Maturity Randomization Approach to the Pricing of American Options under Constant and Stochastic Volatility

Candidate Number: 

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A thesis submitted in partial fulfillment of the

M.Sc. Mathematical Finance

September 29, 2016
Abstract

The largest part of listed options traded on the global financial markets are American-styled. Still, except for some special cases no analytic closed-form solution for the American option value exists. Therefore American options are usually valued by means of numerical algorithms. However, in particular if stochastic volatility models are considered, those numerical methods tend to be computationally expensive. In this dissertation we study an alternative analytic approach to the valuation of American options: maturity randomization. The core of this approach is to replace the constant maturity of an American option by a random variable with a suitable distribution in order to reduce the complexity of the resulting pricing equations. We apply this method to the valuation of American puts and calls in the constant elasticity of variance model and a fast mean-reverting stochastic volatility model. To this end, we derive analytic pricing functions for maturity-randomized American option contracts which approximate the equivalent fixed-maturity American option. Comparing our results with those obtained from popular numerical approaches we find that maturity randomization yields very good approximations for the American option value and exercise boundary. Besides, since the pricing function is derived in an analytic form, computing numerical results hardly requires any computation time.
Contents

1 Introduction ................................................. 4

2 Maturity Randomization ................................. 6
  2.1 General Approach ...................................... 6
  2.2 First-Order Randomization: Exponentially Distributed Maturity ................. 7
  2.3 Example: American Put in the Black-Scholes Model ............................... 8
  2.4 Higher-Order Randomization: Erlang Distributed Maturity ....................... 10

3 American Options in the CEV Model ................. 12
  3.1 First-Order Randomization ............................ 12
    3.1.1 Numerical Analysis ................................ 19
  3.2 Higher-Order Randomization ............................ 24
    3.2.1 Numerical Analysis ................................ 31

4 American Options in a Fast Mean-Reverting Stochastic Volatility Model ....... 34
  4.1 Maturity-Randomized American Put .......................... 35
  4.2 Maturity-Randomized American Call .......................... 40
  4.3 Higher-Order Maturity-Randomized American Options ......................... 42
  4.4 Numerical Analysis ..................................... 46

5 Summary and Conclusion ................................... 51

Appendices .............................................. 53

A Pricing Functions ........................................ 54
  A.1 CEV Model ............................................. 54
  A.2 Fast Mean-Reverting Stochastic Volatility Model ............................... 57
Chapter 1

Introduction

In their seminal papers Black, Scholes, and Merton derived analytical closed form solutions to the problem of pricing European put and call options on equities following a log-normal stochastic diffusion process [2, 18]. However, the larger part of listed options on stocks, futures, bonds, or FX rates are American-styled, i.e. they can be exercised at any time prior to the predefined fixed maturity of the option. Due to the early-exercise feature of American options it is in general not possible to find analytic solutions to the pricing problems. There are only a few special cases for which the pricing problem can be solved analytically, for example perpetual American options for which the maturity is infinite. Instead, the value of American options is in practice usually approximated by numerical or perturbative methods, most of which make use of the fact that the free-boundary pricing problem can be reformulated as a linear complementarity problem.

Out of the large number of numerical methods one of the most popular approaches is the finite differences method developed by Brennan and Schwartz [4]. In this approach the time and asset price dimension is discretized and the derivatives in the Black-Scholes partial differential equation are replaced by finite difference quotients. Also the binomial options pricing model proposed by Cox et al. [7] makes use of a discretized time dimension and solves the pricing problem by simulating the path of the underlying stock price by means of a binomial tree. These methods have proven to be very powerful within the Black-Scholes framework. However, empirical analysis of option prices observed in the market indicates that the implied volatility for a given stock is not a constant, as suggested by the Black-Scholes model, but instead is a function of the moneyness and maturity of the option. This observation lead to the development of stochastic volatility models, e.g. the Heston model [10] or the Hull-White model [12]. For such multi-factor stochastic volatility models the finite differences and binomial approach are usually not applicable. Instead, the least-squares Monte Carlo approach proposed by Longstaff and Schwartz [17] is to date the most popular method for numerical valuation of American options in multi-factor settings.

