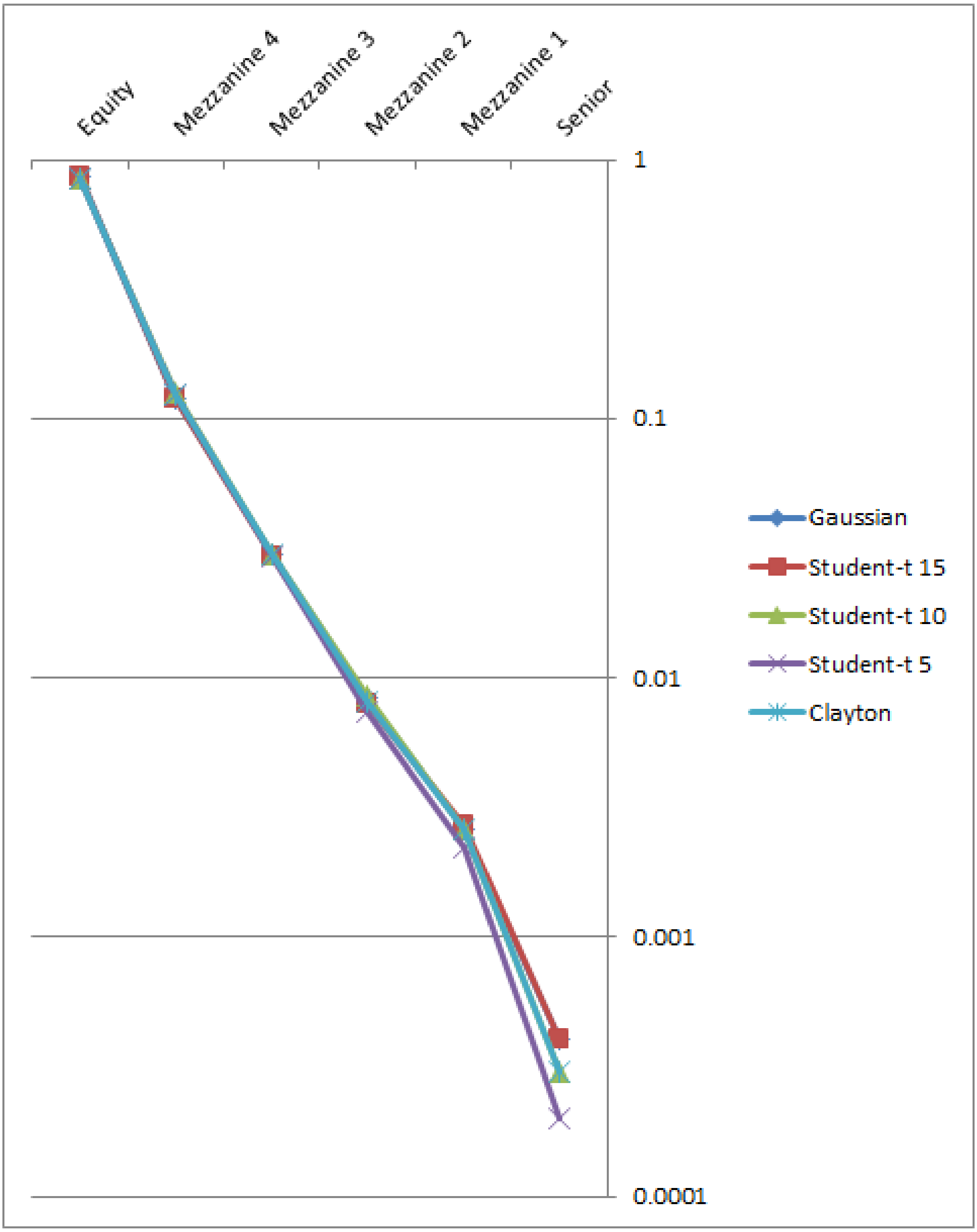

Figure 2: Probability of default for each tranche of the example CDO, as calculated using different copulas with the same equity tranche premium.

