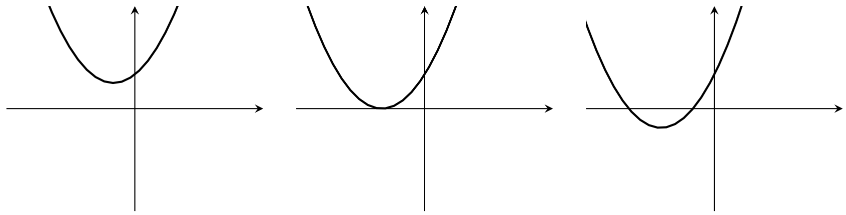
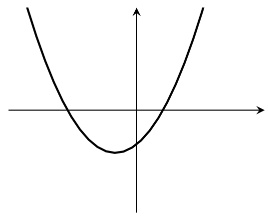
Warm-up

1. Sketch $y = ax^2 + bx + c$ in each of the following eight cases;

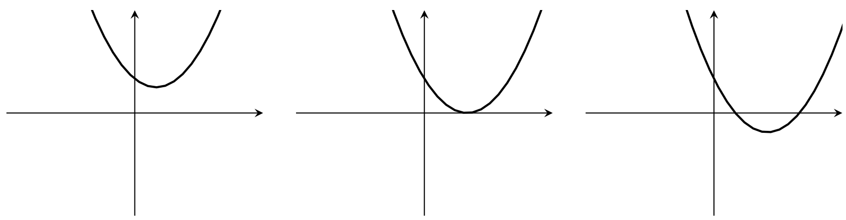
   - $a > 0$, $b > 0$, $c > 0$. There could be 0 or 1 or 2 roots.



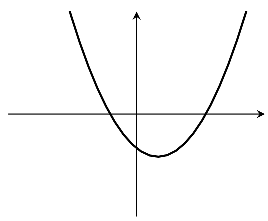
   - $a > 0$, $b > 0$, $c < 0$. Must have two roots.



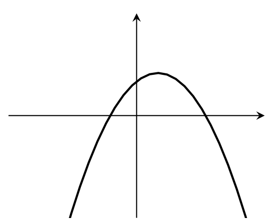
   - $a > 0$, $b < 0$, $c > 0$. There could be 0 or 1 or 2 roots.



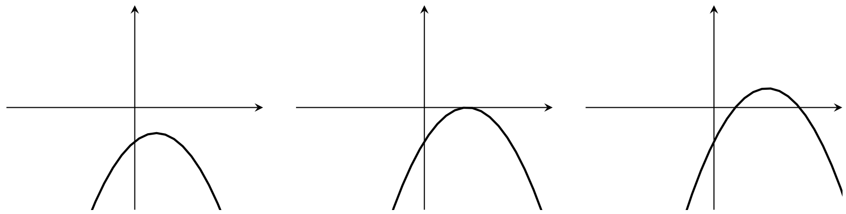
   - $a > 0$, $b < 0$, $c < 0$. Must have two roots.



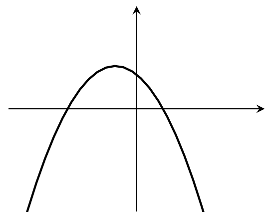
   - $a < 0$, $b > 0$, $c > 0$. Must have two roots.



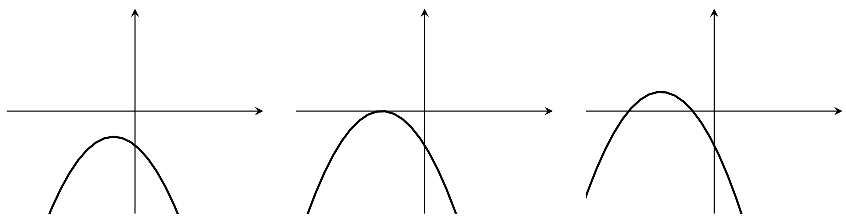
   - $a < 0$, $b > 0$, $c < 0$. There could be 0 or 1 or 2 roots.



- $a < 0$, $b < 0$, $c > 0$. Must have two roots.

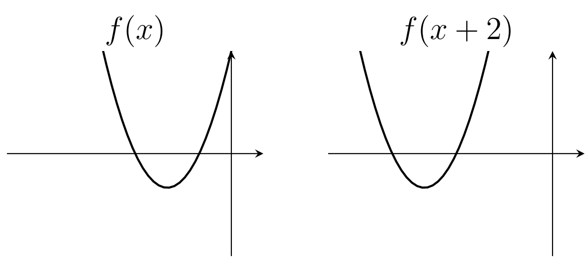


- $a < 0$, $b < 0$, $c < 0$. There could be 0 or 1 or 2 roots.