In this dissertation we want to study an alternative analytic method introduced by Carr [5] for the valuation of American options, namely maturity randomization also known as “Canadization”. In contrast to the numerical finite differences methods and binomial pricing models, which solve the linear complementarity problem approximately by discretizing the time dimension, in the maturity randomization approach the features of the option contract itself are approximated such that the resulting pricing equations can be solved analytically. More precisely the fixed maturity of the American option contract is replaced by a random maturity which is determined by the arrival time of a certain number of jumps of a Poisson process. In [3] a formal proof for the convergence of the obtained series of maturity-randomized American option prices to the true American option value was provided. The approach has also already successfully been applied to the pricing of American-styled options in general Levy models [15, 16, 13] and American-styled options...
with path-dependent pay-offs, such as Russian options [14, 9]. The scope of this dissertation is to study the maturity randomization approach for the pricing of American puts and calls for two different volatility models. To this end, we will first review the discussion presented in the paper by Carr for the constant volatility case of the Black-Scholes model. Then, we will apply the method to American options in the constant elasticity of variance (CEV) model. In this case, the resulting differential equations cannot be solved analytically; instead we will use asymptotic expansions (in a suitable small parameter) and derive the option’s value in terms of a perturbative expansion. As another application of the method, we will study maturity randomization for American options in a fast mean-reverting stochastic volatility model. Maturity randomization for this particular model has been examined in the recent paper by Agarwal et al. [1]. In this article a perturbative expansion of the value of a maturity-randomized American put on a non-dividend paying stock is derived, where it is assumed that the option matures at the first jump of a Poisson process, i.e. the random maturity has an exponential distribution. As an extension to the discussion in the paper, we introduce a constant dividend yield in the underlying stock process and derive the respective pricing formulas for American puts and calls. Besides, we consider options maturing at multiple jumps of the Poisson process for which the resulting distribution of the random maturity is narrower and hence better approximates the fixed maturity of the true American option contract.

The dissertation is organized as follows. In chapter 2 the basic concept of maturity randomization is outlined. It is demonstrated how pricing formulas are derived for randomized option contracts with the random maturity determined by an arbitrary number of jumps of a Poisson process. As an example, we discuss the derivation of the randomized American put value in the Black-Scholes framework. In chapter 3 maturity randomization is applied to the pricing of American puts in the CEV model. We derive American put values for random maturities determined by the first, second and third jump of a Poisson process. The obtained exercise boundaries and pricing functions for the randomized American put are compared with numerical results obtained from a finite differences scheme. Chapter 4 deals with maturity randomization in the context of a fast mean-reverting stochastic volatility model. We include a constant dividend yield in the diffusion process for the underlying stock and study both randomized American put and call options. Again, exercise boundary and option prices are derived for options maturing at the first, second and third jump of a Poisson process. In this case, we compare the option values obtained from maturity randomization with respective results from Monte-Carlo simulations obtained in [1].
Chapter 2

Maturity Randomization

2.1 General Approach

As mentioned above, no closed-form analytic solution exists for the valuation of the American put. In this chapter, we review the maturity randomization (or Canadization) approach developed by Carr [5] which allows one to derive a semi-explicit valuation formula for American Put options. By contrast to other approaches where the space or time dimensions are discretized to obtain approximate solutions to the American option problem, in Carr’s approach the American option contract is approximated by random-maturity options.

A random-maturity American option is an option contract which can be exercised at any time up to and including a random maturity date. This contract is valued by assuming a certain distribution for the random maturity date and then calculating the expected value of the contract’s price. By choosing a suitable distribution, the complexity of the random-maturity problem can be significantly reduced compared to the original problem. In the particular case of the American option, the time-dependence of the option’s value and hence also the exercise boundary can be removed by choosing an exponential distribution for the random maturity date, i.e. the maturity date of the option is determined as the $n$th jump of a Poisson process.

In this setting, it is possible to determine the exact value of the random-maturity American put option in the Black-Scholes model in a closed analytic form. Besides, it is possible to derive a closed-form solution for the optimal exercise boundary (which is time-independent and hence only a function of the initial stock price).

However, our objective is not the valuation of a random-maturity American option, since this is a very rare kind of financial derivative, but the valuation of a fixed-maturity American option. Hence, we have to make the transition from the random-maturity setting to the fixed maturity of the considered American option. To this end, we have to fix the expected value of the maturity distribution at the maturity date of the American option and then let the variance of the distribution approach zero. For the Poisson distribution this means that we have to let the number of jumps which determine the option’s maturity date approach infinity. As we will see in the following review of Carr’s derivation, the larger the number of jumps the more involved is the derivation of the randomized option’s value. However, we will also see that even for a rather small number of jumps the randomization approach yields a good approximation of the American option’s value.

In general, the value of an American put option can be written as a function $P(t, S_t, T)$ of time $t$, the current stock price $S_t$, and the maturity date $T$. It is given by the solution to a linear complementarity problem, or optimal stopping problem, with the objective to optimize the value of the option. The optimal stopping time is the first time the stock price passes the so-called critical stock price $S(t, T)$. In the Black-Scholes model, the
critical stock price $S$ for a given time $t$ is the highest value of the stock price for which the value of the put is equal to its payoff $K - S$. While the value of the put decreases towards maturity, the critical stock price increases as a function of time. This function is called the optimal exercise boundary of the American put.

