Two sides of the Camassa–Holm equation: A Lipschitz metric and a noisy version

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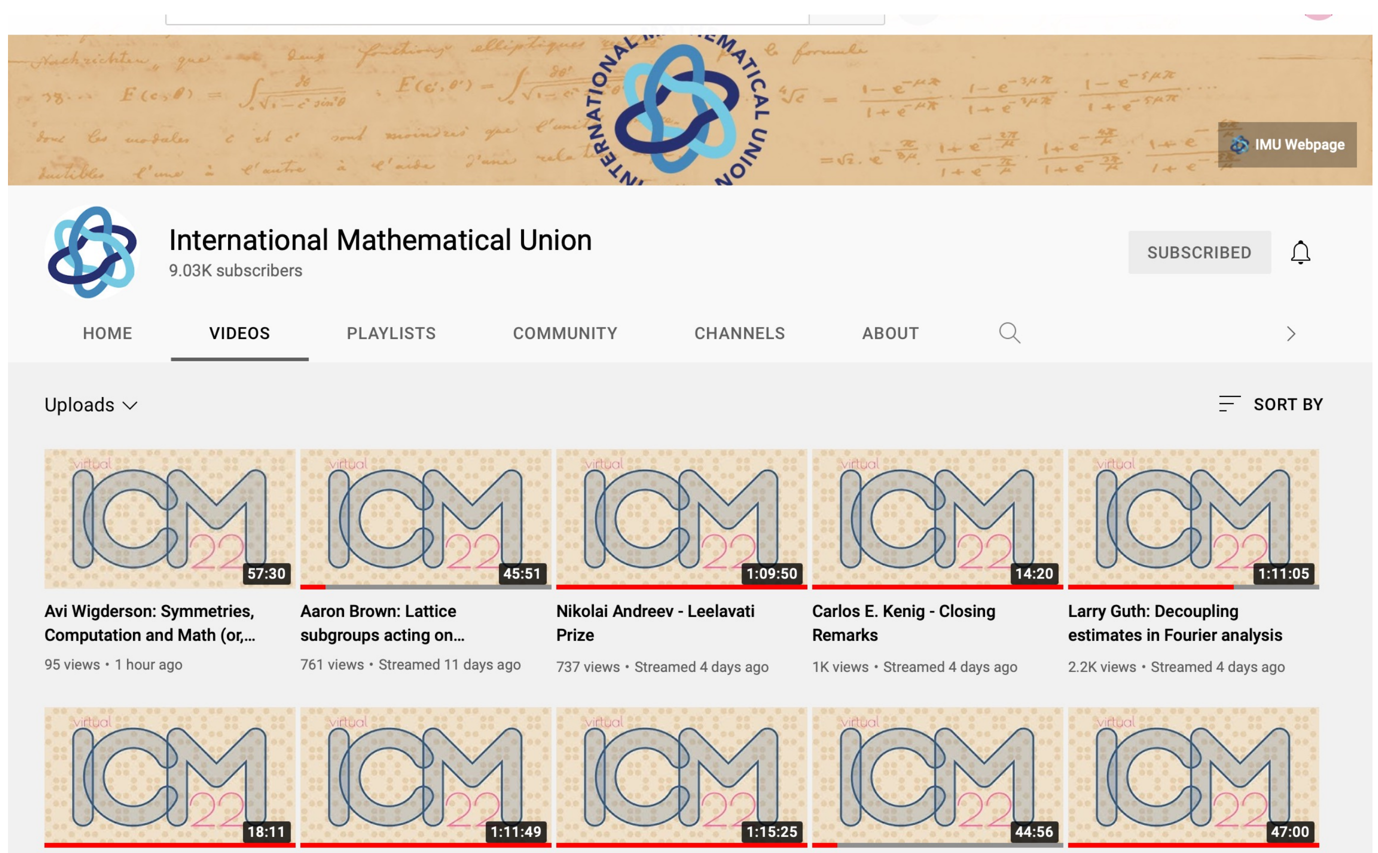
Joint work with J. A. Carrillo (Oxford) and K. Grunert (NTNU)

A Lipschitz metric for the Camassa–Holm equation
Forum Math. Sigma 8 (2020), Paper No. e27, 292 pp.

Joint work with Kenneth H. Karlsen (Oslo) and Peter H.C. Pang (NTNU)

Global well-posedness of the viscous Camassa--Holm equation with gradient noise (In prep.)

virtual ICM 2022



All talks available on IMU’s YouTube channel
https://www.youtube.com/c/InternationalMathematicalUnion

The Camassa–Holm equation

$$u_t - u_{xxt} + 2\kappa u_x + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, \quad u|_{t=0} = \bar{u}$$

Camassa–Holm: *Phys. Rev. Lett.* (1993)

Some properties:

- A model for water waves and hyperelastic rods
- Completely integrable (Lax pair); inverse scattering transform
- bi-Hamiltonian
- Algebro-geometric solutions, infinite hierarchy of integrable equations
- Infinitely many conserved quantities
- Soliton-like solution («multi-peakons»)
- Extensions to systems and multidimensions
- Wave breaking in finite time with steep gradients while keeping finite Sobolev norm

Rewrite of the Camassa–Holm equation

$$\boxed{\begin{aligned} u_t + uu_x + p_x &= 0, \\ \mu_t + (u\mu)_x &= (u^3 - 2pu)_x, \\ p - p_{xx} &= \frac{1}{2}(u^2 + d\mu) \end{aligned}}$$

$$u(t,\cdot) \in H^1(\mathbb{R}), \quad \mu(t,\cdot) \in \mathcal{M}_+(\mathbb{R}), \quad \text{and} \quad d\mu_{\text{ac}} = (u^2 + u_x^2)dx$$

The goal

To determine a metric such that

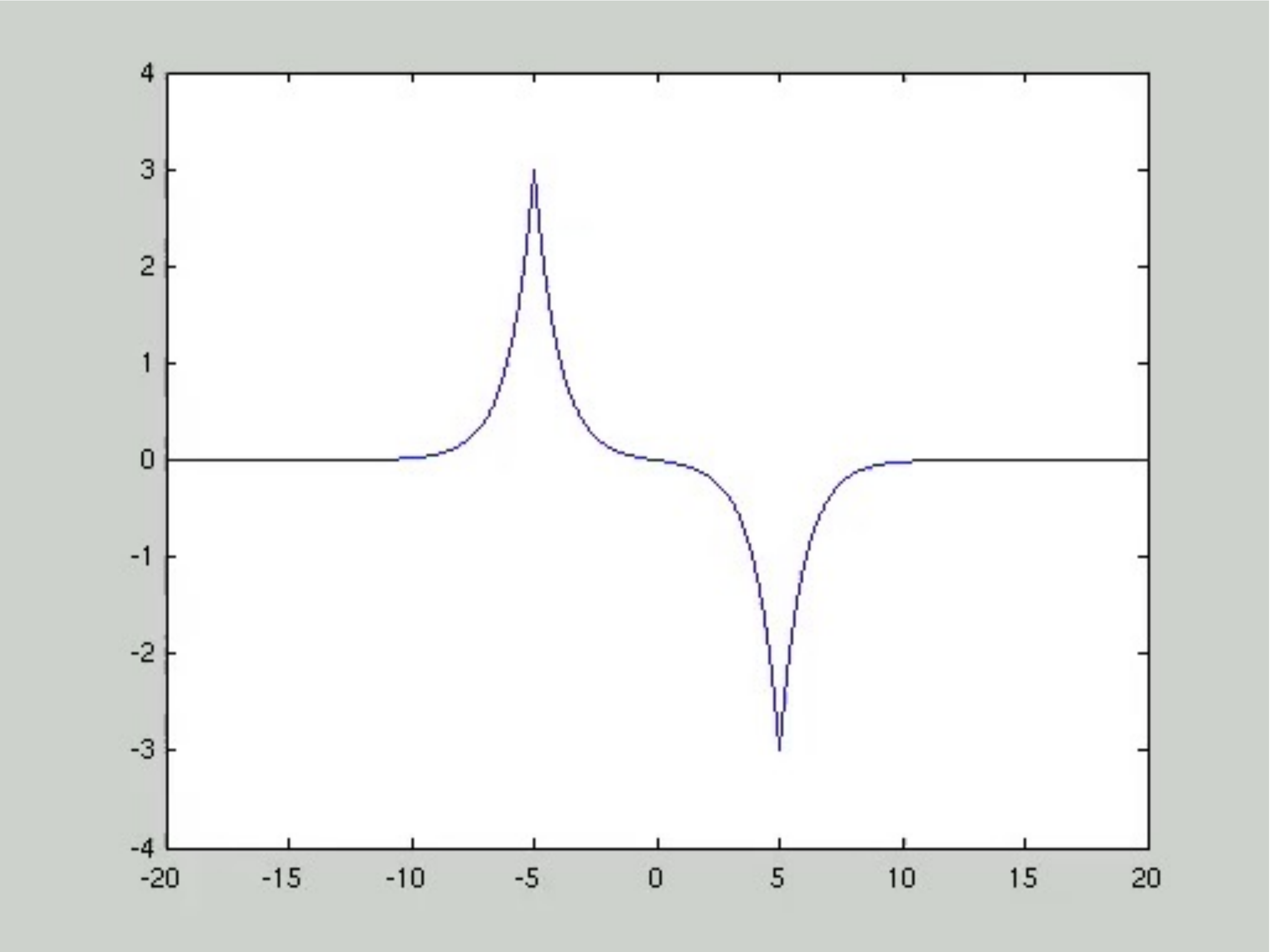
$$d((u_1(t),\mu_1(t)),(u_2(t),\mu_2(t))) \leq \alpha(t) d((u_{1,0},\mu_{1,0}),(u_{2,0},\mu_{2,0}))$$

$$\alpha(0)=1$$

comparing two weak conservative solutions $(u_j(t),\mu_j(t))$

Earlier results by Bressan and Fonte (2005) and Grunert, H., Raynaud (2011, 2013)

