Logarithms and powers – Solutions

Revision Questions

1. $(2^3)^4 = 2^{12}$. $(2^4)^3 = 2^{12}$. $2^{4\cdot2^3} = 2^7$. $2^3\cdot2^4 = 2^7$.

2. This is a quadratic for $\frac{1}{x}$ with solutions $\frac{1}{x} = -1$ or $\frac{1}{x} = -3$. So $x = -1$ or $x = -\frac{1}{3}$. Alternatively, we could multiply both sides by $x^2$ and solve the quadratic that we get.

3. This is $\log_{10} 12$.

4. The quadratic inside the brackets factorises, and this is $\log_3(x + 2) + \log_2(x + 1)$. Other answers are possible, such as $\log_3(2(x + 2)) + \log_3((x + 1)/2)$.

5. The left-hand side is just 2 so we want $2 = x^3$. So $x = \sqrt[3]{2}$.

6. The left-hand side is $1 + \log_x 2$ so we want $\log_x 2 = 2$. So $x^2 = 2$ and $x = \sqrt{2}$ (not $-\sqrt{2}$ because we’re told that $x > 0$).

7. Take $(x + 5)$ to the power of each side to get $6x + 22 = (x + 5)^2$. Expand the square and rearrange for $x^2 + 4x + 3 = 0$. The solutions are $x = -1$ or $x = -3$. Check these solutions; $\log_4(16) = 2$ and $\log_2(4) = 2$.

8. • $\ln 1024 = \ln (2^{10}) = 10 \ln 2 = 10a$.
   • $\ln 40 = \ln 8 + \ln 5 = 3a + b$.
   • $\ln \sqrt{2/5} = \frac{1}{2} \ln 2/5 = \frac{1}{2} (a - b)$.
   • $\ln(1/10) = -\ln 10 = -\ln 2 - \ln 5 = -a - b$.
   • $\ln 1.024 = \ln 1024 + \ln 1/1000 = 10a + 3(-a - b) = 7a - 3b$.
   • There are other solutions, partly because $b = a \times \log_2 5$.

9. $e^{x+y} + e^{y-x} - e^{-x-y} + e^{x+y} - e^{y-x} + e^{-x+y} - e^{-x-y}$. That’s $2e^{x+y} - 2e^{-x-y}$.
   $e^{x+y} + e^{y-x} + e^{-x-y} + e^{x+y} - e^{y-x} - e^{-x+y} + e^{-x-y}$. That’s $2e^{x+y} + 2e^{-x-y}$.

10. $2^x = 3$ is what it means for $x$ to be $\log_2 3$.
   If $0.5^x = 3$ then $2^{-x} = 3$ so $x = -\log_2 3$. Alternatively, just write down $x = \log_{0.5} 3$.
   If $4^x = 3$ then $2^{2x} = 3$ so $x = \frac{1}{2} \log_2 3$. Alternatively, just write down $x = \log_4 3$.

11. $1^x = 1$ is true for all real $x$. $1^x$ is never equal to 3 for real $x$.

12. $0^b = 0$ for any real $b > 0$. $a^0$ is never 0.

13. We have $\log_{10} x = 10^6$, which is a million. So $x$ is ten to the power of a million. That’s got a million zeros at the end.

For more see www.maths.ox.ac.uk/r/matlive
14. Multiply both sides by $e^x$ and rearrange to get $e^{2x} - 4e^x + 1 = 0$. This is a quadratic for $e^x$. Solve it for $e^x = 2 \pm \sqrt{3}$. So $x = \ln(2 \pm \sqrt{3})$.

Following the previous working, we can see that we’ll get two roots for $e^x$ if $c^2 - 4 > 0$. But we need these to be positive roots, so we need $c > 2$. If $c = 2$ there’s a repeated root. If $c < 2$ there are no roots.

15. Move both terms onto the left-hand side and use the fact that $(N + \sqrt{N^2 - 1})(N - \sqrt{N^2 - 1}) = N^2 - (N^2 - 1) = 1$; that’s the difference of two squares. Remember that $\ln 1 = 0$.

As a result, our solutions above $x = \ln(2 \pm \sqrt{3})$ are actually $x = \pm \ln(2 + \sqrt{3})$, revealing a lovely symmetry. But you could have spotted that from the equation, of course!

16. We have $\ln(x^y) = \ln(y^x)$ which we can simplify down to $y \ln x = x \ln y$. Now rearrange to get $\frac{\ln x}{x} = \frac{\ln y}{y}$. You might choose to put this fraction the other way up, or to square both sides or something, so your $f(x)$ might not be the same as mine. Here’s a sketch of $y = \frac{\ln(x)}{x}$.

17. This is $(a^{\log_a b})^k = b^k$.

18. We can use a similar bit of algebra to the previous question with $k = \log_b c$.

$$a^x = a^{\log_a b \log_b c} = (a^{\log_a b})^{\log_b c} = (b)^{\log_b c} = c.$$ 

So $a^x = c$ and therefore $x = \log_a c$.

19. If we relabel the previous result, we can write $\log_c a \log_a b = \log_c b$. Now divide by $\log_c a$ to get

$$\log_a b = \frac{\log_c b}{\log_c a}$$

In particular, if we take $c$ to be the number $e$, then we can write that fraction with $\ln$ instead of $\log_e$. This is handy because it shows that we can always write logarithms like $\log_a x$ in terms of $\ln$. All other $\log_a x$ graphs are just simple transformations of the $\ln x$ graph.

