Warm-up

- Simplify $\log_{10} 3 + \log_{10} 4$ into a single term.
  - This is $\log_{10} 12$.
- If we write $a = \ln 2$ and $b = \ln 5$, then write the following in terms of a and b.

$$\ln 1024, \quad \ln 40, \quad \ln \sqrt{2/5}, \quad \ln 1/10, \quad \ln 1.024.$$

  - $\ln 1024 = \ln\left(2^{10}\right) = 10 \ln 2 = 10a$.
  - $\ln 40 = \ln 8 + \ln 5 = 3a + b$.
  - $\ln \sqrt{2/5} = \frac{1}{2} \ln 2/5 = \frac{1}{2}(a - b)$.
  - $\ln 1/10 = -\ln 10 = -\ln 2 - \ln 5 = -a - b$.
  - $\ln 1.024 = \ln 1024 + \ln 1/1000 = 10a + 3(-a - b) = 7a - 3b$.
  - There are other solutions, partly because $b = a \times \log_2 5$.
- Expand $\left(e^x + e^{-x}\right)\left(e^y - e^{-y}\right) + \left(e^x - e^{-x}\right)\left(e^y + e^{-y}\right)$.
  - $e^{x+y} + e^{y-x} - e^{x-y} - e^{-x-y} + e^{x+y} - e^{y-x} + e^{x-y} - e^{-x-y}$.
  - That's $2e^{x+y} - 2e^{-x-y}$.
- Expand $\left(e^x + e^{-x}\right)\left(e^y + e^{-y}\right) + \left(e^x - e^{-x}\right)\left(e^y - e^{-y}\right)$.
  - $e^{x+y} + e^{y-x} + e^{x-y} + e^{-x-y} + e^{x+y} - e^{y-x} - e^{x-y} + e^{-x-y}$.
  - That's $2e^{x+y} + 2e^{-x-y}$.
- Solve $2^x = 3$. Solve $0.5^x = 3$. Solve $4^x = 3$.
  - $2^x = 3$ is what it means for x to be $\log_2 3$.
  - If $0.5^x = 3$ then $2^{-x} = 3$ so $x = -\log_2 3$.
    Alternatively, just write down $x = \log_{0.5} 3$.
  - If $4^x = 3$ then $2^{2x} = 3$ so $x = \frac{1}{2} \log_2 3$.
    Alternatively, just write down $x = \log_4 3$.
- For which values of x (if any) does $1^x = 1$? For which values of x (if any) does $1^x = 3$?
  - $1^x = 1$ is true for all real x.
  - 1^x is never equal to 3 for real x.

MAT questions

MAT 2015 Q1H

- That's a very strange logarithm. The only way I can think of to get rid of it is to write

$$4 - 5x^2 - 6x^3 = (x^2 + 2)^2.$$

- Now this is a polynomial question (but we should be careful to go back and check our solutions in the original equation).
- The polynomial rearranges to $x^4 + 6x^3 + 9x^2 = 0$.
- Either $x = 0$ or $x^2 + 6x + 9 = 0$ which has a repeated root $x = -3$.
- Let's quickly check these solutions $\log_2(4) = 2$ and $\log_{11}(4 - 45 + 162) = \log_{11} 121 = 2$.
- There are two distinct solutions.
- The answer is (c).

MAT 2017 Q1I

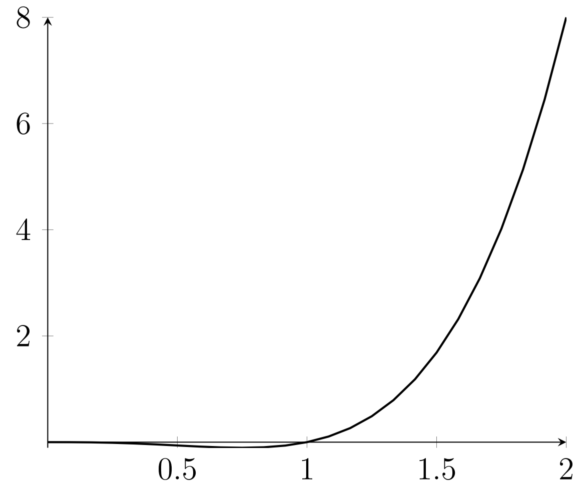
- Let's simplify each term. $\log_b\left((b^x)^x\right) = x \log_b(b^x) = x^2 \log_b b = x^2$.
- Also, $\log_a\left(\frac{c^x}{b^x}\right) = x \log_a(c/b) = x\left(\log_a c - \log_a b\right)$.
- And the last term is $-\log_a b \log_a c$.
- So this is a quadratic. Even better, we can factorise it