The problem I

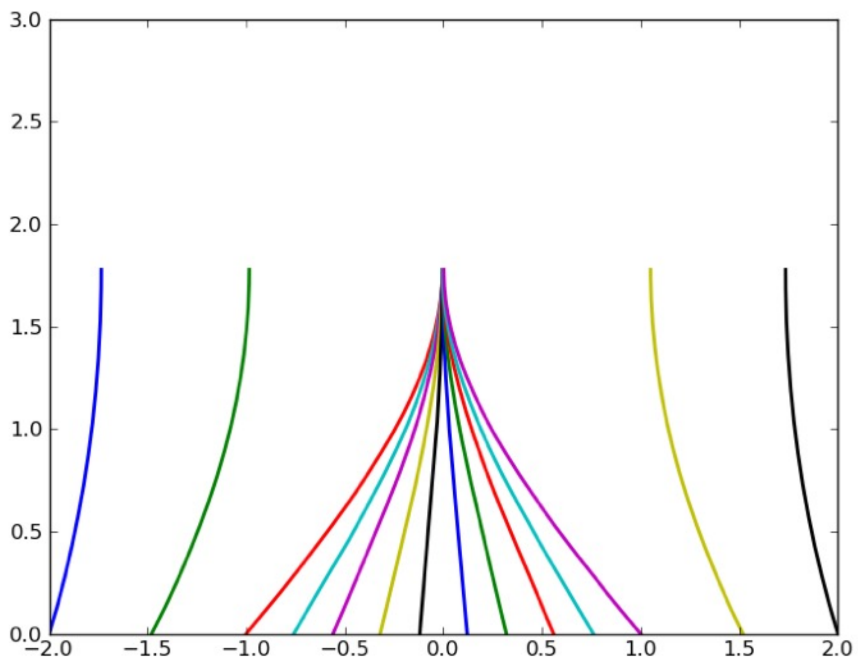


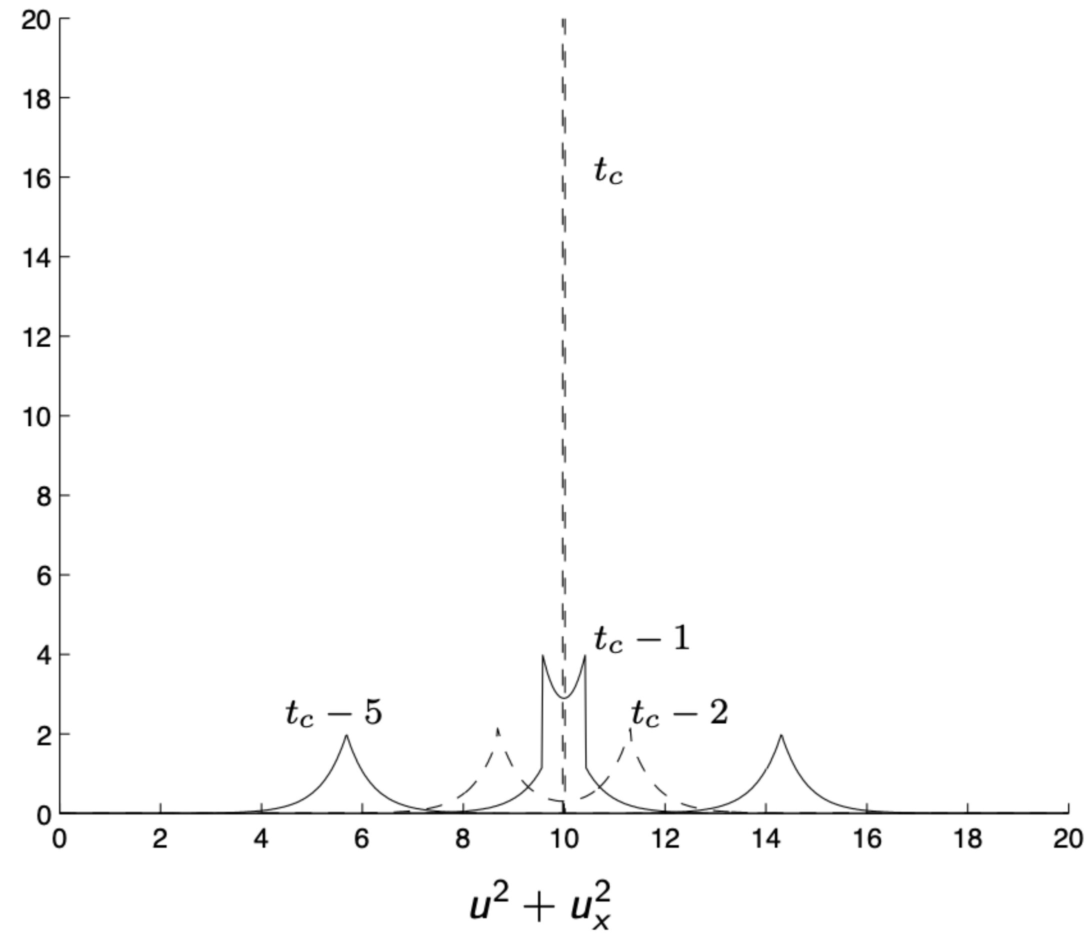
As $t \nearrow T$:

$u_x(x_0, t) \to -\infty$ $u \to 0$ almost everywhere $\|u\|_{H^1} < \infty$

The problem II

Characteristics





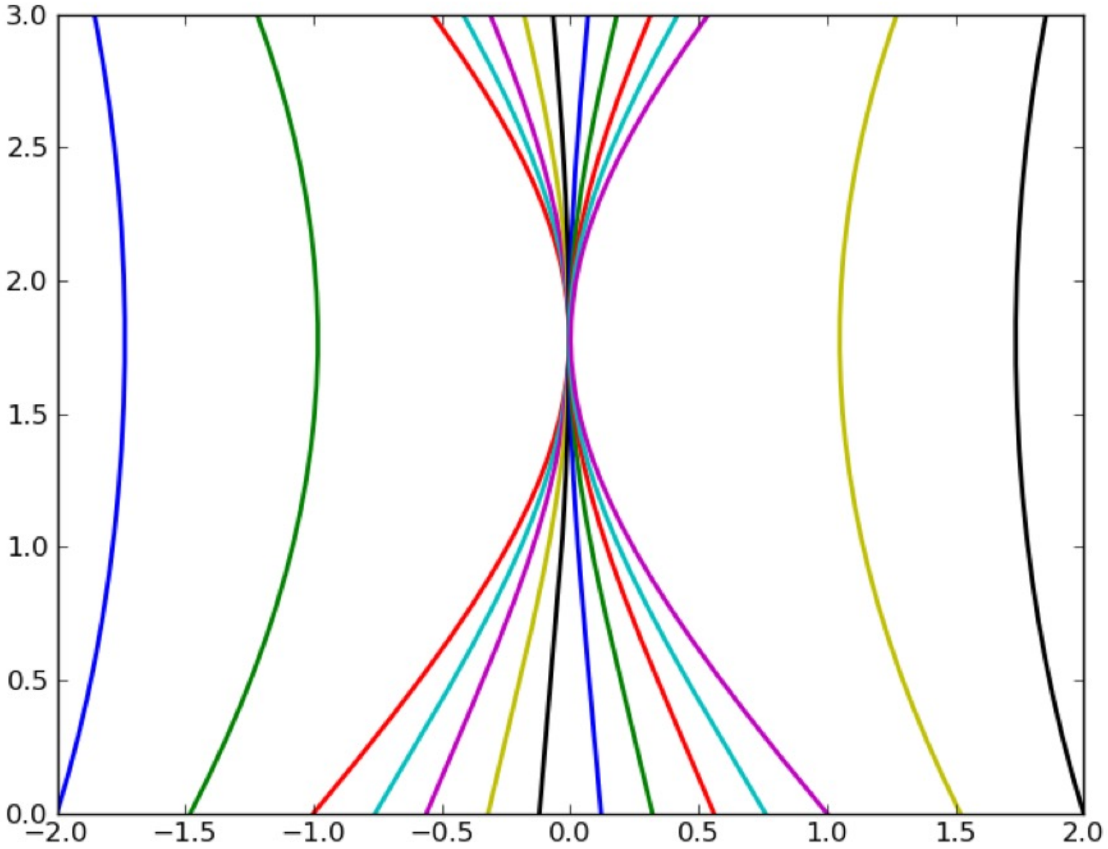
$$\mu$$

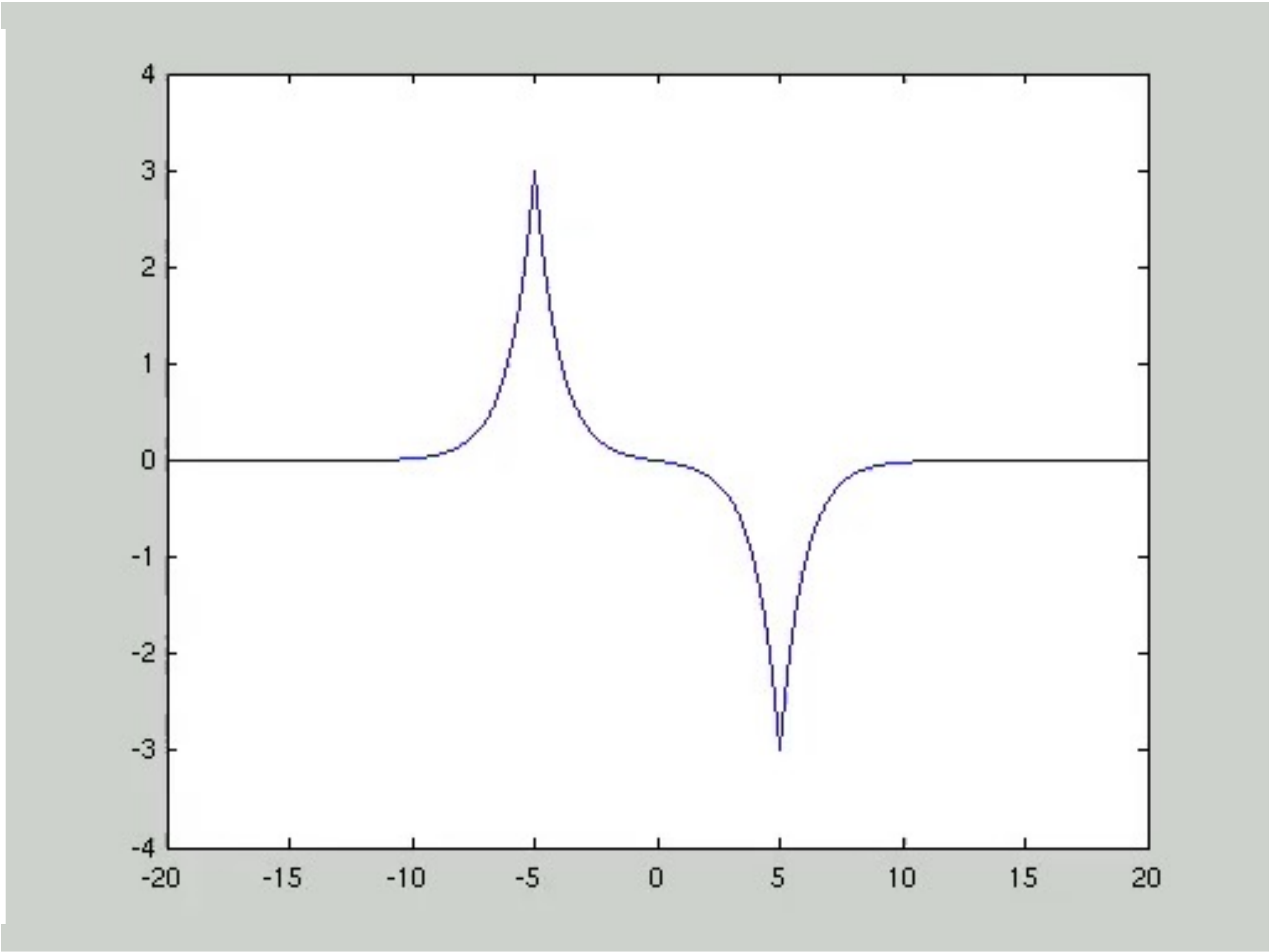
Wellposedness: uniqueness and stability?