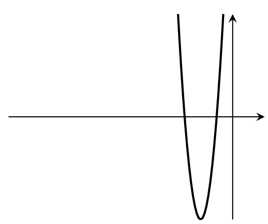
2. Let $f(x) = x^2 + 4x + 3$. Sketch the graph of $y = f(x + 2)$.

   - Note that $x^2 + 4x + 3 = (x + 3)(x + 1)$. The graph of $y = f(x + 2)$ is the graph of $y = f(x)$ after it has been translated two units to the left.

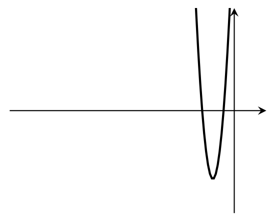


3. Sketch the graph of $y = 3f(2x)$.

   - The graph is "squashed" by a factor of 2 parallel to the x-axis, then "stretched" by a factor of 3 parallel to the y-axis.

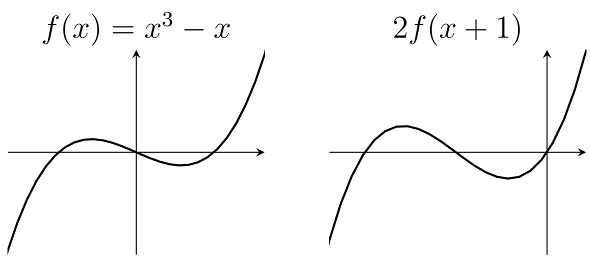


4. Sketch the graph of $y = 2f(3x)$. Is that the same as the previous graph?

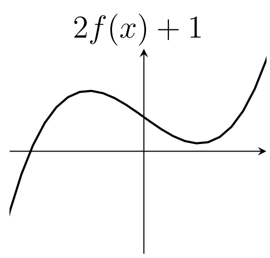


- No. For example, the roots aren't in the same places. We could also check that $3\left((2x)^2 + 4(2x) + 3\right)$ isn't the same thing as $2\left((3x)^2 + 4(3x) + 3\right)$, but that feels like a lot of work.

5. Let $f(x) = x^3 - x$. Sketch the graph of $y = 2f(x+1)$.



6. Sketch the graph of $y = 2f(x) + 1$. Is that the same as the previous graph?

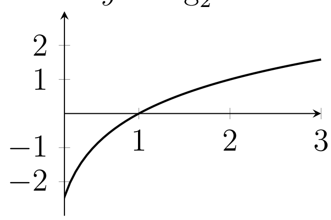


- No. We could compare, for example, the values of $2f(x+1)$ and $2f(x) + 1$ when $x = 0$. Those are $2f(1)$ and $2f(0) + 1$, which are 0 and 1, so not the same. Note that even if these values were the same, then the graphs might still be different (this is just a spot-check at $x = 0$).

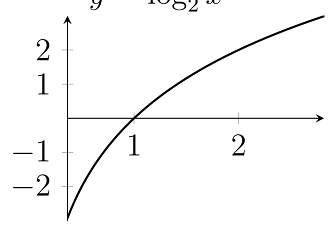
7. Sketch the graph of $y = \log_2 x$. Sketch the graph of $y = \log_2(x^2)$.

- Note that $\log_2(x^2) = 2\log_2 x$

$y = \log_2 x$

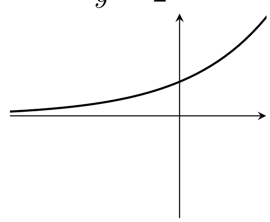


$y = \log_2 x^2$

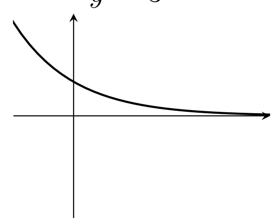


8. Sketch the graph of $y = 2^x$. Sketch the graph of $y = \left(\frac{1}{3}\right)^x$.

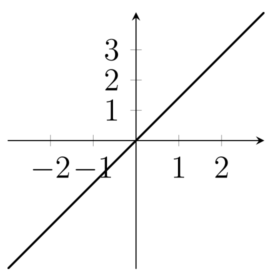
$y = 2^x$



$y = 3^{-x}$



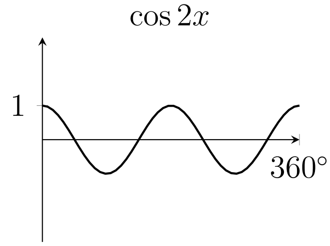
9. Sketch the graph of $y = x\sqrt{2}$. Does the graph go through the point $\left(\dfrac{p}{q}, \dfrac{r}{s}\right)$ for any positive whole numbers p, q, r, s?