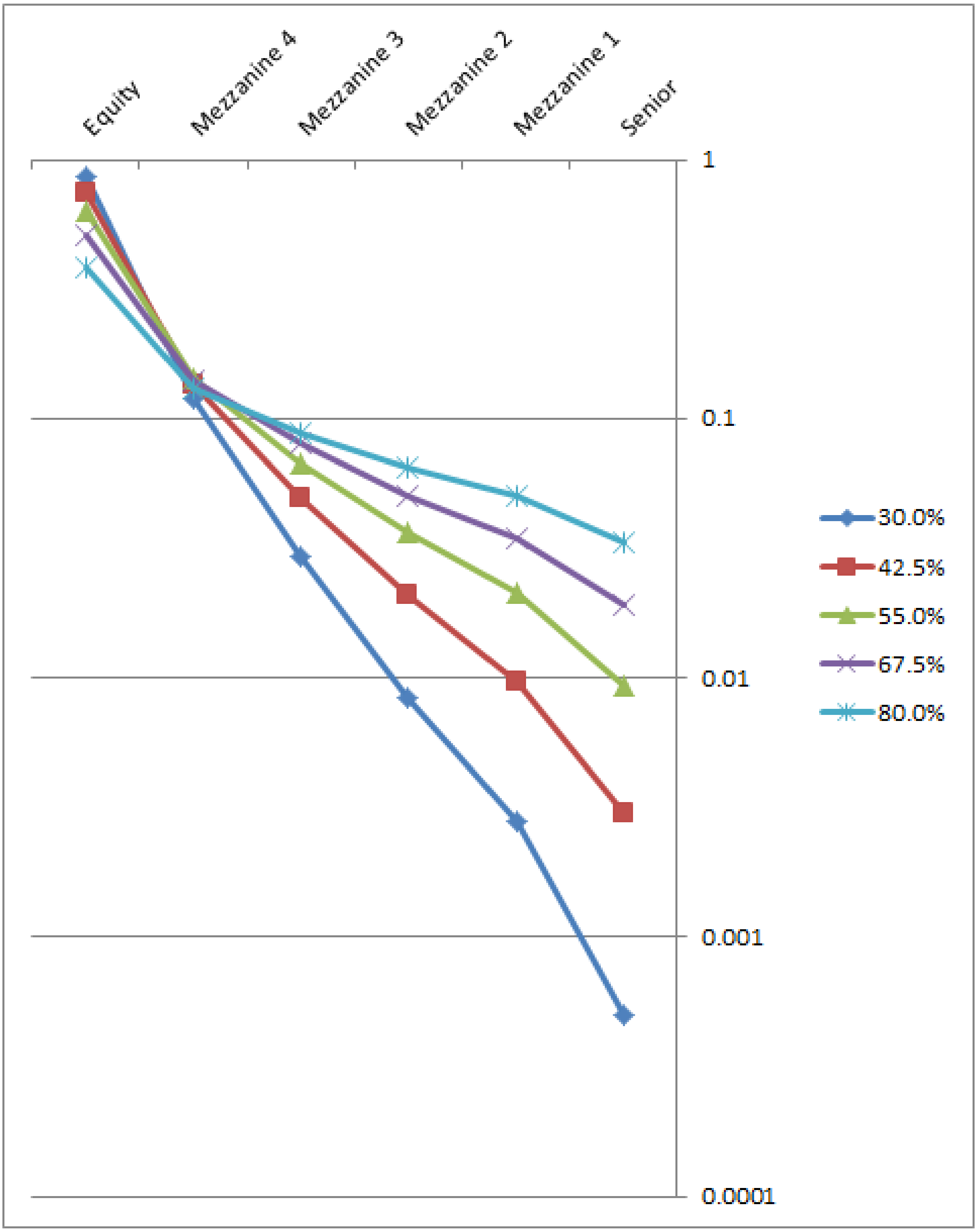

Figure 3: Probability of default for each tranche of the example CDO, as calculated by the Gaussian copulas with varying correlation.

Tranche	**Gaussian**	**Student-t 15**	**Student-t 10**	**Student-t 5**	**Clayton**
Senior	0.0004 AAA	0.0009 AA+	0.0013 AA+	0.0025 AA	0.0082 A-
Mezzanine 1	0.0029 AA-	0.0047 A+	0.0058 A	0.0094 BBB+	0.0211 BBB-
Mezzanine 2	0.0080 A-	0.0127 BBB	0.0136 BBB	0.0205 BBB-	0.0369 BB+
Mezzanine 3	0.0297 BBB-	0.0382 BB+	0.0397 BB+	0.0508 BB+	0.0704 BB
Mezzanine 4	0.1200 BB-	0.1286 B+	0.1296 B+	0.1372 B+	0.1467 B+
Equity	0.8566 NA	0.8191 NA	0.7991 NA	0.7409 NA	0.6056 NA

Table 4: Probability of default and Fitch rating of tranches calculated using different copulas with the same Kendall's tau.