Independent from the underlying stochastic process for the stock price evolution, the initial value of the American put can be written in terms of the following expectation under the risk-neutral probability measure:

$$P(0, S, T) = \sup_{\tau_S \in [0, T]} \mathbb{E}\left\{ e^{-r\tau_S} [K - S_{\tau_S}]^+ | t = 0, S_0 = S \right\}.$$  \hspace{1cm} (2.1.1)

The option value is maximized over stopping times $\tau_S$. Since the optimal stopping time is the minimum of the first time the stock price passes the exercise boundary and the maturity of the option, the initial value of the American put can also be written as an optimization problem over exercise boundaries $B(t)$ instead of stopping times $\tau_S$:

$$P(0, S, T) = \sup_{B(t), t \in [0, T]} \mathbb{E}\left\{ e^{-r\min(\tau_B, T)} [K - S_{\min(\tau_B, T)}]^+ | t = 0, S_0 = S \right\}, \quad S > S(0, T),$$  \hspace{1cm} (2.1.2)

where $\tau_B$ is the first passage time of the stock price $S_t$ to the exercise boundary $B(t)$. This representation of the optimal stopping problem is particularly useful for the maturity randomization approach, since a sensible choice of the maturity’s probability distribution will remove the time dependence from the expectation and hence we only have to optimize the expectation over constant exercise boundaries. We will explain this point in detail in the following section 2.2.

Let us now derive the equivalent pricing formula for a random-maturity American option which approximates the fixed-maturity American option contract. To this end, we replace the fixed maturity $T$ by a random maturity $\tau$. As already discussed, our objective is to remove the time dependence from the option’s value and optimal exercise boundary. This is achieved by assuming that the random maturity is given by the $n$th jump of a Poisson process. Besides, it is assumed that the Poisson process is independent from the stochastic process of the underlying stock price.

For brevity, we will in the following refer to a random-maturity American option which matures at the $n$th jump of a Poisson process as $n$th-order randomized American option.

### 2.2 First-Order Randomization: Exponentially Distributed Maturity

As a first approximation let $n = 1$. In this case, the random maturity $\tau$ is exponentially distributed:

$$\Pr(\tau \in dt) = \lambda e^{-\lambda t} dt.$$  \hspace{1cm} (2.2.1)

To approximate an American option with fixed maturity $T$ the parameter $\lambda$ is chosen such that the mean of the distribution is equal to $T$, i.e.

$$\lambda = \frac{1}{T}.$$  \hspace{1cm} (2.2.2)

The exponential distribution is memoryless, meaning that, if the jump has not happened until time $t_1$, the probability for the jump to happen in the time interval $t_1 + \Delta t$ is the same as the initial probability for the jump to happen in the time interval $t_0 + \Delta t$, or, in terms of conditional probabilities,

$$\Pr(\tau \in t_1 + \Delta t | \tau > t_1) = \Pr(\tau \in \Delta t | \tau > t_0 = 0).$$  \hspace{1cm} (2.2.3)
Due to this property the value of the active randomized American put and its exercise boundary do not depend on time. Therefore the value of the American put maturing at the first jump time of the Poisson process, denoted by $P^{(1)}(S)$, is given by the following optimization problem:

$$P^{(1)}(S) = \sup_B \mathbb{E} \left\{ e^{-r \min(\tau_B, \tau)} [K - S_{\min(\tau_B, \tau)}]^+ \mid t = 0, S_0 = S \right\}, \quad S > S_1,$$

with $S_1$ the time-independent critical stock price.

Following the discussion in [5] we rewrite the expectation as an iterated expectation:

$$P^{(1)}(S) = \sup_B \mathbb{E} \left\{ \mathbb{E} \left\{ e^{-r \min(\tau_B, \tau_M)} [K - S_{\min(\tau_B, \tau_M)}]^+ \mid \tau = t_M \right\} \mid t = 0, S_0 = S \right\}, \quad S > S_1$$

(2.2.5)

with the first expectation over the random maturity and the second over the future stock price at a given maturity $t_M$. The value of the second expectation is equivalent to the value of a fixed-maturity down-and-out put with maturity $t_M$, barrier $B$ and rebate $K - B$, given by

$$D(0, S, t_M, B) = \mathbb{E} \left\{ e^{-r \min(\tau_B, \tau_M)} [K - S_{\min(\tau_B, \tau_M)}]^+ \mid t = 0, S_0 = S \right\}.$$

(2.2.6)

Replacing the first expectation with an integral over the exponential distribution of the random maturity we obtain

$$P^{(1)}(S) = \sup_B \lambda \int_0^\infty e^{-\lambda t_M} D(0, S, t_M, B) dt_M.$$

(2.2.7)

The value of the randomized American put is hence given by the Laplace-Carson transform of the down-and-out barrier put value with fixed maturity maximized over constant barriers $B$. In the following, we will frequently make use of this relation to derive partial differential equations (PDEs) for the considered random-maturity options. If we know the PDE which is satisfied by the value of a fixed-maturity barrier put in the given volatility model, we can simply derive the PDE obeyed by the corresponding random-maturity American put option by applying the Laplace-Carson transform to both sides of the equation. In particular, due to the integral over time the transformation removes any time-dependence from the differential equation and hence reduces the complexity of the resulting equation.

### 2.3 Example: American Put in the Black-Scholes Model

As an example, let us now briefly review the derivation of the random-maturity American put value in the standard Black-Scholes model. We thereby follow the discussion in [5] and also apply a similar notation.

In the Black-Scholes model it is assumed that the price $S_t$ of a stock follows a log-normal diffusion process and hence obeys the stochastic differential equation

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t,$$

(2.3.1)where $\mu$ is the drift and $\sigma$ the volatility of the stock, and $W_t$ denotes a standard Brownian motion. The parameters $\mu$ and $\sigma$ are taken to be positive real-valued constants. Besides, we assume that the stock does not pay any dividends.