$$\mu_{\mathrm{ac}} = (u^2 + u_x^2)dx$$

Encode the energy concentration in the measure

The problem III





Conservative solution

Main alternative: Dissipative solution

All intermediate («alpha-dissipative») solutions possible; Grunert, H., and Raynaud (2015)

Towards the Lipschitz metric I

Formal calculations, assuming smooth solution $(u(t,\,\cdot\,),\mu(t,\,\cdot\,))$

$$F(t,x)=\mu(t,(-\infty,x))=\int_{-\infty}^{x}d\mu(t),$$

$$G(t,x)=\int_{-\infty}^{x}(2P-u^2)(t,y)dy+F(t,x)=2P_x(t,x)+2F(t,x)$$

$$\mathcal{Y}(t,\eta)=\sup\{x\mid G(t,x)<\eta\}.$$

(Domain of definition depends on the total energy)

Formally $\quad \mathcal{Y}(t,G(t,x))=x \qquad G(t,\mathcal{Y}(t,\eta))=\eta$

$$\mathcal{Y}_t+(\frac{2}{3}\mathcal{U}^3+\mathcal{S})\mathcal{Y}_\eta(t,\eta)=\mathcal{U}$$

$$\mathcal{U}(t,\eta)=u(t,\mathcal{Y}(t,\eta))$$

Towards the Lipschitz metric II

$$\mathcal{Y}(t,\eta) = \sup\{x \mid G(t,x) < \eta\}.$$

$$\mathcal{Y}_t + (\frac{2}{3}\mathcal{U}^3 + \mathcal{S})\mathcal{Y}_\eta(t,\eta) = \mathcal{U}$$

$$\mathcal{U}(t,\eta) = u(t,\mathcal{Y}(t,\eta))$$

$$\mathcal{U}_t + (\frac{2}{3}\mathcal{U}^3 + \mathcal{S})\mathcal{U}_\eta = -\mathcal{Q}$$

$$\mathcal{Q}(t,\eta) = P_x(t,\mathcal{Y}(t,\eta)) \qquad \mathcal{P}(t,\eta) = P(t,\mathcal{Y}(t,\eta))$$

$$\mathcal{P}_t + (\frac{2}{3}\mathcal{U}^3 + \mathcal{S})\mathcal{P}_\eta = \mathcal{Q}\mathcal{U} + \mathcal{R}$$

($\mathcal{R}$ and $\mathcal{S}$ explicit and complicated)

The new system of equations

The same system holds for weak solutions:

$$\mathcal{Y}_t + (\frac{2}{3}\mathcal{U}^3 + \mathcal{S})\mathcal{Y}_\eta = \mathcal{U},$$
$$\mathcal{U}_t + (\frac{2}{3}\mathcal{U}^3 + \mathcal{S})\mathcal{U}_\eta = -\mathcal{Q},$$
$$\mathcal{P}_t + (\frac{2}{3}\mathcal{U}^3 + \mathcal{S})\mathcal{P}_\eta = \mathcal{Q}\mathcal{U} + \mathcal{R},$$

Technical problem that the domain depends on the total energy.
Introduce a suitable scaling of functions:

$$\tilde{\mathcal{Y}}(t,\eta) = (2C)^{1/2}\mathcal{Y}(t, 2C\eta)$$
$$\mu(t,\mathbb{R}) = C > 0$$

Estimate time development:

$$\frac{d}{dt}\left\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\right\|^2_{L^2([0,1])} \leq \mathcal{O}(1)\Big(\left\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\right\|^2_{L^2([0,1])} + \left\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\right\|^2_{L^2([0,1])} + \left\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\right\|^2_{L^2([0,1])} + |\sqrt{C_1} - \sqrt{C_2}|^2\Big)$$

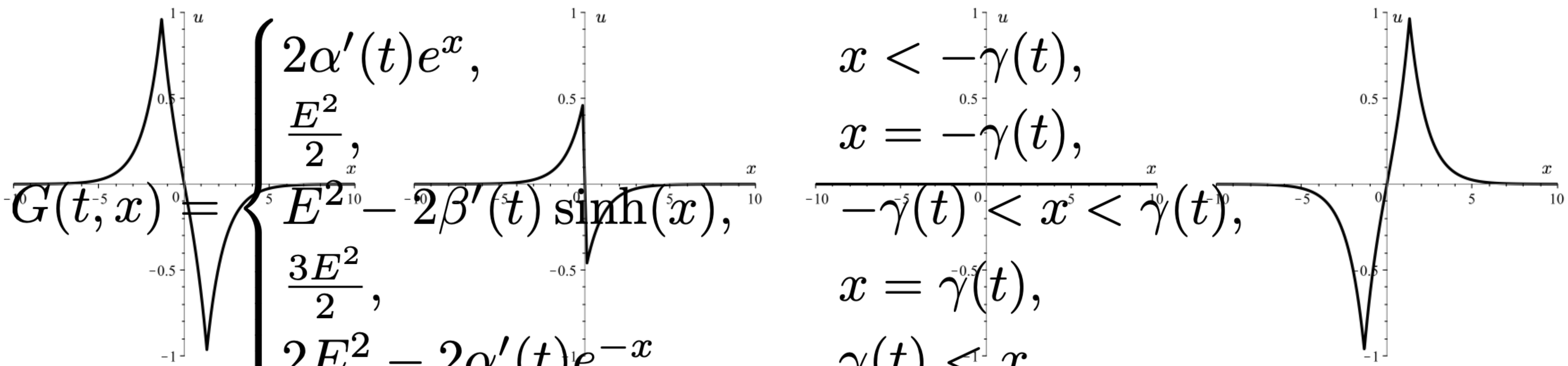
The two-peakon example

$$u(t,x) = \begin{cases} -\alpha(t)e^{x}, & x \le -\gamma(t), \\ \beta(t)\sinh(x), & -\gamma(t) \le x \le \gamma(t), \\ \alpha(t)e^{-x}, & \gamma(t) \le x, \end{cases}$$

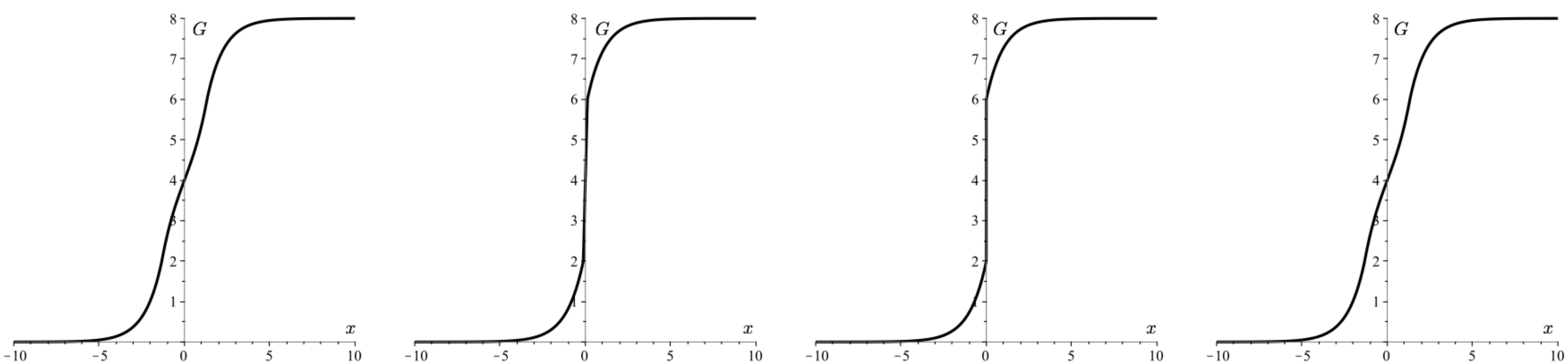
$$\alpha(t) = \frac{E}{2}\sinh(\frac{E}{2}(t-t_0)), \quad \beta(t) = E\frac{1}{\sinh(\frac{E}{2}(t-t_0))}, \quad \gamma(t) = \ln(\cosh(\frac{E}{2}(t-t_0)))$$