Tranche	**Gaussian**	**Student-t 15**	**Student-t 10**	**Student-t 5**	**Clayton**
Senior	0.0004 AAA	0.0004 AAA	0.0003 AAA	0.0002 AAA	0.0003 AAA
Mezzanine 1	0.0029 AA-	0.0027 AA-	0.0026 AA-	0.0022 AA-	0.0026 AA-
Mezzanine 2	0.0080 A-	0.0080 A-	0.0086 A-	0.0074 A-	0.0081 A-
Mezzanine 3	0.0297 BBB-	0.0295 BBB-	0.0301 BBB-	0.0295 BBB-	0.0299 BBB-
Mezzanine 4	0.1200 BB-	0.1197 BB-	0.1211 BB-	0.1247 BB-	0.1230 BB-
Equity	0.8566 NA	0.8580 NA	0.8504 NA	0.8385 NA	0.8482 NA

Table 5: Probability of default and Fitch rating of tranches calculated using different copulas with the same equity tranche premium.

Tranche	$\rho = 30\%$	$\rho = 42.5\%$	$\rho = 55\%$	$\rho = 67.5\%$	$\rho = 80\%$
Senior	0.0004 AAA	0.0030 AA-	0.0093 BBB+	0.0192 BBB-	0.0335 BBB-
Mezzanine 1	0.0029 AA-	0.0097 BBB+	0.0214 BBB-	0.0345 BBB-	0.0503 BB+
Mezzanine 2	0.0080 A-	0.0210 BBB-	0.0366 BB+	0.0505 BB+	0.0646 BB
Mezzanine 3	0.0297 BBB-	0.0497 BB+	0.0674 BB	0.0797 BB	0.0883 BB-
Mezzanine 4	0.1200 BB-	0.1350 B+	0.1421 B+	0.1391 B+	0.1316 B+
Equity	0.8566 NA	0.7503 NA	0.6341 NA	0.5101 NA	0.3796 NA

Table 6: Probability of default and Fitch rating of tranches calculated using the Gaussian copula with different correlation.