As discussed above, the starting point for the derivation is the initial value of a down-and-out barrier put option $D(0, S, t_M, B)$. The put value is given by the solution to the
Black-Scholes equation
\[
\frac{1}{2} \sigma^2 S^2 \frac{\partial^2 D}{\partial S^2} + r S \frac{\partial D}{\partial S} - rD = \frac{\partial D}{\partial t_M}
\]  
(2.3.2)

with the following terminal and boundary conditions:
\[
D(t, S, t_M, B) = K - B, \quad \text{for } S \leq B,
\]
\[
D(t, S, t_M, B) = [K - S]^+, \quad \text{for } S > B,
\]
\[
\lim_{S \to \infty} D(t, S, t_M, B) = 0.
\]  
(2.3.3)

Note that the time derivative on the right-hand side of the equation is with respect to the time to maturity. Now, we apply the Laplace-Carson transform with parameter \(\lambda\) to both sides of the equation and maximize over barriers \(B\) to obtain a differential equation for the randomized American put value \(P^{(1)}(S)\):
\[
\lambda \int_0^\infty e^{-\lambda t_M} \left( \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 D}{\partial S^2} + r S \frac{\partial D}{\partial S} - rD \right) dt_M = \lambda \int_0^\infty e^{-\lambda t_M} \left( \frac{\partial D}{\partial t_M} \right) dt_M
\]  
(2.3.4)

\[
\Rightarrow \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 P^{(1)}(S)}{\partial S^2} + r S \frac{\partial P^{(1)}(S)}{\partial S} - rP^{(1)}(S) = \lambda \int_0^\infty e^{-\lambda t_M} \left( \frac{\partial D}{\partial t_M} \right) dt_M,
\]

where we made use of Eq. (2.2.7) on the left-hand side of the equation. For the term on the right-hand side we use integration by parts to obtain
\[
\lambda \int_0^\infty e^{-\lambda t_M} \left( \frac{\partial D}{\partial t_M} \right) dt_M = \lambda \left[ e^{-\lambda t_M} D(0, S, t_M, B) \right]_0^\infty + \lambda^2 \int_0^\infty e^{-\lambda t_M} D(0, S, t_M, B) dt_M
\]  
(2.3.5)

\[
= -\lambda [K - S]^+ + \lambda P^{(1)}(S).
\]

Plugging Eq. (2.3.5) into the right-hand side of Eq. (2.3.2) we get the following partial-differential equation for the value of the first-order randomized American put option:
\[
\frac{1}{2} \sigma^2 S^2 \frac{\partial^2 P^{(1)}(S)}{\partial S^2} + r S \frac{\partial P^{(1)}(S)}{\partial S} - rP^{(1)}(S) = \lambda \left( P^{(1)}(S) - [K - S]^+ \right), \quad \text{for } S > S_1
\]  
(2.3.6)

with boundary conditions
\[
\lim_{S \to \infty} P^{(1)}(S) = 0,
\]
\[
\lim_{S \downarrow S_1} P^{(1)}(S) = K - S_1,
\]
\[
\lim_{S \downarrow S_1} \frac{\partial P^{(1)}(S)}{\partial S} = -1,
\]  
(2.3.7)

where \(S_1\) is the optimal exercise boundary for the American put. The optimal exercise boundary is the optimized value of the barrier \(B\) in Eq. (2.2.7). This ordinary differential equation can be solved analytically for \(P^{(1)}(S)\). The optimal exercise boundary \(S_1\) can also be determined in closed form. In this context, however, we only want to note that there are separate solutions for in-the-money options \((K > S > S_1)\) and out-of-the-money options \((S > K)\) which are matched at \(S = K\). For \(S < S_1\) the option should be exercised early and therefore its value is equal to the payoff \(K - S\). For the full analytic solution we refer to [5].
2.4 Higher-Order Randomization: Erlang Distributed Maturity

In [5] it is shown that the value of an American options maturing at the 1st jump of a Poisson process gives a fairly good approximation for the value of an American option with fixed maturity. As already discussed, the accuracy of the approximation is determined by the variance of the distribution underlying the random maturity variable. For the exponential distribution used in the previous example the variance is $T^2$.

Hence, the objective is to reduce the variance of the distribution but thereby retain the memoryless property (in order to remove time-dependence from the resulting partial differential equation). To this end, we again assume that the randomized maturity of the American option is represented by a Poisson process, but with the maturity date given by the $n$th jump of the Poisson process instead of the first jump. The waiting time until $n$ jumps of the Poisson process have occurred, which is equivalent to the maturity $\tau$ of the option, is Erlang distributed. The probability for the maturity to be contained in an infinitesimal time interval $dt$ is hence given by

$$
\Pr \{ \tau \in dt \} = \frac{\lambda^n}{(n-1)!} t^{n-1} e^{-\lambda t} dt.
$$

The mean of this distribution is $n/\lambda$ and the variance is $n/\lambda^2$. By letting $\lambda = n/T$ the mean of the distribution becomes $T$ while the variance is $T^2/n$. The variance is hence reduced by a factor of $1/n$ compared to the simpler “first-jump” approach. In the following we denote the value of an American put which can be exercised up to and including the $n$th jump time of the Poisson process described above by $P^{(n)}(S)$.