$$E = \|u(t)\|_{H^1}$$

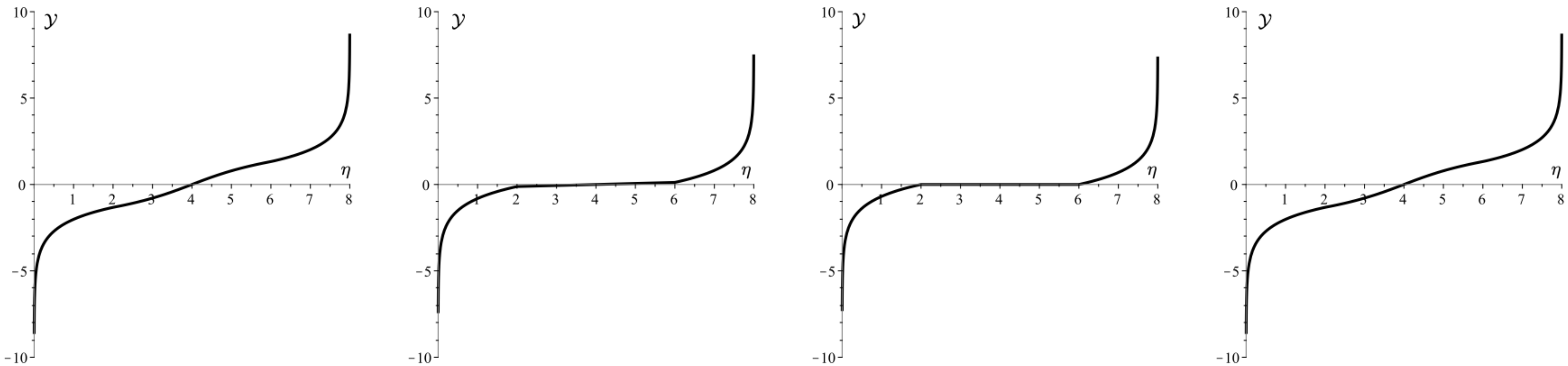
The two-peakon example



$$G(t,x) = \begin{cases} 2\alpha'(t)e^{x}, & x < -\gamma(t), \\ \frac{E^2}{2}, & x = -\gamma(t), \\ E^2 - 2\beta'(t)\sinh(x), & -\gamma(t) < x < \gamma(t), \\ \frac{3E^2}{2}, & x = \gamma(t), \\ 2E^2 - 2\alpha'(t)e^{-x}, & \gamma(t) < x. \end{cases}$$

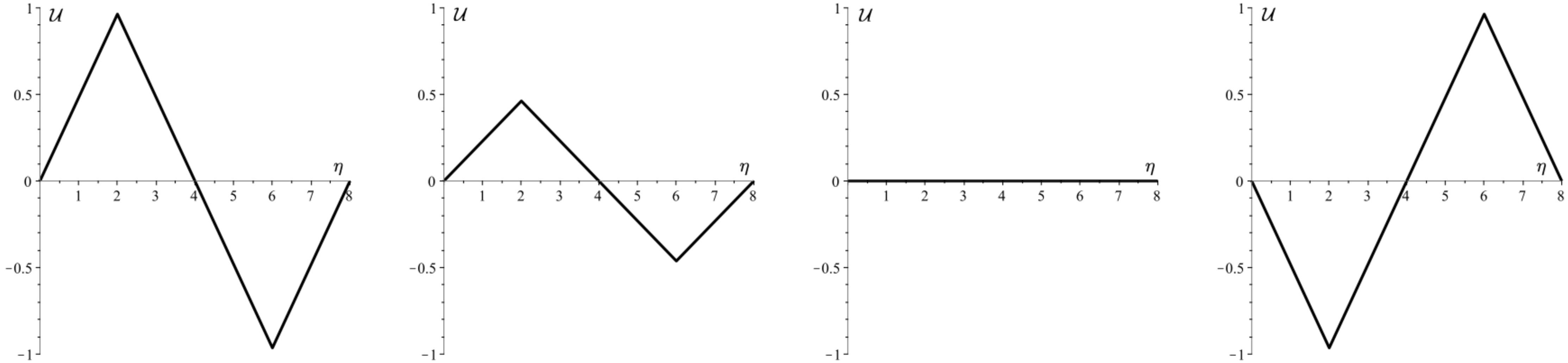


The two-peakon example

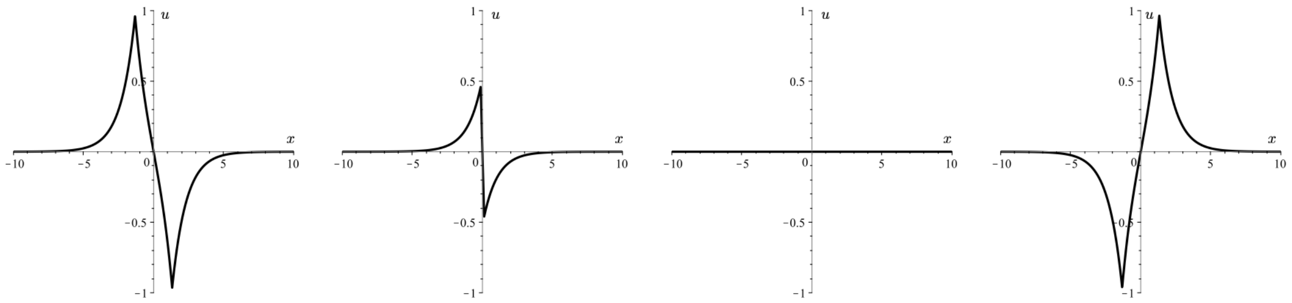


$$
\mathcal{Y}(t,\eta)=\begin{cases}\ln\left(\frac{\eta}{2\alpha'(t)}\right), & 0<\eta\le\frac{E^2}{2},\\ \sinh^{-1}\left(\frac{E^2-\eta}{2\beta'(t)}\right), & \frac{E^2}{2}\le\eta\le\frac{3E^2}{2},\\ \ln\left(\frac{2\alpha'(t)}{2E^2-\eta}\right), & \frac{3E^2}{2}\le\eta<2E^2,\end{cases}
$$

The two-peakon example



$$\mathcal{U}(t,\eta) = \begin{cases} -\dfrac{\alpha(t)}{2\alpha'(t)}\eta, & 0 < \eta \leq \dfrac{E^2}{2}, \\ \dfrac{\beta(t)}{2\beta'(t)}(E^2 - \eta), & \dfrac{E^2}{2} \leq \eta \leq \dfrac{3E^2}{2}, \\ \dfrac{\alpha(t)}{2\alpha'(t)}(2E^2 - \eta), & \dfrac{3E^2}{2} \leq \eta < 2E^2, \end{cases}$$



Definition of metric

$$d((u_1,\mu_1),(u_2,\mu_2)) = \left\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\right\|_{L^2([0,1])} + \left\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\right\|_{L^2([0,1])} + \left\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\right\|_{L^2([0,1])} + |\sqrt{2C_1} - \sqrt{2C_2}|$$

Theorem:

$$d((u_1(t),\mu_1(t)),(u_2(t),\mu_2(t))) \le e^{\mathcal{O}(1)t} d((u_{1,0},\mu_{1,0}),(u_{2,0},\mu_{2,0}))$$

$\mathcal{O}(1) = \mathcal{O}_{\max_j(C_j)}(1)$ as $\max_j(C_j) \to 0$

An estimate

$$\tilde{\mathcal{Y}}_{i,t} + \Big(\frac{2}{3}\frac{1}{A_i^5}\tilde{\mathcal{U}}_i^3 + \frac{1}{A_i^6}\tilde{\mathcal{S}}_i\Big)\tilde{\mathcal{Y}}_{i,\eta} = \tilde{\mathcal{U}}_i,$$

$$\begin{aligned}
\frac{d}{dt}\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t,\eta)d\eta &= 2\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(\tilde{\mathcal{Y}}_{1,t} - \tilde{\mathcal{Y}}_{2,t})(t,\eta)d\eta \\
&= 2\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t,\eta)d\eta \\
&\quad + \frac{4}{3}\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)\Big(\frac{1}{A_2^5}\tilde{\mathcal{U}}_2^3\tilde{\mathcal{Y}}_{2,\eta} - \frac{1}{A_1^5}\tilde{\mathcal{U}}_1^3\tilde{\mathcal{Y}}_{1,\eta}\Big)(t,\eta)d\eta \\
&\quad + 2\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)\Big(\frac{1}{A_2^6}\tilde{\mathcal{S}}_2\tilde{\mathcal{Y}}_{2,\eta} - \frac{1}{A_1^6}\tilde{\mathcal{S}}_1\tilde{\mathcal{Y}}_{1,\eta}\Big)(t,\eta)d\eta \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

The nonlinear variational wave equation

Joint with S. Galtung and K. Grunert and (in progress)

$$u_{tt} - c(u)\big(c(u)u_x\big)_x = 0, \quad u|_{t=0} = u_0, \quad u_t|_{t=0} = u_1$$

$$0 < c^{-1} \leq c(u) \leq c$$

$$R = u_t + c(u)u_x, \quad S = u_t - c(u)u_x$$

$$\begin{aligned} R_t - cR_x &= 0, \\ S_t + cS_x &= 0, \\ (R^2)_t - (cR^2)_x &= 0, \\ (S^2)_t + (cS^2)_x &= 0. \end{aligned}$$

The Camassa–Holm equation with noise

The regularized CH equation with transport (or gradient) noise on Stratonovich form:

$$0 = \mathrm{d}u + \left[u\,\partial_x u + \partial_x P - \varepsilon \partial_x^2 u\right]\,\mathrm{d}t + \sigma\,\partial_x u \circ \mathrm{d}W,$$

$$-\partial_x^2 P + P = u^2 + \frac{1}{2}\left(\partial_x u\right)^2, \quad \text{for } (t,x) \in (0,T) \times \mathbb{S}^1,$$

Deterministic equation: $\sigma = 0$ Inviscid equation $\varepsilon = 0$

Noise on Itô form:

$$\begin{aligned} 0 = \mathrm{d}u &+ \left[u\,\partial_x u + \partial_x P - \varepsilon \partial_x^2 u\right]\,\mathrm{d}t \\ &- \frac{1}{2}\sigma(x)\partial_x\left(\sigma(x)\partial_x u\right)\,\mathrm{d}t + \sigma(x)\partial_x u\,\mathrm{d}W, \end{aligned}$$

$$-\partial_x^2 P + P = u^2 + \frac{1}{2}\left(\partial_x u\right)^2, \quad \text{for } (t,x) \in (0,T) \times \mathbb{S}^1,$$

Former work on the noisy CH equation

Earlier work by (incomplete):

Albeverio, Brzezniak, and Daletskii;
Crisan and Holm;
Y. Chen, J. Duan, and H. Gao;
Y. Chen, H. Gao, and B. Guo;
Rohde and H. Tang
H. Tang

Main theorem in H^1

Suppose $\sigma \in W^{2,\infty}(\mathbb{S}^1)$, $p_0 > 4$, and $u_0 \in L^{p_0}(\Omega; H^1(\mathbb{S}^1))$.