For more see [www.maths.ox.ac.uk/r/matlive](http://www.maths.ox.ac.uk/r/matlive)
MAT Questions

MAT 2014 Q1B

• The function \( y = x^2 - 2x + 2 = (x - 1)^2 + 1 \) if we complete the square.
• The minimum occurs when \( x = 1 \) and the value at that minimum is \( y = 1 \).
• Consider now the function \( y = \log_{10}(f(x)) \) where \( f(x) \) is the quadratic above. This graph has a local minimum at \( x = 1 \), and the value there will be \( \log_{10} 1 = 0 \). Only one of the graphs looks like that.
• The answer is (e).

MAT 2015 Q1H

• We can start by using the fact that if \( \log_b a = c \) then \( a = b^c \).
• Here \( (4 - 5x^2 - 6x^3) = (x^2 + 2)^2 \).
• That polynomial simplifies to \( x^4 + 6x^3 + 9x^2 = 0 \).
• There’s a repeated root at \( x = 0 \), and the polynomial factorises as \( x^2(x + 3)^2 = 0 \).
• We should check our solutions; \( \log_2(4) = 2 \) and \( \log_{11}(4-5 \times 9+6 \times 27) = \log_{11}(121) = 2 \).
• So there are two distinct solutions.
• The answer is (c).

MAT 2017 Q1I

• The equation simplifies to

\[
x^2 + x \log_a \left( \frac{c}{b} \right) + \log_a \left( \frac{1}{b} \right) \log_a (c) = 0.
\]

• This quadratic factorises as

\[
(x - \log_a (b)) (x + \log_a (c)) = 0,
\]

where I’ve used the law of logarithms that \( \log_a c - \log_a b = \log_a \left( \frac{c}{b} \right) \).
• This quadratic has two distinct roots, unless \( - \log_a b = \log_a c \). That happens precisely when \( c = \frac{1}{b} \).
• The answer is (d).
MAT 2008 Q1B
Let \( x = \log_{10} \pi \). Note that \( 0 < x < 1 \) because \( 1 < \pi < 10 \). The four values are

\[
x, \quad \sqrt{2x}, \quad x^{-3}, \quad \frac{2}{x}.
\]

Now we want to compare these terms. We have \( x < \sqrt{2x} \) if \( x^2 < 2x \) which happens if \( x < 2 \). Also \( x < x^{-3} \) because \( x < 1 \). Also \( x < 2/x \) because \( x < \sqrt{2} \).

The answer is (a).

MAT 2008 Q1E
Ignore small powers of \( x \) inside each pair of round brackets. We get something like

\[
\left\{ (2x^6 + \ldots)^3 + (3x^8 + \ldots)^4 \right\}^5 + \left\{ (3x^5 + \ldots)^5 + (x^7 + \ldots)^4 \right\}^6 \bigg\}^3
\]

Now apply the powers on the round brackets and compare terms again

\[
\left\{ \left[ 2^3 x^{18} + 3^4 x^{32} + \ldots \right]^5 + \left[ 3^5 x^{25} + x^{28} + \ldots \right]^6 \right\}^3
\]

Take the largest power inside each square bracket and apply the power on that bracket

\[
\left\{ 3^{20} x^{160} + x^{168} + \ldots \right\}^3
\]

Take the largest power inside the curly brackets and apply the power on that bracket

\[
x^{504} + \ldots
\]

The answer is (d).
MAT 2010 Q1E
First note that \( \log_4 8 = \frac{3}{2} \) because \( 4 = 2^2 \) and \( 8 = 2^3 \), so \( 4^{3/2} = 8 \).

Is \( \log_2 3 > \frac{3}{2} \)? Only if \( 3 > 2^{3/2} \) so only if \( 9 > 8 \). Yes!

Is \( \log_3 2 > \frac{3}{2} \)? No, it’s less than one.

Is \( \log_5 10 > \frac{3}{2} \)? Only if \( 10 > 5^{3/2} \), so only if \( 100 > 125 \). No!

So only \( \log_2 3 \) is larger than \( \frac{3}{2} \).

The answer is (a).

MAT 2012 Q1C
Simplify \( (\sqrt{3})^3 = 3\sqrt{3} \).

Simplify \( \log_3(9^2) = 4 \).

Simplify \( (3 \sin 60^\circ)^2 = (3\sqrt{3}/2)^2 = \frac{27}{4} \).

Simplify \( \log_2(\log_2 8^5) = \log_2(\log_2 2^{15}) = \log_2(15) \).

The next thing I notice is that \( 15 < 16 \) so \( \log_2(15) < 4 \). Aha, that means that it’s less than \( \log_3(9^2) \).

Then perhaps I could notice that \( \frac{27}{4} > 4 \). So my remaining candidates for the smallest of the numbers are \( 3\sqrt{3} \) and \( \log_2(15) \). But \( 3\sqrt{3} = \sqrt{27} > 4 \). So option (d) is the only one that’s less than 4.

The answer is (d).

Extension
- We’ve seen above that for \( 0 < \alpha < 1 \), the smallest is (a). For \( \alpha > 1 \), it turns out that \( \alpha^{-3} \) is the smallest.
- \( (8!)^9 = (8!)^8 \times 8! \) and \( (9!)^8 = (8!)^8 \times 9^8 \). Now \( 9^8 \) is clearly larger than \( 8! \) (each is the product of eight things, and in the case of \( 9^8 \), each of those eight things is larger!). So \( (9!)^8 > (8!)^9 \).