To valuate this higher-order randomized American option contract with Erlang distributed maturity an iterative approach has to be followed: Assuming that $n - 1$ jumps have already occurred, the value of the active American put is given by the “first-jump” valuation problem described in Eq. (2.2.7) where $\lambda$ is replaced by $n/T$ and the solution is $P^{(1)}(S)$. Going back one time period (i.e. $n - 2$ jumps have occurred), the randomized American put is equivalent to an option contract paying $P^{(1)}(S)$ at the next jump or $K - S$ on early exercise. Hence, for arbitrary stochastic processes of the underlying stock price the value of an American option maturing at the second jump of a Poisson process $P^{(2)}(S)$ can be expressed in terms of the value of an option maturing at the first jump of a Poisson process:

$$
P^{(2)}(S) = \sup_{B > 0} \mathbb{E} \left\{ e^{-rt_2} [K - B]^+ \mathcal{H}(\tau_2 - \tau_B) + e^{-rt_2} P^{(1)}(S_{\tau_2}) \mathcal{H}(\tau_B - \tau_2) \right\}, \quad S > S_2,
$$

where $\tau_2$ denotes the time until the next jump of the Poisson process occurs and $\mathcal{H}(x)$ is the Heaviside step function. $S_2$ denotes the critical stock price for the period until the next jump of the Poisson process occurs, i.e. the period from $t = 0$ to $\tau_2$. The first term on the right-hand side of the equation corresponds to the payoff in case the option is exercised before the next jump occurs. The second term corresponds to the value of the option, if it is not exercised before the next jump.

In general, the value of the randomized American option maturing after $m$ jumps of the Poisson process can hence be expressed as

$$
P^{(m)}(S) = \sup_{B > 0} \mathbb{E} \left\{ e^{-rt_m} [K - B]^+ \mathcal{H}(\tau_m - \tau_B) + e^{-rt_m} P^{(m-1)}(S_{\tau_m}) \mathcal{H}(\tau_B - \tau_m) \right\}, \quad S > S_m,
$$

10
with \( \tau_m \) the length of the time period until the next jump of the Poisson process occurs and \( S_m \) the critical stock price for this time period. In analogy to the case of exponentially distributed maturity, it is possible to write the above equation as an iterated expectation over the random time \( \tau_m \) until the next jump occurs and the future stock price:

\[
P^{(m)}(S) = \sup_{B > 0} \mathbb{E} \left\{ e^{-r\tau_m} \mathcal{H}(\tau_B) \mathcal{H}(t_M - \tau_B) \right\} \\
+ e^{-rt_M} P^{(m-1)}(S_{t_M}) \mathcal{H}(\tau_B - t_M) | \tau_m = t_M \right\} | t = 0, S_0 = S, S > S_m.
\]

In this case, the value of the expectation over the future stock price is equivalent to the value of a fixed-maturity down-and-out barrier put with maturity \( t_M \), barrier \( B \) and rebate \( K - B \), which pays \( P^{(m-1)}(S_{t_M}) \) at \( t_M \):

\[
\bar{D}(0, S, t_M, B) = \mathbb{E} \left\{ e^{-r\tau_B} [K - B]^+ \mathcal{H}(t_M - \tau_B) \right\} \\
+ e^{-rt_M} P^{(m-1)}(S_{t_M}) \mathcal{H}(\tau_B - t_M) | t = 0, S_0 = S, S > B.
\]

The first expectation over the random time until the next jump of the Poisson process can again be expressed as an integral over the exponential distribution of the individual jump time. The value of the random-maturity American put maturing after \( m \) jumps of the Poisson process is hence given by

\[
P^{(m)}(S) = \sup_{B > 0} \lambda \int_0^\infty e^{-\lambda t} \bar{D}(0, S, t_M, B) dt_M,
\]

where the value of the fixed-maturity barrier option \( \bar{D}(0, S, t_M, B) \) depends on the value of the randomized American put maturing after \( m - 1 \) jumps of the Poisson process \( P^{(m-1)}(S) \) with \( \lambda = m/T \).

By iteratively solving this equation it is possible to derive the value of an American option maturing at an arbitrary number \( m \) of jumps of the Poisson process. For the Black-Scholes model an explicit solution for arbitrary \( m \) is derived in [5]. In the particular case of the Black-Scholes model without dividends, it is also possible to find an analytic solution for the optimal exercise boundary for arbitrary \( m \).
Chapter 3

American Options in the CEV Model

In the previous chapter we gave a general overview of the maturity randomization approach to the valuation of American options. As an example, we reviewed the derivations for the Black-Scholes model presented in [5]. We now want to study a more general volatility model, the constant-elasticity-of-variance (CEV) model.