There exists a unique strong H^1 solution to

$$
\begin{aligned}
0 = \mathrm{d}u + \left[u\,\partial_x u + \partial_x P - \varepsilon \partial_x^2 u\right]\,\mathrm{d}t \\
- \frac{1}{2}\sigma(x)\partial_x\left(\sigma(x)\partial_x u\right)\,\mathrm{d}t + \sigma(x)\partial_x u\,\mathrm{d}W, \\
-\partial_x^2 P + P = u^2 + \frac{1}{2}\left(\partial_x u\right)^2, \quad \text{for } (t,x) \in (0,T)\times\mathbb{S}^1,
\end{aligned}
$$

with initial condition $u|_{t=0} = u_0$.

Main theorem in H^m

Fix $m \geq 2$. Suppose $\sigma \in W^{m+1,\infty}(\mathbb{S}^1)$, $p_0 > 4$, and $u_0 \in L^{p_0}(\Omega; H^m(\mathbb{S}^1))$.

There exists a unique strong H^m solution to

$$
\begin{aligned}
0 = \mathrm{d}u + \left[u\,\partial_x u + \partial_x P - \varepsilon \partial_x^2 u\right]\,\mathrm{d}t \\
- \frac{1}{2}\sigma(x)\partial_x\left(\sigma(x)\partial_x u\right)\,\mathrm{d}t + \sigma(x)\partial_x u\,\mathrm{d}W, \\
-\partial_x^2 P + P = u^2 + \frac{1}{2}\left(\partial_x u\right)^2, \quad \text{for } (t,x) \in (0,T) \times \mathbb{S}^1,
\end{aligned}
$$

with initial condition $u|_{t=0} = u_0$.

Solution concepts – weak H^m solution

Fix $m \in \mathbb{N}$ and $p_0 > 4$. Stochastic basis $\mathcal{S} := \left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$

The triple $\left(\mathcal{S}, u, W\right)$ is a weak H^m solution to the noisy CH equation with Λ the law of $u_0 := u(0)$ if the following conditions hold:

$$\int_{H^m(\mathbb{S}^1)} \|v\|_{H^m(\mathbb{S}^1)}^{p_0} \, \Lambda(\mathrm{d}v) < \infty.$$

$u \colon \Omega \times [0,T] \to H^1(\mathbb{S}^1)$ is adapted, with $u \in L^{p_0}\left(\Omega; C([0,T]; H^1(\mathbb{S}^1)\right)$.

$$\begin{cases} u \in L^2\left(\Omega; L^2([0,T]; H^2(\mathbb{S}^1)\right), & \textit{if } m = 1, \\ u \in_{\mathrm{sb}} L^2([0,T]; H^{m+1}(\mathbb{S}^1)) \cap L^\infty([0,T]; H^m(\mathbb{S}^1)), & \textit{if } m \geq 2, \end{cases}$$

$$\begin{aligned} &\int_{\mathbb{S}^1} u(t)\varphi \, \mathrm{d}x - \int_{\mathbb{S}^1} u_0 \varphi \, \mathrm{d}x \\ &\qquad = \int_0^t \int_{\mathbb{S}^1} \left(-u \, \partial_x u \, \varphi + (P - \varepsilon \partial_x u) \, \partial_x \varphi\right) \mathrm{d}x \, \mathrm{d}s \\ &\qquad\quad - \frac{1}{2} \int_0^t \int_{\mathbb{S}^1} \sigma \partial_x u \, \partial_x \left(\sigma \varphi\right) \mathrm{d}x \, \mathrm{d}s - \int_0^t \int_{\mathbb{S}^1} \varphi \sigma \, \partial_x u \, \mathrm{d}x \, \mathrm{d}W(s), \end{aligned}$$

Solution concept – strong H^m solution

Fix a stochastic basis $\mathcal{S} = \big(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\big)$, and a Wiener process W defined on $\mathcal{S}$.

Fix $m \in \mathbb{N}$ and $p_0 > 4$, and consider a random variable $u_0 \in L^{p_0}(\Omega; H^1(\mathbb{S}^1))$.

A process u, defined relative to $\mathcal{S}$, is a strong H^m solution

if $\big(\mathcal{S}, u, W\big)$ is a weak H^m solution with initial distribution $\Lambda := (u_0)_* \mathbb{P}$

Galerkin approximation

Define $\mathbf{\Pi}_n : \left(H^1(\mathbb{S}^1)\right)^* \to H_n$ by

$$u_n = \mathbf{\Pi}_n u = \sum_{i=1}^{n} \langle u, e_i \rangle_{L^2(\mathbb{S}^1)} e_i,$$

with $e_{2j}(x) = \cos(2\pi jx)$ and $e_{2j+1}(x) = \sin(2\pi jx)$, $x \in [0,1]$.

$$u_n(\omega, t, x) = \sum_{i=1}^{n} w_i(\omega, t) e_i(x)$$

$$\begin{aligned} 0 &= \mathrm{d}u_n - \varepsilon \partial_x^2 u_n \,\mathrm{d}t + \mathbf{\Pi}_n \left(u_n \partial_x u_n + \partial_x P[u_n]\right) \mathrm{d}t \\ &\quad - \frac{1}{2} \mathbf{\Pi}_n \left(\sigma \partial_x \left(\sigma \partial_x u_n\right)\right) \mathrm{d}t + \mathbf{\Pi}_n \left(\sigma \partial_x u_n\right) \mathrm{d}W, \end{aligned}$$

$$u_n(0) = \mathbf{\Pi}_n u_0.$$

Galerkin approximation – existence theorem

For any fixed n, there exists a unique $C([0,T];H_n)$-valued adapted process u_n that is a strong solution to

$$
\begin{aligned}
0 = \mathrm{d}u_n - \varepsilon\partial_x^2 u_n \,\mathrm{d}t + \mathbf{\Pi}_n \left(u_n\partial_x u_n + \partial_x P[u_n]\right) \mathrm{d}t \\
- \frac{1}{2}\mathbf{\Pi}_n \left(\sigma\partial_x \left(\sigma\partial_x u_n\right)\right) \mathrm{d}t + \mathbf{\Pi}_n \left(\sigma\partial_x u_n\right) \mathrm{d}W,
\end{aligned}
$$

$$u_n(0) = \mathbf{\Pi}_n u_0.$$

A priori estimates – Galerkin approximation

$$\mathbb{E}\left\|u_n\right\|^2_{L^\infty([0,T];H^1(\mathbb{S}^1))} + \varepsilon\mathbb{E}\int_0^T \left\|\partial_x u_n(t)\right\|^2_{H^1(\mathbb{S}^1)}\,\mathrm{d}t \le C.$$

$$\frac{1}{2}\mathbb{E}\left\|u_n\right\|^{2p}_{L^\infty([0,T];H^1(\mathbb{S}^1))} + \varepsilon p\mathbb{E}\int_0^T \left\|u_n(t)\right\|^{2p-2}_{H^1(\mathbb{S}^1)}\left\|\partial_x u_n(t)\right\|^2_{H^1(\mathbb{S}^1)}\,\mathrm{d}t \le C\mathbb{E}\left\|u_0\right\|^{2p}_{H^1(\mathbb{S}^1)}.$$

Tightness of Galerkin approximation

For each $n \in \mathbb{N}$, let u_n be a solution with $\mathbb{E}\,\|u_0\|^p_{H^1(\mathbb{S}^1)} < \infty$ for $p > 2$.

Strong convergence in (t, x) variables is secured by a priori estimates.

We will replace $\{u_n\}$ by Jakubowski–Shorokhod representations $\{\tilde{u}_n\}$.

Tightness of approximation

$$\begin{aligned}
\mathcal{X}_u &:= L^2([0,T];H^1(\mathbb{S}^1)) \cap C([0,T];H^1_w(\mathbb{S}^1)),\\
\mathcal{X}_W &:= C([0,T]),\\
\mathcal{X}_0 &:= H^1(\mathbb{S}^1),
\end{aligned}$$

$$\mathcal{X} := \mathcal{X}_u \times \mathcal{X}_W \times \mathcal{X}_0$$

Let μ^n denote the (joint) law of the

$(\mathcal{X}, \mathcal{B}(\mathcal{X}))$-valued random variable $\left(u_n, W, \mathbf{\Pi}_n u_0\right)$.

The laws $\{\mu^n\}$ are tight.

The Jakubowski–Skorokhod representation

There exist a probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$

and $\mathcal{X}$-valued random variables $\big\{\big(\tilde{u}_n, \tilde{W}_n, \tilde{u}_{0,n}\big)\big\}_{n\in\mathbb{N}}$, $\big(\tilde{u}, \tilde{W}, \tilde{u}_0\big)$,

defined on $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$, such that along a subsequence (not relabelled),

$$\tilde{u}_n \sim u_n, \quad \tilde{W}_n \sim W, \quad \tilde{u}_{0,n} \sim \mathbf{\Pi}_n u_0$$

and, $\tilde{\mathbb{P}}$-almost surely,

$$\big(\tilde{u}_n, \tilde{W}_n, \tilde{u}_{0,n}\big) \xrightarrow{n\uparrow\infty} \big(\tilde{u}, \tilde{W}, \tilde{u}_0\big) \quad \text{in } \mathcal{X}$$

Weak H^1 solution of noisy CH equation

Suppose $\sigma \in W^{2,\infty}(\mathbb{S}^1)$, $p > 2$, and $\mathbb{E}\,\|u_0\|^p_{H^1(\mathbb{S}^1)} < \infty$.

Let $\tilde{u}$, $\tilde{W}$, $\tilde{u}_0$ be the Jakubowski–Skorokhod a.s. representations, and let $\tilde{\mathcal{S}}$ be the corresponding stochastic basis.

