Stable and finite Morse index solutions to semilinear elliptic equations

Alessio Figalli

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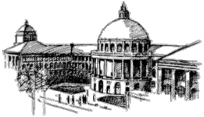

ETH Zürich

FIM
INSTITUTE FOR MATHEMATICAL RESEARCH

Overview of the talk

1. Semilinear elliptic PDEs
2. Regularity of critical points
3. Smoothness of stable solutions
   1. Known results and Brezis' problem
   2. New results
4. Smoothness of finite Morse index solutions

Part 1: Semilinear elliptic PDEs

Semilinear elliptic PDEs

Given $\Omega \subset \mathbb{R}^n$, one looks at $u : \Omega \to \mathbb{R}$ solving

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

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Some natural structural assumptions on f:

$f \geq 0$, $\quad f$ smooth, convex, increasing, and superlinear (at $+\infty$).

The energy functional

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Set

$$F(t) := \int_0^t f(s)\,ds.$$

Then our PDE corresponds to the Euler-Lagrange equation for the energy functional

$$\mathcal{E}[v] := \int_\Omega \Big(\frac{|\nabla v|^2}{2} - F(v)\Big)\,dx.$$

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$$\mathcal{E}[v] := \int_\Omega \Big(\frac{|\nabla v|^2}{2} - F(v) \Big)\, dx.$$

In other words, u solves $-\Delta u = f(u)$ if and only if

$$0 = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathcal{E}[u+\epsilon\xi] = \int_\Omega \nabla u \cdot \nabla \xi - f(u)\xi \qquad \forall\, \xi \in C_0^1(\Omega).$$

Minimizers vs stationary vs stable

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Thus, given $v : \Omega \to \mathbb{R}$ and $M > 0$,

$$\mathcal{E}[Mv] = M^2 \int_\Omega \frac{|\nabla v|^2}{2} - \int_\Omega F(Mv) \to -\infty$$

as $M \to +\infty$. Hence, $\mathcal{E}$ has no global minimizers.

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Then, one can look at:

1. critical points: $u \in W_0^{1,2}(\Omega)$ s.t. $\frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathcal{E}[u + \epsilon\xi] = 0 \ \forall\, \xi \in C_0^1$

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2. stable critical points: in addition, $\frac{d^2}{d\epsilon^2}\Big|_{\epsilon=0} \mathcal{E}[u + \epsilon\xi] \geq 0$

Part 2: Regularity of critical points

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Once $u \in L^\infty$, one uses classical elliptic regularity (ER):

$$\begin{aligned} u \in L^\infty \quad & \Rightarrow \; f(u) \in L^\infty \Rightarrow \Delta u \in L^\infty \\ & \overset{ER}{\Rightarrow} \; u \in C^{1,\alpha}, \Rightarrow \; f(u) \in C^{1,\alpha} \Rightarrow \Delta u \in C^{1,\alpha} \\ & \overset{ER}{\Rightarrow} \; u \in C^{3,\alpha} \Rightarrow \ldots \end{aligned}$$

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$$u(x) = |x|^{-\alpha} - 1 \quad \text{in } B_1, \qquad \alpha > 0.$$

Then

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Also,

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Thus

$$\int_{B_1} |\nabla u|^2 < \infty \quad \Leftrightarrow \quad 2(\alpha+1) < n \quad \Leftrightarrow \quad \frac{\alpha+2}{\alpha} > \frac{n+2}{n-2}.$$

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No regularity!!!

Part 3: Smoothness of stable solutions

Stable solutions

A critical point is stable if

$$0 \leq \frac{d^2}{d\epsilon^2}\Big|_{\epsilon=0}\mathcal{E}[u+\epsilon\xi] = \int_\Omega |\nabla\xi|^2 - f'(u)\xi^2 \qquad \forall\,\xi \in C_0^1(\Omega).$$

Remark: to prove smoothness, it suffices to prove boundedness.

A motivating problem (Barenblatt-Gelfand 1963)

Given $\lambda \geq 0$, let u solve the Barenblatt-Gelfand problem

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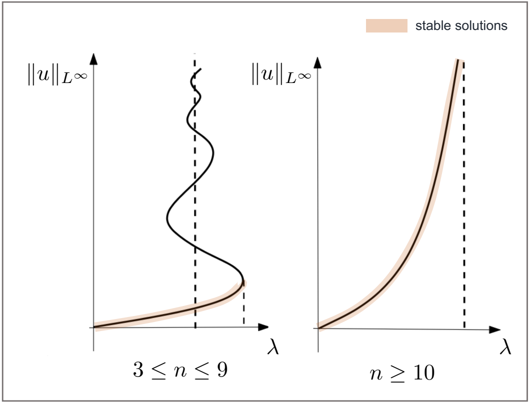
When $f(u) = e^u$, this is a model for combustion of solid fuel, where:

- $\lambda \longleftrightarrow$ amount of combustible;
- boundedness of stable solutions $\longleftrightarrow$ pointwise bounds for the thermodynamic quantities.