In the CEV model the risk-neutral stochastic process of the stock price evolution is governed by

\[
\frac{dS_t}{S_t} = r dt + \sigma \left( \frac{S_t}{S} \right)^\alpha dW_t. \tag{3.0.1}
\]

The model was first considered by Cox [6] and applied to describe the leverage effect observed in the stock market. When the price of a stock falls, its volatility tends to increase. In the CEV model this inverse relationship of stock prices and volatilities is reproduced for negative values of \( \alpha \). For \( \alpha > 0 \), instead, the model describes an increase of volatility with rising stock prices. However, since such an effect is rarely observed in the stock market, we will not consider this particular case in the following. For \( \alpha = 0 \) the usual log-normal price process from the Black-Scholes model is retained.

3.1 First-Order Randomization

As in the case of the standard Black-Scholes model, no analytic closed form solution exists for the value of an American put in the CEV model. In the following we will therefore apply maturity randomization to approximate the American put by a random-maturity American put. To this end, we start with considering the value of a randomized American put maturing at the first jump time of a Poisson process. From the discussion in section 2.2 we know that in this case the value of the American put is given by eq. (2.2.7):

\[
P^{(1)}(S) = \sup_B \lambda \int_0^\infty e^{-\lambda t} D(0, S, t, B) dt, \tag{3.1.1}
\]

i.e. the value is given by the Laplace-Carson transform of the value of a down-and-out barrier option with barrier \( B \), fixed maturity \( t_M \) and rebate \( K - B \), maximized over barriers. As also show in section 2.2, the differential equation for the American option’s value \( P^{(1)}(S) \) can hence be derived by applying the Laplace-Carson transform to the differential equation satisfied by the barrier option’s value.

Similar to the Black-Scholes equation, the partial differential equation for the value of a barrier option in the CEV model can be derived by means of non-arbitrage arguments and application of Ito’s lemma. The resulting Black-Scholes like PDE for the value of a
Barrier option in the CEV model is

\[
\frac{1}{2} \sigma^2 S^2 \left( \frac{S}{\bar{S}} \right)^{2\alpha} \frac{\partial^2 D}{\partial S^2} + r S \frac{\partial D}{\partial S} - r D = \frac{\partial D}{\partial t_M}
\]

(3.1.2)

with the time derivative with respect to the maturity \( t_M \) of the option.

After applying the Laplace-Carson transform to both sides of the equation we obtain the following ordinary differential equation for the value of the first-order randomized American put \( P^{(1)}(S) \) in the CEV model:

\[
\frac{1}{2} \sigma^2 S^2 \left( \frac{S}{\bar{S}} \right)^{2\alpha} \frac{\partial^2 P^{(1)}(S)}{\partial S^2} + r S \frac{\partial P^{(1)}(S)}{\partial S} - r P^{(1)}(S) = \lambda \left( P^{(1)}(S) - [K - S]^+ \right), \quad S > S_1
\]

(3.1.3)

and boundary conditions

\[
\lim_{S \to \infty} P^{(1)}(S) = 0,
\]

\[
\lim_{S \downarrow S_1} P^{(1)}(S) = K - \bar{S}_1,
\]

\[
\lim_{S \downarrow S_1} \frac{\partial P^{(1)}(S)}{\partial S} = -1.
\]

For \( \alpha = 0 \) the problem is equivalent to the valuation problem in the standard Black-Scholes model and can be solved analytically. For arbitrary values of \( \alpha \), however, no simple closed-form analytic solution to the free-boundary problem exists. Therefore we try to find a solution to the problem in terms of an asymptotic expansion for small volatilities \( \sigma \).

Let us first introduce dimensionless variables for the stock price and the option’s value

\[
s = \frac{S}{K}, \quad p^{(1)}(s) = \frac{P^{(1)}(S)}{K}.
\]

(3.1.5)

With these variables the pricing problem becomes

\[
\frac{1}{2} \sigma^2 s^{2+2\alpha} \left( \frac{K}{\bar{S}} \right)^{2\alpha} \frac{\partial^2 p^{(1)}(s)}{\partial s^2} + r s \frac{\partial p^{(1)}(s)}{\partial s} - r p^{(1)}(s) = \lambda \left( p^{(1)}(s) - [1 - s]^+ \right), \quad s > S_1
\]

(3.1.6)

In the limit \( \sigma^2 \to 0 \) we are faced with a singular perturbation problem, since \( \sigma^2 \) multiplies the highest derivative with respect to \( s \) in the differential equation (and hence the highest derivative will not be present in the leading-order order differential equation). To solve the problem, we therefore introduce an inner region around the strike \( K \), i.e. \( s = 1 \). To this end, we adapt the definition of the inner region from [11] where asymptotic expansions for European plain-vanilla options are studied. We perform a perturbative expansion for the inner region where

\[
|s - 1| = \mathcal{O}(\sigma \sqrt{T})
\]

(3.1.7)

and later match the obtained results with the result obtained for the outer region.

