Logarithms and powers – Solutions

Revision Questions

1. $(2^3)^4 = 2^{12}$. $(2^4)^3 = 2^{12}$. $2^4 2^3 = 2^7$. $2^3 2^4 = 2^7$.
2. This is a quadratic for $\frac{1}{x}$ with solutions $\frac{1}{x} = -1$ or $\frac{1}{x} = -3$. So $x = -1$ or $x = -\frac{1}{3}$. Alternatively, we could multiply both sides by x^2 and solve the quadratic that we get.
3. This is $\log_{10} 12$.
4. The quadratic inside the brackets factorises, and this is $\log_3(x+2) + \log_3(x+1)$. Other answers are possible, such as $\log_3(2(x+2)) + \log_3((x+1)/2)$.
5. The left-hand side is just 2 so we want $2 = x^3$. So $x = \sqrt[3]{2}$.
6. The left-hand side is $1 + \log_x 2$ so we want $\log_x 2 = 2$. So $x^2 = 2$ and $x = \sqrt{2}$ (not $-\sqrt{2}$ because we're told that $x > 0$).
7. Take $(x+5)$ to the power of each side to get $6x + 22 = (x+5)^2$. Expand the square and rearrange for $x^2 + 4x + 3 = 0$. The solutions are $x = -1$ or $x = -3$. Check these solutions; $\log_4(16) = 2$ and $\log_2(4) = 2$.
8.
   - $\ln 1024 = \ln\left(2^{10}\right) = 10 \ln 2 = 10a$.
   - $\ln 40 = \ln 8 + \ln 5 = 3a + b$.
   - $\ln\sqrt{2/5} = \frac{1}{2}\ln 2/5 = \frac{1}{2}(a - b)$.
   - $\ln(1/10) = -\ln 10 = -\ln 2 - \ln 5 = -a - b$.
   - $\ln 1.024 = \ln 1024 + \ln 1/1000 = 10a + 3(-a-b) = 7a - 3b$.
   - There are other solutions, partly because $b = a \times \log_2 5$.
9. $e^{x+y} + e^{y-x} - e^{x-y} - e^{-x-y} + e^{x+y} - e^{y-x} + e^{x-y} - e^{-x-y}$. That's $2e^{x+y} - 2e^{-x-y}$.

   $e^{x+y} + e^{y-x} + e^{x-y} + e^{-x-y} + e^{x+y} - e^{y-x} - e^{x-y} + e^{-x-y}$. That's $2e^{x+y} + 2e^{-x-y}$.
10. $2^x = 3$ is what it means for x to be $\log_2 3$.

    If $0.5^x = 3$ then $2^{-x} = 3$ so $x = -\log_2 3$. Alternatively, just write down $x = \log_{0.5} 3$.

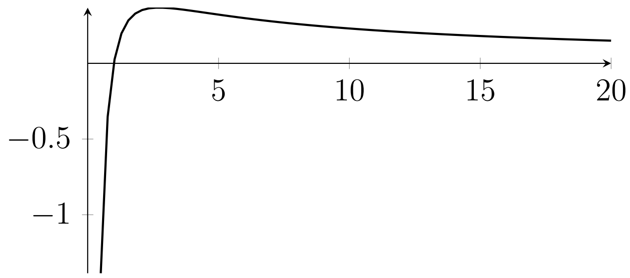
    If $4^x = 3$ then $2^{2x} = 3$ so $x = \frac{1}{2}\log_2 3$. Alternatively, just write down $x = \log_4 3$.
11. $1^x = 1$ is true for all real x. 1^x is never equal to 3 for real x.
12. $0^b = 0$ for any real $b > 0$. a^0 is never 0.
13. We have $\log_{10} x = 10^6$, which is a million. So x is ten to the power of a million. That's got a million zeros at the end.
14. Multiply both sides by e^x and rearrange to get $e^{2x} - 4e^x + 1 = 0$. This is a quadratic for e^x. Solve it for $e^x = 2 \pm \sqrt{3}$. So $x = \ln(2 \pm \sqrt{3})$.

    Following the previous working, we can see that we'll get two roots for e^x if $c^2 - 4 > 0$. But we need these to be positive roots, so we need $c > 2$. If $c = 2$ there's a repeated root. If $c < 2$ there are no roots.

15. Move both terms onto the left-hand side and use the fact that $(N+\sqrt{N^2-1})(N-\sqrt{N^2-1}) = N^2-(N^2-1) = 1$; that's the difference of two squares. Remember that $\ln 1 = 0$.

    As a result, our solutions above $x = \ln(2 \pm \sqrt{3})$ are actually $x = \pm\ln(2+\sqrt{3})$, revealing a lovely symmetry. But you could have spotted that from the equation, of course!

16. We have $\ln(x^y) = \ln(y^x)$ which we can simplify down to $y \ln x = x \ln y$. Now rearrange to get $\dfrac{\ln x}{x} = \dfrac{\ln y}{y}$. You might choose to put this fraction the other way up, or to square both sides or something, so your $f(x)$ might not be the same as mine. Here's a sketch of $y = \dfrac{\ln(x)}{x}$.



17. This is $(a^{\log_a b})^k = b^k$.

18. We can use a similar bit of algebra to the previous question with $k = \log_b c$.

$$a^x = a^{\log_a b \log_b c} = \left(a^{\log_a b}\right)^{\log_b c} = (b)^{\log_b c} = c.$$

    So $a^x = c$ and therefore $x = \log_a c$.