Appendix B MATLAB Code

```
function results = CDOmonte(n,p,cop,para,s,delta,tranches,T)
% Function to run Monte Carlo simulations of CDO tranche prices
%
% Inputs: cop - desired copula:
%             0: Gaussian, 1: t 15 degree freedom,
%             2: t 10 degree freedom, 3: t 5 degree freedom,
%             4: Clayton
%         : para - copula parameter
%         : n - number of underlyings. Assume each of notional 1
%         : p - number of Monte Carlo runs
%         : s - spread of underlying, constant & homogeneous
%         : delta - recovery rate. Assume constant & homogeneous
%         : tranches - attachment of tranche points incl. 0 & 1
%         : T - length of contract in years
%
% Output: results - array containing for each tranche:
%         detachment point, prob of default, std of pd,
%         expected payoff, std of payoff
%

% Number of tranches
n_tran = length(tranches) - 1;

% Generate p sets of default times for each underlying
tau = GenerateDefaultTimes(n,p,cop,para,s,delta);

% Number of defaults during life of contract for each run
defaults = sum((tau<=T),2);

% Since each underlying has notional of 1, total underlying
    notional is n
notional = n;

% Results for total CDO for each run
cdo_loss = (1-delta)*defaults;
cdo_payoff = notional - cdo_loss;

% Initialise results arrays
results = zeros(1+n_tran,4);
cdo_tran_default = zeros(p, n_tran);
cdo_tran_payoff = zeros(p, n_tran);
```

```
cdo_tran_payoff_fraction = zeros(p, n_tran);

% Functions for calculating tranche payoffs
absoluteLoss = @(loss, K1, K2) (K2 - K1) - (max(loss - K1, 0) -
    max(loss - K2, 0));
fractionLoss = @(payoff, K1, K2) payoff / (K2 - K1);

% Actually calculate payoffs
for i=1:n_tran
    % Tranche defaults if any losses at all
    cdo_tran_default(:,i) = (cdo_loss > (tranches(i) * notional)
        );
    % Actual payoff of tranche
    cdo_tran_payoff(:,i) = absoluteLoss(cdo_loss, tranches(i)*
        notional, tranches(i+1)*notional);
    % Tranche payoff as fraction of tranche
    cdo_tran_payoff_fraction(:,i) = fractionLoss(cdo_tran_payoff
        (:,i), tranches(i)*notional, tranches(i+1)*notional);



    % Detachment point
    results(1+i, 1) = tranches(1+i) * notional;
    % Mean number of defaults across all runs gives prob of
        default
    results(1+i, 2) = mean(cdo_tran_default(:,i));
    % Std of pd
    results(1+i, 3) = std(cdo_tran_default(:,i));
    % Expected payoff
    results(1+i, 4) = mean(cdo_tran_payoff_fraction(:,i));
    % Std of payoff
    results(1+i, 5) = std(cdo_tran_payoff_fraction(:,i));
end

% Expected payoff for total CDO
results(1,1) = NaN;
% Don't care about 'default' of whole CDO as almost certain
results(1,2) = NaN;
results(1,3) = NaN;
results(1,4) = mean(cdo_payoff) / notional;
results(1,5) = std(cdo_payoff / notional);

end
```

```
function tau = GenerateDefaultTimes(n,p,cop,para,s,delta)
% Function to generate default times according to desired copula
%       Assumes homogeneous underlying spread and correlation.
%
% Inputs: cop - desired copula:
%             0: Gaussian, 1: t 15 degree freedom,
%             2: t 10 degree freedom, 3: t 5 degree freedom,
%             4: Clayton
%         : para - copula parameter
%         : n - number of default times
%         : p - number of samples
%         : s - spread of underlying
%         : delta - recovery rate
%
% Output: tau - a p x n matrix containing p columns of
% n default times
%

% If not Clayton copula, parameter is correlation and inbuilt
% MATLAB function used
if cop < 4
    % Generate n x n correlation matrix
    r = para * ones(n) + diag(ones(1,n)*(1-para));

    % If Gaussian
    if cop == 0
        fprintf('Copula: Gaussian\n');
        U = copularnd('Gaussian',r,p);
    else
        % Student-t copula in descending order of degrees of freedom
        fprintf('Copula: student-t with %d degrees of freedom\n',(4-cop)*5)
        U = copularnd('t',r,(4-cop)*5,p);
    end
elseif cop == 4
    % Using algorithm from p98 of Structure Credit Portfolio Analysis,
    % Baskets & CDOS
    fprintf('Copula: Clayton\n')

    % Generate uniformly distributed variables
    U = rand(p,n);
```

```
    % Generate gamma variable
    X = gamrnd(1/para,1,p,1);
    % Divide element by element
    v = bsxfun(@rdivide, -log(U),X);
    % Convert to uniformly distributed
    U = (1+v).^(-1/para);
end

% Hazard rate
lambda = s / (1 - delta);

% Convert uniformly distributed results into equivalent default
    times
tau = -(1/lambda) * log(ones(p,n)-U) ;

end
```

```
function prem = tranchePremium(n,p,tau,delta,K1,K2,T)
% Function to calculate fair premium for CDO tranche
%
% Inputs: n - number of underlyings. Assume each of notional 1.
%       : p - number of Monte Carlo runs
%       : tau - n x p matrix of default times
%       : delta - recovery rate. Assume constant & homogeneous
%       : K1 - tranche lower attachment point
%       : K2 - tranche upper attachment point
%       : T - length of contract in years
%
% Output: prem - fair premium in bps per year
%

% How many coupons (quarters) need to be paid?
q = 4 * T;

% Calculate total loss at t
L = zeros(p,q);
Ltr = zeros(p,q);

% Find number of defaults up until each quarter
for i = 1: q
    L(:,i) = (1-delta) * sum((tau<=i*0.25),2);
end

% If senior tranche reduce notional on default
```

```
% Assume if detachment point above maximum loss K2 = 100%
if K2 > n * (1 - delta)
    Notional = 100 - L * delta / (1 - delta);
    for i = 1: q
        Ltr(:,i) = bsxfun(@rdivide, max(L(:,i) - K1,0),Notional(:,i)-K1);
    end
else
    Ltr = (max(L - K1,0) - max(L - K2,0))/(K2-K1);
end

% Pay spread on average tranche notional since previous coupon date
premleg = (1 - Ltr(:,1)/2);
for i = 2: 20
    premleg = premleg + (1 - (Ltr(:,i-1) + Ltr(:,i))/2);
end

% Find expectation and multiply by year fraction between payments (quarter)
premleg = 0.25 * mean(premleg);

% If senior tranche recalculate actual tranche loss
if K2 > n * (1 - delta)
    Ltr = (max(L - K1,0) - max(L - K2,0))/(K2-K1);
end

% Calculate time-zero value of protection leg
protleg = mean(Ltr(:,20));

% Infer fair premium to have paid on premium leg
prem = 10000 * protleg / premleg;
% Only round if more than 2 s.f.
if prem >= 10
    prem = round(prem);
end
```