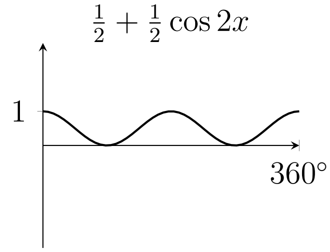
- Let's suppose that $\dfrac{r}{s} = \sqrt{2} \times \dfrac{p}{q}$. Then we would have $\sqrt{2} = \dfrac{qr}{ps}$. But $\sqrt{2}$ is irrational, so we can't have p, q, r, s all be positive whole numbers, or we would have written $\sqrt{2}$ as a fraction.

10. Sketch the graph of $y = \cos 2x$. Sketch the graph of $y = \frac{1}{2} + \frac{1}{2}\cos 2x$.

$\cos 2x$

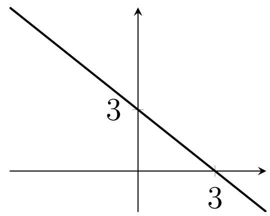


$\frac{1}{2} + \frac{1}{2}\cos 2x$



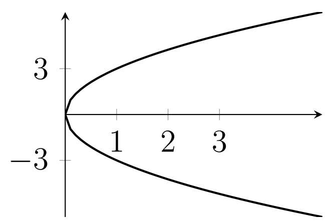
11. Sketch all the points (x, y) that satisfy $x + y = 3$.

    - That's the straight line $y = 3 - x$.



12. Sketch all the points (x, y) that satisfy $y^2 = 9x$.

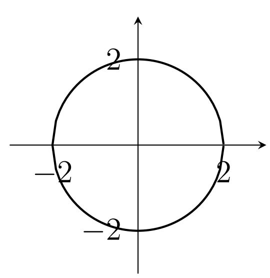
    - We could have $y = 3\sqrt{x}$ or $y = -3\sqrt{x}$.



    - Alternatively, if you switch x and y you get a parabola. Switching x and y is a reflection in the line $y = x$ (not on the list of transformations on the MAT syllabus, but you might have come across it).

13. Sketch all the points (x, y) that satisfy $x^2 = 4 - y^2$.

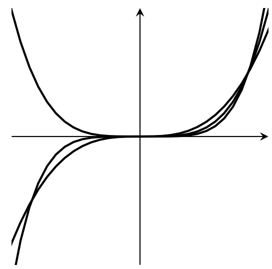
    - That's the circle $x^2 + y^2 = 4$.



    - Note that for these last two, there is no function $y = f(x)$ that covers all of the points in the sketch, because there are some values of x where there are two or more points that satisfy the equation. A classic question that's like this is "sketch all the points (x, y) that satisfy $\cos(x) = \cos(y)$".

MAT questions

MAT 2015 Q1I



- First draw a rough sketch. When $x > 0$ all the values are positive. When $x < 0$ two of them are negative and one is positive. All the graphs go through $(0, 0)$ and $(1, 1)$ and two of them go through $(-1, -1)$. There are no other points of intersection.
- Now count the regions; there are some big regions like the one above all the graphs and the region below all the graphs. Over in $x < 0$ there are three more regions. For $x > 0$ it's a bit more complicated; we've counted the region below all three graphs and the region above all three graphs, but there are four more regions in between the graphs (two in $0 < x < 1$ and then two regions in $x > 1$).
- Overall that adds up to nine regions.
- The answer is (d).

MAT 2014 Q1B

- The function inside the brackets is $(x-1)^2+1$. That's always positive (so this logarithm is always defined), and in fact it's always bigger or equal to one, so this logarithm will only give us positive values.
- The particular value $x = 1$ is interesting, because that's the location of the minimum of the quadratic inside the brackets. At that value of x, we have $y = \log_{10} 1 = 0$. So the graph includes the point $(1, 0)$. Only one of the options does that.
- The answer is (e).