19. If we relabel the previous result, we can write $\log_c a \log_a b = \log_c b$. Now divide by $\log_c a$ to get

$$\log_a b = \frac{\log_c b}{\log_c a}$$

    In particular, if we take c to be the number e, then we can write that fraction with ln instead of $\log_e$. This is handy because it shows that we can always write logarithms like $\log_a x$ in terms of ln. All other $\log_a x$ graphs are just simple transformations of the $\ln x$ graph.

MAT Questions

MAT 2014 Q1B

- The function $y = x^2 - 2x + 2 = (x-1)^2 + 1$ if we complete the square.
- The minimum occurs when $x = 1$ and the value at that minimum is $y = 1$.
- Consider now the function $y = \log_{10}(f(x))$ where $f(x)$ is the quadratic above. This graph has a local minimum at $x = 1$, and the value there will be $\log_{10} 1 = 0$. Only one of the graphs looks like that.
- The answer is (e).

MAT 2015 Q1H

- We can start by using the fact that if $\log_b a = c$ then $a = b^c$.
- Here $(4 - 5x^2 - 6x^3) = \left(x^2 + 2\right)^2$.
- That polynomial simplifies to $x^4 + 6x^3 + 9x^2 = 0$.
- There's a repeated root at $x = 0$, and the polynomial factorises as $x^2(x+3)^2 = 0$.
- We should check our solutions; $\log_2(4) = 2$ and $\log_{11}(4 - 5\times 9 + 6\times 27) = \log_{11}(121) = 2$.
- So there are two distinct solutions.
- The answer is (c).

MAT 2017 Q1I

- The equation simplifies to

$$x^2 + x\log_a\left(\frac{c}{b}\right) + \log_a\left(\frac{1}{b}\right)\log_a(c) = 0.$$

- This quadratic factorises as

$$(x - \log_a(b))\,(x + \log_a(c)) = 0,$$

  where I've used the law of logarithms that $\log_a c - \log_a b = \log_a\left(\frac{c}{b}\right)$.
- This quadratic has two distinct roots, unless $-\log_a b = \log_a c$. That happens precisely when $c = \frac{1}{b}$.
- The answer is (d).

MAT 2008 Q1B

Let $x = \log_{10} \pi$. Note that $0 < x < 1$ because $1 < \pi < 10$. The four values are

$$x, \qquad \sqrt{2x}, \qquad x^{-3} \qquad \frac{2}{x}.$$

Now we want to compare these terms. We have $x < \sqrt{2x}$ if $x^2 < 2x$ which happens if $x < 2$. Also $x < x^{-3}$ because $x < 1$. Also $x < 2/x$ because $x < \sqrt{2}$.
The answer is (a).

MAT 2008 Q1E

Ignore small powers of x inside each pair of round brackets. We get something like

$$\left\{\left[\left(2x^6+\dots\right)^3+\left(3x^8+\dots\right)^4\right]^5+\left[\left(3x^5+\dots\right)^5+\left(x^7+\dots\right)^4\right]^6\right\}^3$$

Now apply the powers on the round brackets and compare terms again

$$\left\{\left[2^3x^{18}+3^4x^{32}+\dots\right]^5+\left[3^5x^{25}+x^{28}+\dots\right]^6\right\}^3$$

Take the largest power inside each square bracket and apply the power on that bracket

$$\left\{3^{20}x^{160}+x^{168}+\dots\right\}^3$$

Take the largest power inside the curly brackets and apply the power on that bracket

$$x^{504}+\dots$$

The answer is (d).

MAT 2010 Q1E

First note that $\log_4 8 = \dfrac{3}{2}$ because $4 = 2^2$ and $8 = 2^3$, so $4^{3/2} = 8$.

Is $\log_2 3 > \dfrac{3}{2}$? Only if $3 > 2^{3/2}$ so only if $9 > 8$. Yes!

Is $\log_3 2 > \dfrac{3}{2}$? No, it's less than one.

Is $\log_5 10 > \dfrac{3}{2}$? Only if $10 > 5^{3/2}$, so only if $100 > 125$. No!

So only $\log_2 3$ is larger than $\dfrac{3}{2}$.

The answer is (a).

MAT 2012 Q1C

Simplify $(\sqrt{3})^3 = 3\sqrt{3}$.
Simplify $\log_3(9^2) = 4$.
Simplify $(3 \sin 60°)^2 = (3\sqrt{3}/2)^2 = \dfrac{27}{4}$.
Simplify $\log_2(\log_2 8^5) = \log_2(\log_2 2^{15}) = \log_2(15)$.
The next thing I notice is that $15 < 16$ so $\log_2(15) < 4$. Aha, that means that it's less than $\log_3(9^2)$.
Then perhaps I could notice that $\frac{27}{4} > 4$. So my remaining candidates for the smallest of the numbers are $3\sqrt{3}$ and $\log_2(15)$. But $3\sqrt{3} = \sqrt{27} > 4$. So option (d) is the only one that's less than 4.
The answer is (d).

Extension

- We've seen above that for $0 < \alpha < 1$, the smallest is (a). For $\alpha > 1$, it turns out that α^{-3} is the smallest.
- $(8!)^9 = (8!)^8 \times 8!$ and $(9!)^8 = (8!)^8 \times 9^8$. Now 9^8 is clearly larger than $8!$ (each is the product of eight things, and in the case of 9^8, each of those eight things is larger!). So $(9!)^8 > (8!)^9$.