For the inner region we define a new dimensionless variable \( x \) and a dimensionless parameter \( \epsilon \)

\[
x = \frac{s - 1}{\sigma \sqrt{T}}, \quad \epsilon = \sigma \sqrt{T}.
\]

(3.1.8)

Besides, we define a new pricing function \( \hat{p}^{(1)} \) by a change of variable and rescaling with
a factor of $1/\epsilon$

$$\hat{p}^{(1)}(x) = \frac{p^{(1)}(s)}{\epsilon}.$$  \hspace{1cm} (3.1.9)

For better readability of the following equations we introduce the parameters

$$\beta = 2\alpha + 2, \quad \hat{K} = \left(\frac{K}{S}\right)^{2\alpha}, \quad \gamma^2 = \frac{2\lambda T}{\hat{K}}.$$ \hspace{1cm} (3.1.10)

Besides, for our choice of $\epsilon$ we assume that the ratio $\sigma^2/2r = \mathcal{O}(\epsilon)$. This assumption can be justified by looking at the solution to the similar pricing problem of a perpetual American put in the Black-Scholes model. In this case the pricing problem can be solved analytically and the critical stock price is given by

$$S_p = K \left(1 + \frac{\sigma^2}{2r}\right) = K - K\sigma^2 \left(1 + \frac{\sigma^2}{2r}\right).$$ \hspace{1cm} (3.1.11)

For our inner expansion we have $S - K = \mathcal{O}(\epsilon) \times K$ and hence the critical stock price for the maturity randomized American put is $S_1 - K = \mathcal{O}(\epsilon) \times K$. The critical stock price for the maturity-randomized American put can only be of the same order of magnitude as the critical stock price of the perpetual American option, if we suppose that $\sigma^2/2r = \mathcal{O}(\epsilon)$. Hence, we define the $\mathcal{O}(1)$ parameter

$$k = \frac{r\epsilon}{\sigma^2 K}.$$ \hspace{1cm} (3.1.12)

With these new variables and parameters the original free-boundary valuation problem for the randomized American option becomes

$$\frac{1}{2}(1 + \epsilon x)^{\beta} \frac{\partial^2 \hat{p}^{(1)}(x)}{\partial x^2} + k(1 + \epsilon x) \frac{\partial \hat{p}^{(1)}(x)}{\partial x} - \epsilon k \hat{p}^{(1)}(x)(x - [-x]^+) = \frac{1}{2} \gamma^2 (\hat{p}(x) - [-x]^+) , \quad x > 0_1$$ \hspace{1cm} (3.1.13)

subject to the boundary conditions

$$\lim_{x \to \infty} \hat{p}^{(1)}(x) = 0,$$

$$\lim_{x \downarrow 0_1} \hat{p}^{(1)}(x) = -0_1,$$

$$\lim_{x \downarrow 0_1} \frac{\partial \hat{p}^{(1)}(x)}{\partial x} = -1.$$ \hspace{1cm} (3.1.14)

We note that due to the term proportional to $[-x]^+$ the solution to the differential equation will have a different form for $x \geq 0$ and $x < 0$. Therefore, in the following we will search for separate solutions for both regimes and later match the two solutions by prescribing that the values and first derivatives of both solutions match at $x = 0$.

Let us now expand the option’s value in powers of the small parameter $\epsilon = \sigma \sqrt{T}$

$$\hat{p}^{(1)}(x) = p^{(1)}_0(x) + \epsilon p^{(1)}_1(x) + \epsilon^2 p^{(1)}_2(x) + \ldots$$ \hspace{1cm} (3.1.15)

and perform a Taylor expansion for the term $(1 + \epsilon x)^{\beta}$ around $\epsilon = 0$

$$(1 + \epsilon x)^{\beta} = 1 + \epsilon \beta x + \frac{1}{2} \epsilon^2 (\beta - 1) \beta x^2 + \ldots$$ \hspace{1cm} (3.1.16)

By means of these expansions, we can collect terms of the same power in $\epsilon$ in the partial differential equation (3.1.13) and separately solve the problem for each power in $\epsilon$. 

14
To the zeroth order we obtain the following differential equation for $\hat{p}_0^{(1)}(x)$

$$
\frac{1}{2} \partial_x^2 \hat{p}_0^{(1)}(x) + k \frac{\partial \hat{p}_0^{(1)}(x)}{\partial x} - \frac{1}{2} \gamma^2 \hat{p}_0^{(1)}(x) = -\frac{1}{2} \gamma^2 [-x]^+. \tag{3.1.17}
$$