Then $\left(\tilde{\mathcal{S}}, \tilde{u}, \tilde{W}\right)$ is a weak H^1 solution

of the CH equation with noise with initial law $\tilde{\Lambda} := (\tilde{u}_0)_*\,\tilde{\mathbb{P}}$,

$\tilde{u} : \Omega \times [0,T] \to H^1(\mathbb{S}^1)$ is adapted, with paths

$$\tilde{u}(\omega, \cdot\,) \in C([0,T]; H^1_w(\mathbb{S}^1))$$

for $\tilde{\mathbb{P}}$-almost every $\omega \in \Omega$.

Moreover, $\tilde{u} \in L^p(\tilde{\Omega}; L^\infty([0,T]; H^1(\mathbb{S}^1))) \cap L^2(\tilde{\Omega} \times [0,T]; H^2(\mathbb{S}^1))$.

Pathwise uniqueness in H^1 – given solutions

Let u, v be strong H^1 solutions to the viscous stochastic Camassa–Holm equation

with $\sigma \in W^{2,\infty}(\mathbb{S}^1)$ and initial condition $u_0 \in L^p(\Omega; H^1(\mathbb{S}^1))$ for some $p > 4$.

Then $\mathbb{E}\,\|u - v\|_{L^\infty([0,T];H^1(\mathbb{S}^1))} = 0$

Proof (outline):

Since we do not have estimates on

$L^2\left([0,T]; L^\infty(\Omega; W^{1,\infty}(\mathbb{S}^1))\right)$, we need to replace

T by a stopping time $\eta_R < T$

We need to invoke a stochastic Gronwall inequality

Main theorem in H^1

Suppose $\sigma \in W^{2,\infty}(\mathbb{S}^1)$, $p_0 > 4$, and $u_0 \in L^{p_0}(\Omega; H^1(\mathbb{S}^1))$.

There exists a unique strong H^1 solution to

$$
\begin{aligned}
0 &= \mathrm{d}u + \left[u\,\partial_x u + \partial_x P - \varepsilon \partial_x^2 u\right]\,\mathrm{d}t \\
&\quad - \frac{1}{2}\sigma(x)\partial_x\left(\sigma(x)\partial_x u\right)\,\mathrm{d}t + \sigma(x)\partial_x u\,\mathrm{d}W, \\
-\partial_x^2 P + P &= u^2 + \frac{1}{2}\left(\partial_x u\right)^2, \quad \text{for } (t,x) \in (0,T)\times\mathbb{S}^1,
\end{aligned}
$$

with initial condition $u|_{t=0} = u_0$.

Work on $\varepsilon \to 0$ is in progress.

Thank you for your attention!

The scaled system of equations

$$A = C^{1/2}$$

$$\tilde{\mathcal{Y}}_t + (\frac{2}{3}\frac{1}{A^5}\tilde{\mathcal{U}}^3 + \frac{1}{A^6}\tilde{\mathcal{S}})\tilde{\mathcal{Y}}_\eta = \tilde{\mathcal{U}},$$

$$\tilde{\mathcal{U}}_t + (\frac{2}{3}\frac{1}{A^5}\tilde{\mathcal{U}}^3 + \frac{1}{A^6}\tilde{\mathcal{S}})\tilde{\mathcal{U}}_\eta = -\frac{1}{A^2}\tilde{\mathcal{Q}},$$

$$(\tilde{\mathcal{P}}^{1/2})_t + (\frac{2}{3}\frac{1}{A^5}\tilde{\mathcal{U}}^3 + \frac{1}{A^6}\tilde{\mathcal{S}})(\tilde{\mathcal{P}}^{1/2})_\eta = \frac{1}{2A^2}\frac{\tilde{\mathcal{Q}}\tilde{\mathcal{U}}}{\tilde{\mathcal{P}}^{1/2}} + \frac{1}{2A^3}\frac{\tilde{\mathcal{R}}}{\tilde{\mathcal{P}}^{1/2}}$$

$$\tilde{\mathcal{Y}}(t,\eta) = A\mathcal{Y}(t, A^2\eta), \quad \tilde{\mathcal{U}}(t,\eta) = A\mathcal{U}(t, A^2\eta),$$

$$\tilde{\mathcal{P}}^{1/2}(t,\eta) = A\mathcal{P}^{1/2}(t, A^2\eta), \quad \tilde{\mathcal{H}}(t,\eta) = A^3\mathcal{H}(t, A^2\eta)$$

Strong H^1 solution of noisy CH equation

Suppose $\sigma \in W^{2,\infty}(\mathbb{S}^1)$, $p_0 > 4$, and $u_0 \in L^{p_0}(\Omega; H^1(\mathbb{S}^1))$.

There exists a strong H^1 solution of the stochastic CH equation

with initial condition $u|_{t=0} = u_0$.

$$\mathcal{X}_u := L^2([0,T]; H^1(\mathbb{S}^1)) \cap C([0,T]; H^1_w(\mathbb{S}^1)),$$
$$\mathcal{X}_W := C([0,T]),$$
$$\mathcal{X}_0 := H^1(\mathbb{S}^1),$$

$$\mathcal{Y} := \mathcal{X}_u \times \mathcal{X}_u \times \mathcal{X}_W \times \mathcal{X}_0$$

$$\mu^{m,n} := \mu_u^m \otimes \mu_u^n \otimes \mu_W^m \otimes \mu_0^m.$$

Jakubowski–Skorokhod representation theorem gives convergence of a subsequence.

Gyöngy–Krylov theorem.

Pathwise uniqueness in H^m – given solutions

Let u, v be strong H^m solutions to the viscous stochastic CH equation

with $\sigma \in W^{m+1,\infty}(\mathbb{S}^1)$ and initial condition $u|_{t=0} = v|_{t=0} = u_0 \in L^8(\Omega; H^m(\mathbb{S}^1))$.

Uniqueness holds:

$$\mathbb{E}\,\|u - v\|_{L^\infty([0,T];H^m(\mathbb{S}^1))} = 0.$$

Proof:

Since we do not have estimates on $L^2\left([0,T]; L^\infty(\Omega; W^{1,\infty}(\mathbb{S}^1))\right)$, we need to replace T by a stopping time $\eta_R < T$

We need to invoke a stochastic Gronwall inequality

Even more details

$$\tilde{\mathcal{U}}_i^2 \tilde{\mathcal{Y}}_{i,\eta}(t,\eta) \le A_i^5 \le A^5$$

$$J_2 = \frac{1}{A^5} \Big| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)[\tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{1,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_2^2 \le \tilde{\mathcal{U}}_1^2} + \tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_1^2 < \tilde{\mathcal{U}}_2^2}](\tilde{\mathcal{U}}_2 + \tilde{\mathcal{U}}_1)(\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)(t,\eta)\,d\eta \Big|$$

$$\le \frac{1}{A^5} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2||\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2|(2\tilde{\mathcal{U}}_1^2 \tilde{\mathcal{Y}}_{1,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_2^2 \le \tilde{\mathcal{U}}_1^2} + 2\tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_1^2 < \tilde{\mathcal{U}}_2^2})(t,\eta)\,d\eta$$

$$\le 2\left(\left\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\right\|^2 + \left\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\right\|^2\right)$$

One more thing...

$$
\begin{aligned}
J_3 &= \frac{1}{A^5}\Big|\int_0^1 (\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)\tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{U}}_j^2)(\tilde{\mathcal{Y}}_{2,\eta}-\tilde{\mathcal{Y}}_{1,\eta})(t,\eta)d\eta\Big| \\
&= \Big| -\frac{1}{2A^5}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2\tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{U}}_j^2)(t,\eta)\Big|_{\eta=0}^{1} \\
&\qquad + \frac{1}{2A^5}\int_0^1 (\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2 \frac{d}{d\eta}(\tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{U}}_j^2))(t,\eta)d\eta\Big| \\
&= \frac{1}{2A^5}\Big|\int_0^1 (\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2 \frac{d}{d\eta}(\tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{U}}_j^2))(t,\eta)d\eta\Big| \\
&\leq \frac{3}{4A}\left\|\tilde{\mathcal{U}}_1(t,\,\cdot\,)\right\|_{L^\infty}\int_0^1 (\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\eta)d\eta \\
&\leq \mathcal{O}(1)\left\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\right\|^2.
\end{aligned}
$$

The role of kappa

$$u_t - u_{xxt} + 2\boxed{\kappa} u_x + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, \quad u|_{t=0} = \bar{u}$$

κ positive is required in the water wave setting

If u solves the Camassa–Holm equation with positive κ, then

$$v(t,x) = u(t, x - \kappa t) + \kappa$$

solves the equation with $\kappa = 0$

Only explicit decaying solutions when κ vanishes

The case of non-vanishing asymptotics studied in Grunert, H., and Raynaud (2012 & 2014)

Classical results

Constantin, Escher (*Acta*, 1998): Proof of wavebreaking

Constantin, Escher, and Molinet (1998 & 2000):

