# Warm-up

1. Simplify $(2^3)^4$ and $(2^4)^3$ and $2^4 2^3$ and $2^3 2^4$.
    - $(2^3)^4 = 2^{12}$. $(2^4)^3 = 2^{12}$. $2^4 2^3 = 2^7$. $2^3 2^4 = 2^7$.
2. Solve $x^{-2} + 4x^{-1} + 3 = 0$.
    - This is a quadratic for $\frac{1}{x}$ with solutions $\frac{1}{x} = -1$ or $\frac{1}{x} = -3$.
    - So $x = -1$ or $x = -\frac{1}{3}$.
    - Alternatively, we could multiply both sides by $x^2$ and solve the quadratic that we get.
3. Solve $\log_x(x^2) = x^3$.
    - The left-hand side is just 2 so we want $2 = x^3$.
    - So $x = \sqrt[3]{2}$.
4. Solve $\log_{x+5}(6x + 22) = 2$.
    - Take $(x + 5)$ to the power of each side to get $6x + 22 = (x + 5)^2$.
    - Expand the square and rearrange for $x^2 + 4x + 3 = 0$.
    - The solutions are $x = -1$ or $x = -3$.
    - Check these solutions; $\log_4(16) = 2$ and $\log_2(4) = 2$.
5. Simplify $\log_{10} 3 + \log_{10} 4$ into a single term.
    - This is $\log_{10} 12$.
6. If we write $a = \ln 2$ and $b = \ln 5$, then write the following in terms of $a$ and $b$.

$$\ln 1024, \quad \ln 40, \quad \ln \sqrt{\frac{2}{5}}, \quad \ln \frac{1}{10}, \quad \ln 1.024.$$

    - $\ln 1024 = \ln(2^{10}) = 10 \ln 2 = 10a$.
    - $\ln 40 = \ln 8 + \ln 5 = 3a + b$.
    - $\ln \sqrt{2/5} = \frac{1}{2} \ln 2/5 = \frac{1}{2}(a - b)$.
    - $\ln(1/10) = -\ln 10 = -\ln 2 - \ln 5 = -a - b$.
    - $\ln 1.024 = \ln 1024 + \ln 1/1000 = 10a + 3(-a - b) = 7a - 3b$.
    - There are other solutions, partly because $b = a \times \log_2 5$.
7. Expand $(e^x + e^{-x})(e^y - e^{-y}) + (e^x - e^{-x})(e^y + e^{-y})$.

- $e^{x+y} + e^{y-x} - e^{x-y} - e^{-x-y} + e^{x+y} - e^{y-x} + e^{x-y} - e^{-x-y}$.
- That's $2e^{x+y} - 2e^{-x-y}$.

8. Expand $(e^x + e^{-x})(e^y + e^{-y}) + (e^x - e^{-x})(e^y - e^{-y})$.

- $e^{x+y} + e^{y-x} + e^{x-y} + e^{-x-y} + e^{x+y} - e^{y-x} - e^{x-y} + e^{-x-y}$.
- That's $2e^{x+y} + 2e^{-x-y}$.

9. Solve $2^x = 3$. Solve $0.5^x = 3$. Solve $4^x = 3$.

- $2^x = 3$ is what it means for $x$ to be $\log_2 3$.
- If $0.5^x = 3$ then $2^{-x} = 3$ so $x = -\log_2 3$.
  Alternatively, just write down $x = \log_{0.5} 3$.
- If $4^x = 3$ then $2^{2x} = 3$ so $x = \frac{1}{2}\log_2 3$.
  Alternatively, just write down $x = \log_4 3$.

10. For which values of $x$ (if any) does $1^x = 1$? For which values of $x$ (if any) does $1^x = 3$?

- $1^x = 1$ is true for all real $x$.
- $1^x$ is never equal to 3 for real $x$.

11. For what values of $b$ (if any) does $0^b = 0$? For what values of $a$ (if any) does $a^0 = 0$?

- $0^b = 0$ for any real $b > 0$.
- $a^0$ is never 0.

# MAT questions

## MAT 2015 Q1H

- That's a very strange logarithm. The only way I can think of to get rid of it is to write

$$4 - 5x^2 - 6x^3 = (x^2 + 2)^2.$$

- Now this is a polynomial question (but we should be careful to go back and check our solutions in the original equation).
- The polynomial rearranges to $x^4 + 6x^3 + 9x^2 = 0$.
- Either $x = 0$ or $x^2 + 6x + 9 = 0$ which has a repeated root $x = -3$.
- Let's quickly check these solutions $\log_2(4) = 2$ and $\log_{11}(4 - 45 + 162) = \log_{11} 121 = 2$.
- There are two distinct solutions.
- The answer is (c).