$$x^2 + (\log_a c - \log_a b)\, x - \log_a b \log_a c = (x - \log_a b)\,(x + \log_a c)\,.$$

- Those roots are the same number if $\log_a b = -\log a_c$, which happens when $c = 1/b$.
- The answer is (d).

MAT 2013 Q1F

- The only way that we can have $\log_b a = 2$ is if $a = b^2$. Similarly, we must have $c-3 = b^3$ and $c+5 = a^2$.
- These simultaneous equations aren't linear, but we can do our best to solve them.
- For example, we must have $c+5 = b^4$ and $c-3 = b^3$, so $b^4 - b^3 = 8$. How many solutions does that have?
- Let's sketch $b^3(b-1)$ for $b > 0$ (since we're told that b is positive).



  This is negative in between 0 and 1 and then after that it's positive and increases (because $(b-1)$ and b^3 are both increasing functions). It can only have one solution for $b^3(b-1) = 8$. In fact, that solution is $b = 2$, but what we're interested in for this question is that that's a unique solution.

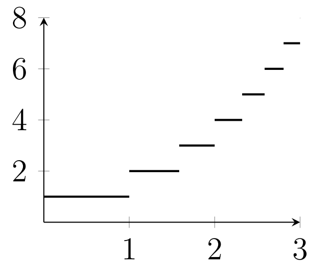
- Now look back at the other equations. We've got $a = b^2 = 4$ and $c = 3 + b^3 = 11$. Let's check the original logarithms.

$$\log_2 4 = 2, \qquad \log_2(11-3) = 3, \qquad \log_4(11+5) = 2$$

- These equations specify a uniquely (it's 4).
- The answer is (a).

MAT 2013 Q1J

- This is some new notation. It appears in the question in the expression $[2^x]$, which mean the largest integer less than or equal to 2^x.
- When $x = 0$, the term $[2^x]$ is 1. It stays at 1 until we get to a point with $2^x = 2$, because then the largest integer less than or equal to 2^x will be 2 instead. That happens when $x = 1$. Then 2^x keeps increasing, but the largest integer under it is still 2, until we get to a point with $2^x = 3$.
- Time for a logarithm; that happens when $x = \log_2 3$. There's a pattern here; the function takes each value between 1 and 2^{n-1} (it doesn't quite get to 2^n until right at the end of the interval when $x = n$).



- When we integrate this function from 0 to n, we're calculating the area under the graph. That's made of a series of rectangles, and the sum of the areas of those rectangles is

  $$1+(\log_2 3-1)\times 2+(\log_2 4-\log_2 3)\times 3+(\log_2 5-\log_2 4)\times 4+\cdots+(n-\log_2(2^n-1))\times(2^n-1)$$

- There's a pattern here (add some $\log_2 k$ then subtract slightly more $\log_2 k$) and it simplifies to

  $$-1-\log_2 3-\log_2 4-\cdots-\log_2(2^n-1)+n(2^n-1)$$

- That's $-\log_2((2^n-1)!)+n(2^n-1)$ which is not quite one of the options.
- But $\log_2 2^n = n$ so we can re-write this as $-\log_2((2^n)!)+n2^n$.
- The answer is (b).

Reflection

- I really like logarithms, but in lots of these questions getting rid of the logarithm is a good thing to do!
- In that last question, we simplified things by using laws of logarithms and also by using the $k!$ notation for $k \times (k-1) \times \cdots \times 2 \times 1$, which is a very compact way to write that product.