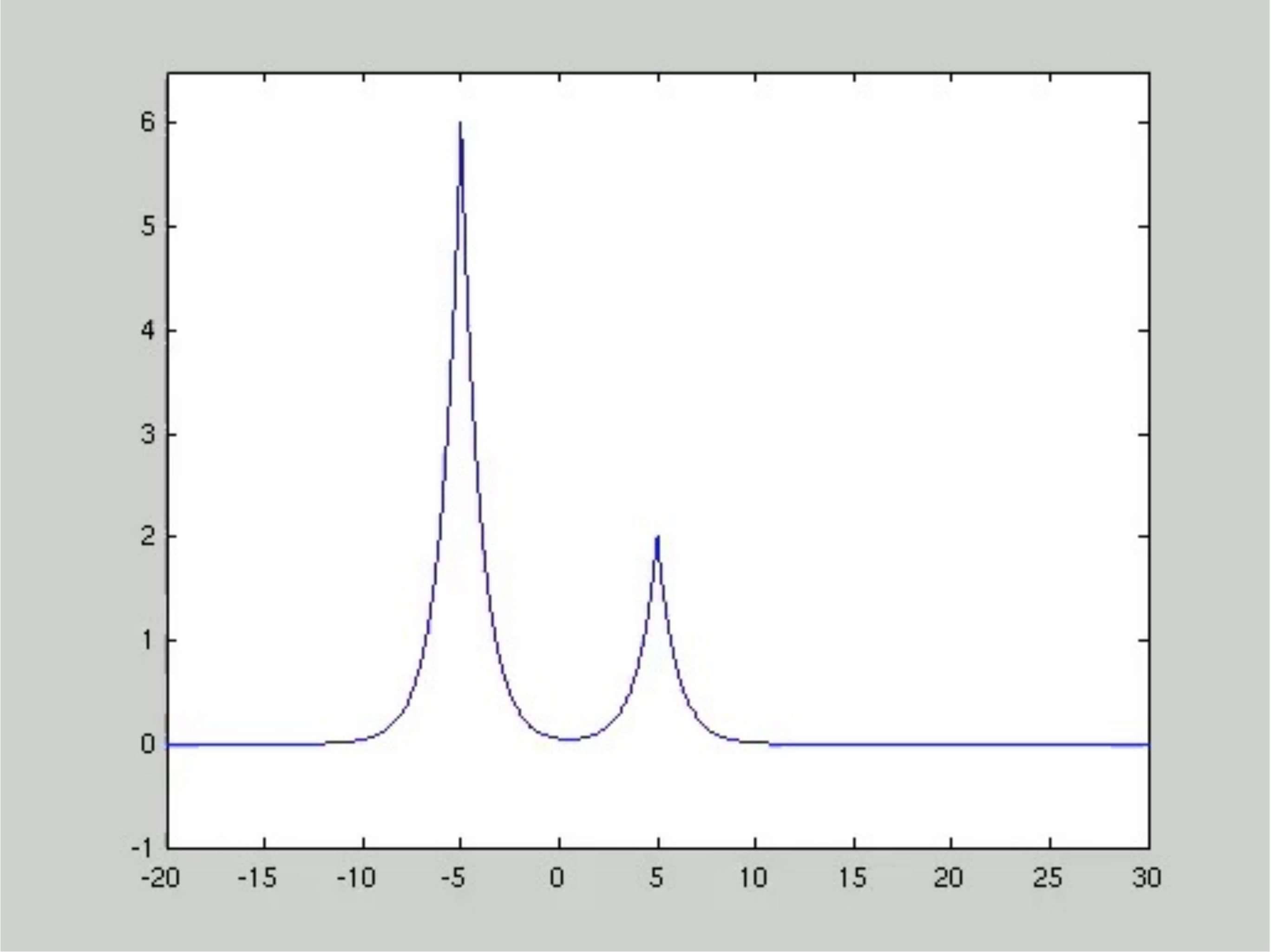
If $u|_{t=0} = \bar{u} \in H^1(\mathbb{R})$ and $\bar{m} := \bar{u} - \bar{u}''$ is a nonnegative Radon measure, then the Camassa–Holm equation has a unique global weak solution $u \in C([0,T], H^1(\mathbb{R}))$ for any $T > 0$.

However, if $\bar{u}$ is odd with $\bar{u}_x(0) < 0$ and $\bar{u} \in H^3(\mathbb{R})$, then T is finite.

Bressan and Constantin, H. and Raynaud (2007 & 2008):

Wellposedness of conservative and dissipative solutions
using a novel change of variables

Multi-peakons



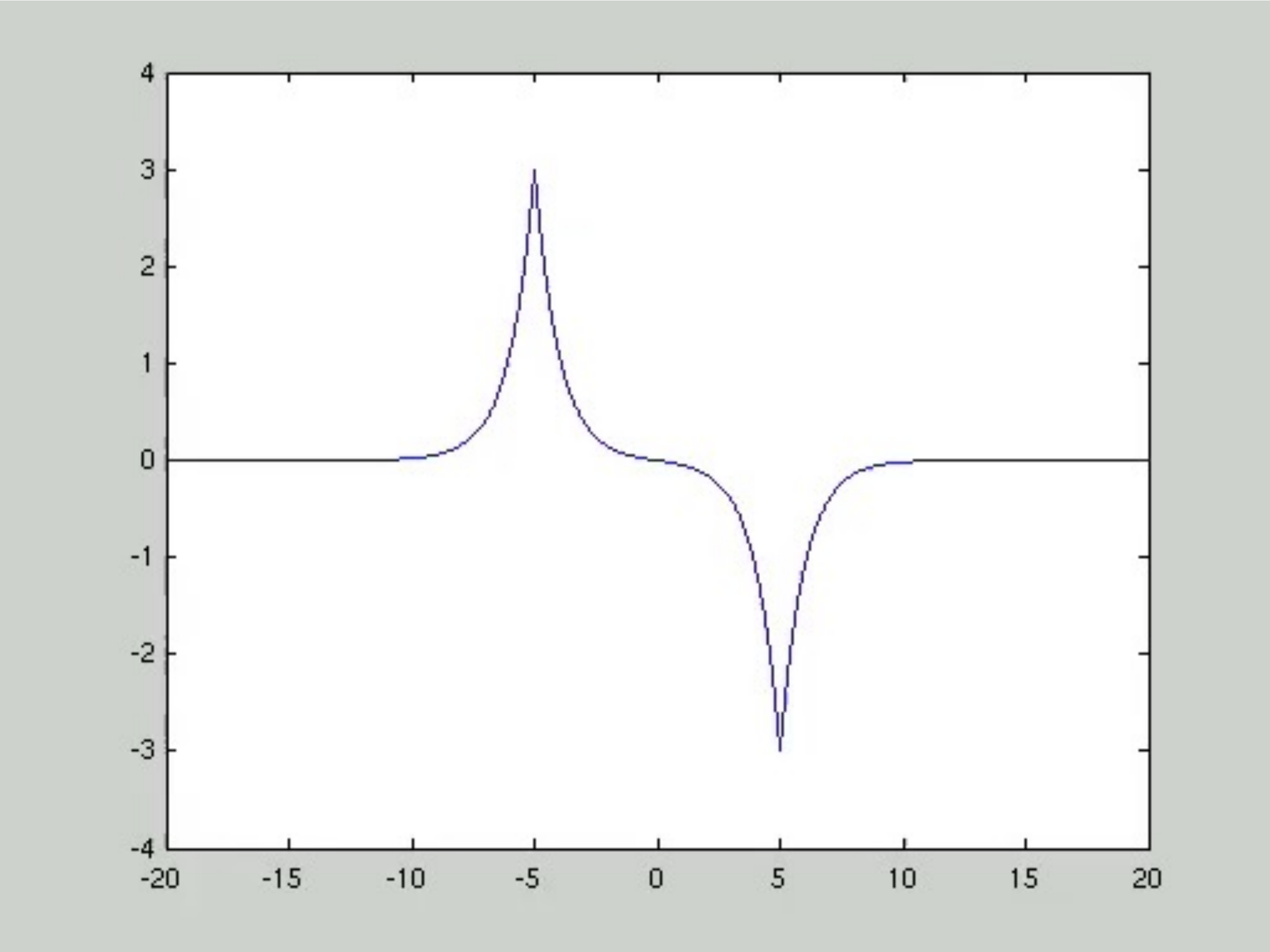
$$H(p,q) = \frac{1}{2}\sum_{i,j=1}^{n} p_i p_j e^{-|q_i - q_j|}$$

$$\dot{q}_i = \frac{\partial H(p,q)}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H(p,q)}{\partial q_i}$$

$$\dot{q}_i = \sum_{j=1}^{n} p_j e^{-|q_i - q_j|}, \quad \dot{p}_i = \sum_{j=1}^{n} p_i p_j \operatorname{sgn}(q_i - q_j) e^{-|q_i - q_j|}.$$

$u(t,x) = \sum_{i=1}^{n} p_i(t) e^{-|x - q_i(t)|}$ is a weak solution of the Camassa–Holm equation

The two-peakon example



Upshot first equation

$$\frac{d}{dt}\left\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\right\|^2_{L^2([0,1])} \leq \mathcal{O}(1)\Big(\left\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\right\|^2_{L^2([0,1])}+\left\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\right\|^2_{L^2([0,1])} + \left\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\right\|^2_{L^2([0,1])}+|\sqrt{C_1}-\sqrt{C_2}|^2\Big)$$