MAT 2017 Q1D

- First, let's look at the main features of $f(x)$. It's got very large absolute value near $x = 1$. For large x (either positive or negative) it looks a bit like the straight line $y = -x$ to me.
- Now let's think about $y = f(-x)$. That would be the result of a reflection in the y-axis, so we would get a graph with big values near $x = -1$ and which looks like $y = x$ for large x.
- Now think about $y = -f(-x)$. That would be a reflection of the previous graph in the x-axis, so we would get a graph with big values near $x = -1$ and which looks like $y = -x$ for large x.
- Only one of the graphs does that.
- The answer is (c).

MAT 2013 Q3

(i) We've got to be careful, because $f(x)$ changes sign in between 0 and 2. We want the integral from 0 to k, then we want to subtract the integral from k to 2 because the function is negative in that region (and areas are positive). So we want

$$A(k) = \int_0^k f(x)\,\mathrm{d}x - \int_k^2 f(x)\,\mathrm{d}x.$$

(ii) We know that $f_k(x)$ is a cubic polynomial, but it's a little bit complicated because that polynomial involves k. If we multiply out the middle bracket of $f_k(x)$, then we get

$$f_k(x) = x^2(x-2) - kx(x-2),$$

so the integral

$$\int f(x)\,\mathrm{d}x = \int x^2(x-2)\,\mathrm{d}x - k\int x(x-2)\,\mathrm{d}x.$$

That's removed k from the function inside each integral. When we do these integrals we'll get a degree-4 polynomial that doesn't depend on k and degree-3 polynomial that's multiplied by k. We have two definite integrals, each of which involves plugging k into those polynomials. So we we'll end up with a degree-4 polynomial in k and a degree-3 polynomial in k that's multiplied by k. Overall, that's a polynomial in k of degree at most 4 ("at most" because the coefficient of k^4 might cancel between the two terms).

(iii) If we replace x with $1+t$ then we get $f_k(1+t) = (1+t)(1+t-k)(1+t-2)$. Now I've got to really concentrate to work out $-f_{2-k}(1-t)$. I need to take $f_k(x) = x(x-k)(x-2)$ and replace every x with $1-t$ and replace every k with a $2-k$. And remember the minus sign out the front. I get

$$-f_{2-k}(1-t) = (1-t)(1-t-(2-k))(1-t-2) = (1-t)(k-1-t)(t+1)$$

where I've done some tidying up to get the last expression. That does match the expression for $f_k(1+t)$.

(iv) Let's interpret the previous result in terms of transformations. On the left, we've got the graph of $f_k(t)$, but it's been translated to the left by a distance of 1. On the right, we've got the graph of $f_{2-k}(t)$, but... it's had the t switched to a $1-t$. What's that? It's what would happen if we replaced t with $1+t$ and then replaced t in that new expression with $-t$. So that's a translation by one unit to the left and then a reflection in the y-axis. We actually have $-f_{2-k}(1-t)$ with a minus sign out the front, which represents a reflection in the x-axis.

So putting this together, if we take the graphs of $f_k(t)$ and $f_{2-k}(t)$ and translate them both one unit to the left, then we reflect the second graph in the y-axis and in the x-axis, we get the same graph (those things are equal by the previous question part). Said differently without the translation, $f_{2-k}(t)$ is what we get if we reflect $f_k(t)$ in the line $x = 1$ and in then in the x-axis.

Those reflections don't affect the area, so $A(k)$, the area for $f_k(x)$, is the same as the area $A(2-k)$ for $f_{2-k}(x)$.

(v) We know that $A(k)$ is a degree-4 polynomial. We can write it as $A(k) = a(k-1)^4 + d(k-1)^3 + b(k-1)^2 + f(k-1) + c$, where I've chosen unusual names for the coefficients so that it matches with the answer I'm working towards. We just need to prove that $d = 0$ and $f = 0$. We've got the fact $A(2-k) = A(k)$. If we plug in $2-k$ into our expression for $A(k)$ and equate coefficients, we get $d = 0$ and $f = 0$. That leaves the expression in the question.

Reflection

- In the long question we thought about some reflections. In general, $f(2-x)$ is the reflection of $f(x)$ in the line $x = 1$ (note; not $x = 2$). We can do something similar for reflections in other lines $x = c$.
- In one of the warm-up problems we thought about reflecting in the line $y = x$ by switching x with y. That's a bit harder to imagine. Reflecting in the line $y = mx$ is very fiddly; there's an equation for it, but you don't need to know it for MAT.