## MAT 2017 Q1I

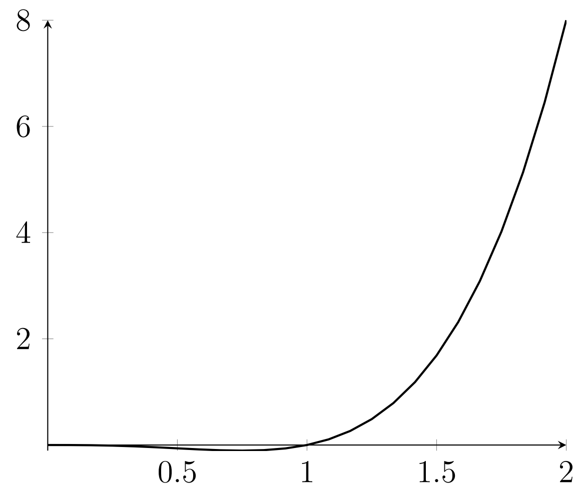
- Let's simplify each term. $\log_b\left((b^x)^x\right) = x \log_b\left(b^x\right) = x^2 \log_b b = x^2$.
- Also, $\log_a\left(\frac{c^x}{b^x}\right) = x \log_a(c/b) = x\left(\log_a c - \log_a b\right)$.
- And the last term is $-\log_a b \log_a c$.
- So this is a quadratic. Even better, we can factorise it

$$x^2 + \left(\log_a c - \log_a b\right) x - \log_a b \log_a c = \left(x - \log_a b\right)\left(x + \log_a c\right).$$

- Those roots are the same number if $\log_a b = -\log a_c$, which happens when $c = 1/b$.
- The answer is (d).

**MAT 2013 Q1F**

- The only way that we can have $\log_b a = 2$ is if $a = b^2$. Similarly, we must have $c-3 = b^3$ and $c+5 = a^2$.
- These simultaneous equations aren't linear, but we can do our best to solve them.
- For example, we must have $c+5 = b^4$ and $c-3 = b^3$, so $b^4 - b^3 = 8$. How many solutions does that have?
- Let's sketch $b^3(b-1)$ for $b > 0$ (since we're told that $b$ is positive).




  This is negative in between 0 and 1 and then after that it's positive and increases (because $(b-1)$ and $b^3$ are both increasing functions). It can only have one solution for $b^3(b-1) = 8$. In fact, that solution is $b = 2$, but what we're interested in for this question is that that's a unique solution.

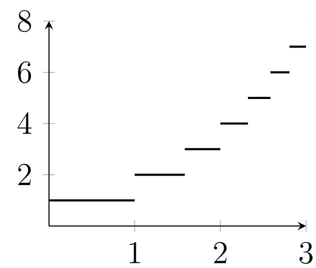
- Now look back at the other equations. We've got $a = b^2 = 4$ and $c = 3 + b^3 = 11$. Let's check the original logarithms.

$$\log_2 4 = 2, \qquad \log_2(11-3) = 3, \qquad \log_4(11+5) = 2$$

- These equations specify $a$ uniquely (it's 4).
- The answer is (a).

**MAT 2013 Q1J**

- This is some new notation. It appears in the question in the expression $[2^x]$, which mean the largest integer less than or equal to $2^x$.
- When $x = 0$, the term $[2^x]$ is 1. It stays at 1 until we get to a point with $2^x = 2$, because then the largest integer less than or equal to $2^x$ will be 2 instead. That happens when $x = 1$. Then $2^x$ keeps increasing, but the largest integer under it is still 2, until we get to a point with $2^x = 3$.
- Time for a logarithm; that happens when $x = \log_2 3$. There's a pattern here; the function takes each value between 1 and $2^{n-1}$ (it doesn't quite get to $2^n$ until right at the end of the interval when $x = n$).




- When we integrate this function from 0 to $n$, we're calculating the area under the graph. That's made of a series of rectangles, and the sum of the areas of those rectangles is

  $$1+(\log_2 3-1)\times 2+(\log_2 4-\log_2 3)\times 3+(\log_2 5-\log_2 4)\times 4+\cdots+(n-\log_2(2^n-1))\times(2^n-1)$$

- There's a pattern here (add some $\log_2 k$ then subtract slightly more $\log_2 k$) and it simplifies to

  $$-1-\log_2 3-\log_2 4-\cdots-\log_2(2^n-1)+n(2^n-1)$$

- That's $-\log_2\left((2^n-1)!\right)+n\left(2^n-1\right)$ which is not quite one of the options.
- But $\log_2 2^n = n$ so we can re-write this as $-\log_2\left((2^n)!\right)+n2^n$.
- The answer is (b).

# Reflection

- I really like logarithms, but in lots of these questions getting rid of the logarithm is a good thing to do!
- In that last question, we simplified things by using laws of logarithms and also by using the $k!$ notation for $k\times(k-1)\times\cdots\times 2\times 1$, which is a very compact way to write that product.