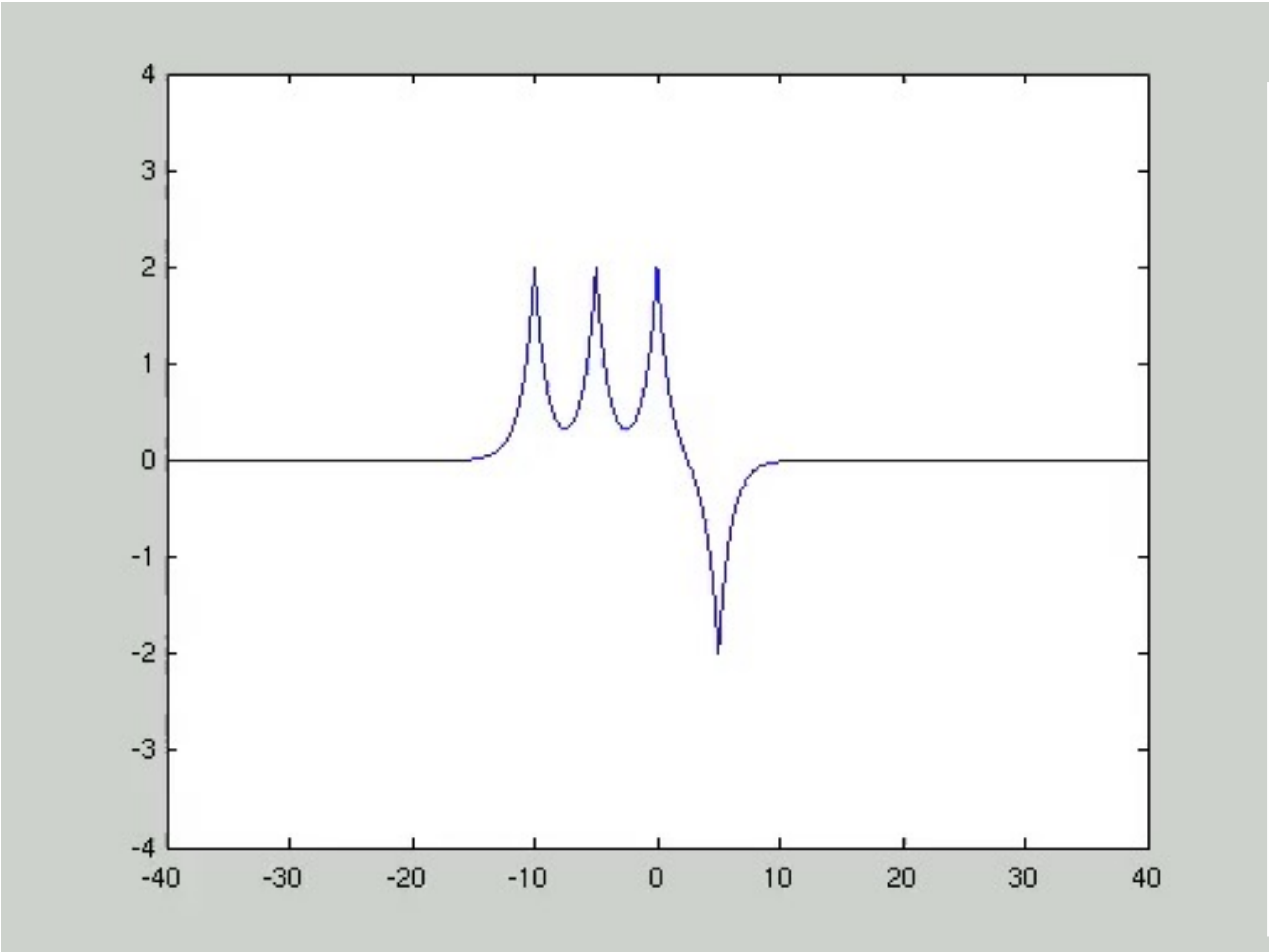
Similar for the other equations

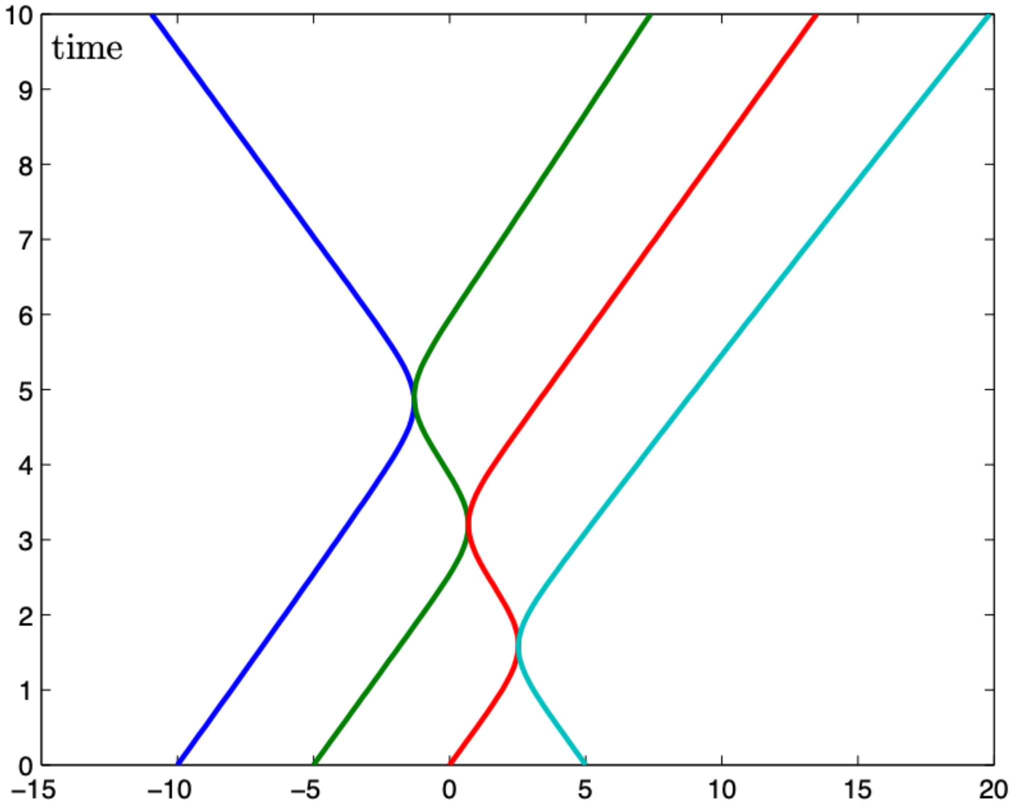
$$d((u_1,\mu_1),(u_2,\mu_2)) = \left\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\right\|_{L^2([0,1])}+\left\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\right\|_{L^2([0,1])} + \left\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\right\|_{L^2([0,1])}+|\sqrt{2C_1}-\sqrt{2C_2}|$$

Theorem:

$$d((u_1(t),\mu_1(t)),(u_2(t),\mu_2(t))) \leq e^{\mathcal{O}(1)t} d((u_{1,0},\mu_{1,0}),(u_{2,0},\mu_{2,0}))$$

Four-multipeakon





Lagrangian variables

Initial data: (u_0, μ_0)

New Lagrangian variables:

$$
\begin{aligned}
y(0,\xi) &= \sup\{x \mid x + F_0(x) < \xi\},\\
H(0,\xi) &= \xi - y(0,\xi),\\
U(0,\xi) &= u(0, y(0,\xi)),
\end{aligned}
$$

$$F_0(x) = \int_{-\infty}^{x} d\mu_0$$

Time evolution:

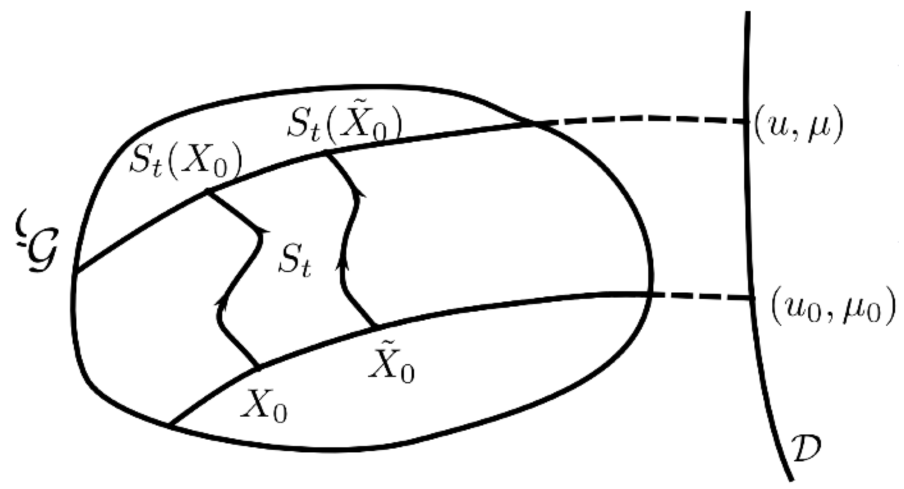
$$
\begin{aligned}
y_t(t,\xi) &= U(t,\xi),\\
U_t(t,\xi) &= -Q(t,\xi),\\
H_t(t,\xi) &= (U^3 - 2PU)(t,\xi)
\end{aligned}
$$

This system has smooth global solution

Relation to other variables: $\mathcal{Y}(t,\eta) = y(t, l(t,\eta))$

Time evolution in Lagrangian variables

$X = (y, U, H) \qquad X(t) = S_t(X_0)$



Problem: Lack of uniqueness in Lagrangian variables – *relabeling*

Recover measure:

$$\mu = y_{\#}(H_\xi d\xi)$$

Further details – the simplest case

$$|I_1| = |2\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t,\eta)d\eta| \leq \int_0^1 ((\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 + (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2)(t,\eta)d\eta$$

$$= \left\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\right\|^2 + \left\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\right\|^2$$

Further details

$$
\begin{aligned}
\frac{3}{4}|I_2| &= \Big|\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)\Big(\frac{1}{A_2^5}\tilde{\mathcal{U}}_2^3\tilde{\mathcal{Y}}_{2,\eta} - \frac{1}{A_1^5}\tilde{\mathcal{U}}_1^3\tilde{\mathcal{Y}}_{1,\eta}\Big)(t,\eta)d\eta\Big| \\
&\leq \frac{1}{(\max_j(A_j))^5}\Big|\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(\tilde{\mathcal{U}}_2^3\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{U}}_1^3\tilde{\mathcal{Y}}_{1,\eta})(t,\eta)d\eta\Big| \\
&\quad + \frac{|A_1^5 - A_2^5|}{A_1^5A_2^5}\Big|\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(\tilde{\mathcal{U}}_1^3\tilde{\mathcal{Y}}_{1,\eta}\mathbb{1}_{A_1\leq A_2} + \tilde{\mathcal{U}}_2^3\tilde{\mathcal{Y}}_{2,\eta}\mathbb{1}_{A_2<A_1})(t,\eta)d\eta\Big| \\
&\leq \frac{1}{A^5}\Big|\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)d\eta\Big| \\
&\quad + \frac{1}{A^5}\Big|\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)[\tilde{\mathcal{U}}_1\tilde{\mathcal{Y}}_{1,\eta}\mathbb{1}_{\tilde{\mathcal{U}}_2^2\leq\tilde{\mathcal{U}}_1^2} + \tilde{\mathcal{U}}_1\tilde{\mathcal{Y}}_{2,\eta}\mathbb{1}_{\tilde{\mathcal{U}}_1^2<\tilde{\mathcal{U}}_2^2}] \\
&\qquad\qquad\qquad\qquad \times (\tilde{\mathcal{U}}_2 + \tilde{\mathcal{U}}_1)(\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)(t,\eta)d\eta\Big| \\
&\quad + \frac{1}{A^5}\Big|\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)\tilde{\mathcal{U}}_1\min_j(\tilde{\mathcal{U}}_j^2)(\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t,\eta)d\eta\Big| \\
&\quad + \frac{|A_1^5 - A_2^5|}{A_1^5A_2^5}\Big|\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(\tilde{\mathcal{U}}_1^3\tilde{\mathcal{Y}}_{1,\eta}\mathbb{1}_{A_1\leq A_2} + \tilde{\mathcal{U}}_2^3\tilde{\mathcal{Y}}_{2,\eta}\mathbb{1}_{A_2<A_1})(t,\eta)d\eta\Big| \\
&= J_1 + J_2 + J_3 + J_4.
\end{aligned}
$$