The radial case

In 1972, Joseph-Lundgren studied the radial case $\Omega = B_1$.

The PDE reduces to a ODE, and they could show the following:



The radial case

Thus, there exists $\lambda^* > 0$ such that:

stable sols exist if and only if $\lambda \in [0, \lambda^*]$

$n \leq 9$: stable sols are bounded for $\lambda \in [0, \lambda^*]$

$n \geq 10$: stable sols are bounded for $\lambda \in [0, \lambda^*)$

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This was extended to any domain if $f(u) \sim u^p$ or $f(u) \sim e^u$ by Crandall-Rabinovitz (1975). Many results followed...

A singular stable solution in dimension $n \geq 10$

$u(x) = -2\log|x| \in W_0^{1,2}(B_1)$ solves

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Hardy's inequality:

$$\int_{B_1} |\nabla \xi|^2 \geq \frac{(n-2)^2}{4}\int_{B_1} \frac{\xi^2}{|x|^2} \qquad \forall\, \xi \in C_0^1(B_1).$$

So **Hardy** $\Rightarrow$ **Stability** if $n \geq 10$.

Open problems and some known results

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- Cabré-Capella, 2005: $u \in L^\infty(\Omega)$ if $n \leq 9$ and $\Omega = B_1$.
- Cabré 2010 - Villegas 2013: $u \in L^\infty(\Omega)$ if $n \leq 4$.

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Theorem (Cabré - Figalli - Serra - Ros-Oton, Acta Math. 2020)

Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be convex, nondecreasing, and superlinear. Let $u \in W_0^{1,2}(\Omega)$ be a stable solution of

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with $\partial\Omega \in C^3$.

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with $\partial\Omega \in C^3$. Then:

1. $\|\nabla u\|_{L^2(\Omega)} \leq C(n,\Omega)\|u\|_{L^1(\Omega)} \leq C(n,f,\Omega)$ *for all* $n \geq 2$.
2. $\|u\|_{C^\alpha(\overline{\Omega})} \leq C(n,\Omega)\|u\|_{L^1(\Omega)} \leq C(n,f,\Omega)$ *for* $n \leq 9$.

Remark: if Ω is convex, then it suffices to assume $f \geq 0$ and locally Lipschitz.

Part 4: Smoothness of finite Morse index solutions

Finite Morse index solutions

Given u solution of $-\Delta u = f(u)$, define

$$Q_u[\xi] = \int_\Omega |\nabla \xi|^2 - f'(u)\xi^2 \qquad \forall\, \xi \in C_0^1(\Omega).$$

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We say that $\mathrm{ind}(u) = k \in \mathbb{N}$ if k is the maximal dimension of a subspace $X \subset C_0^1(\Omega)$ such that

$$Q_u[\xi] < 0 \qquad \forall\, \xi \in X \setminus \{0\}.$$

Boundedness vs Morse index (I)

Let u solve

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Bahri-Lions (1992): let $f(t) \sim t^p$ for $t \gg 1$ with $p < \frac{n+2}{n-2}$. Then

$$\mathrm{ind}(u) \leq C \qquad \Leftrightarrow \qquad \|u\|_{L^\infty(\Omega)} \leq C'.$$

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Let $f(t) \sim t^p$ for $t \gg 1$ with $p = \frac{n+2}{n-2}$. Then $\exists \{u_k\}_{k \geq 1}$ s.t.

$$\mathrm{ind}(u_k) = 1 \qquad \text{but} \qquad \|u_k\|_{L^\infty(\Omega)} \to \infty.$$

Boundedness vs Morse index (II)

Farina (2007), Dancer-Farina (2009): let $f(t) \sim \lambda t^p$ for $t \gg 1$ with $p > \frac{n+2}{n-2}$, or $f(t) \sim \lambda e^t$. Then

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Figalli-Zhang (2021): let $f'(t)f \ge p \int_0^t f(s)ds$ for $t \gg 1$ with $p > \frac{n+2}{n-2}$. Then

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In particular, the same picture as in Joseph-Lundgren holds.

Thanks for your attention!