We first solve the homogeneous part on the l.h.s. of the differential equation and then add a term $[-x]^+$ to the solution to account for the inhomogeneous term on the r.h.s. of the equation. The general solution to this problem is then simply given by

$$
\hat{p}_0^{(1)}(x) = Ae^{(-k-\eta)x} + Be^{(-k+\eta)x} + [-x]^+ - \mathcal{H}(-x) \frac{2k}{\gamma^2} \tag{3.1.18}
$$

with $A$ and $B$ real-valued constants and $\eta$ defined as

$$
\eta = \sqrt{\gamma^2 + k^2}. \tag{3.1.19}
$$

As already mentioned, we are looking for separate solutions for $x < 0$ and $x \geq 0$. Due to the boundary condition $\lim_{x \to -\infty} \hat{p}^{(1)}(x) = 0$ the constant multiplying the term proportional to exp$([-k + \eta]x)$ needs to be zero for the solution for values of $x$ greater than zero. Taking also the boundary conditions at $\xi_1$ into account we therefore obtain the solution to zeroth order in $\epsilon$

$$
\hat{p}_0^{(1)}(x) = \begin{cases} 
C e^{(-k-\eta)x}, & \text{for } x \geq 0 \\
A e^{(-k-\eta)x} + Be^{(-k+\eta)x} - x - \frac{2k}{\gamma^2}, & \text{for } \xi_1 < x < 0 \\
-\frac{1}{k}, & \text{for } x \leq \xi_1
\end{cases} \tag{3.1.20}
$$

with $A$, $B$ and $C$ real-valued constants. We note that after having rescaled the pricing function with a parameter $1/\epsilon$ the free boundary conditions at $\xi_1$ are of zeroth order in $\epsilon$. Otherwise the boundary conditions would have been proportional to $\epsilon$ and hence could not be fulfilled by the leading-order solution.

For now, let us also expand the critical stock price $\xi_1$ in powers of $\epsilon$

$$
\xi_1 = \xi_{1,0} + \epsilon \xi_{1,1} + \epsilon^2 \xi_{1,2} + \ldots. \tag{3.1.21}
$$

The boundary conditions at $\xi_1$ then also have to be expanded and become

$$
\lim_{x \to \xi_1} \left\{ \hat{p}_0^{(1)}(x) + \epsilon \left( \xi_{1,1} \frac{\partial \hat{p}_0^{(0)}(x)}{\partial x} + \hat{p}_1^{(1)}(x) \right) + \ldots \right\} = -\xi_{1,0} - \epsilon \xi_{1,1} + \ldots, \tag{3.1.22}
$$

$$
\lim_{x \to \xi_1} \left\{ \frac{\partial \hat{p}_0^{(1)}(x)}{\partial x} + \epsilon \left( \xi_{1,1} \frac{\partial^2 \hat{p}_0^{(0)}(x)}{\partial x^2} + \frac{\partial \hat{p}_1^{(1)}(x)}{\partial x} \right) + \ldots \right\} = -1.
$$

Note that the expanded boundary conditions are evaluated at the zeroth-order approximation of the exercise boundary $\xi_{1,0}$. Hence, the goodness of the approximation will strongly depend on the accuracy of the zeroth-order approximation.

From the zeroth-order terms of the smooth pasting conditions at $\xi_1$ and by assuming that $\hat{p}_0^{(1)}(x)$ is continuous and differentiable at $x = 0$, we obtain a system of four equations for the four unknown parameters $A$, $B$, $C$ and $\xi_{1,0}$. Solving this system of equations we find that the three parameters $A$, $B$ and $C$ are given by

$$
A = \frac{e^{2\eta \xi_{1,0}}}{2\eta}, \quad B = -\frac{k + \eta}{2k\eta - 2\eta^2}, \quad C = \frac{\eta \left( 1 + e^{2\eta \xi_{1,0}} \right) - k \left( 1 - e^{2\eta \xi_{1,0}} \right)}{2k\eta + \eta^2}. \tag{3.1.23}
$$
and the zeroth-order approximation of the exercise boundary is

$$\xi_{1,0} = \frac{1}{k - \eta} \ln \left( \frac{1}{2} + \frac{\eta}{2k} \right).$$

(3.1.24)

In principle, it is possible to perform this analysis also to higher orders in $\epsilon$, i.e. find a solution to the differential equation for terms proportional to $\epsilon$, determine the constants in the solution and the higher-order contribution to the exercise boundary by applying the expanded boundary conditions at $x = \xi_{1,0}$ and $x = 0$.

However, as we will later see, the complexity of the algebraic equation which have to be solved to determine the constants and contributions to the exercise boundary strongly increases for higher orders in $\epsilon$ and particularly for higher orders in the randomization. Hence, instead of expanding the exercise boundary $\xi_1$ (and the boundary conditions at $\xi_1$) in powers of $\epsilon$, we keep the exercise boundary and constants $A$, $B$ and $C$ undetermined for now and will later solve for their values numerically by applying the boundary conditions to the full solution to a given order in $\epsilon$. The disadvantage of this approach is that the obtained solution will not be fully consistent to a given order in $\epsilon$. Since we do not expand the exercise boundary, its numerically obtained value will implicitly contain contributions from higher-order corrections. Hence, the expansion is hybrid.

An advantage of the approach, however, is that the smooth pasting condition is not evaluated at the zeroth-order approximation of the exercise boundary $\xi_{1,0}$, but at the numerically obtained higher-order value of the exercise boundary. This strongly increases the convergence of the obtained value for the exercise boundary compared to the approach where the exercise boundary is expanded (which is a manifestation of the fact that the numerical approach implicitly takes higher order contributions into account).

Collecting all terms in eq. (3.1.13) which are proportional to $\epsilon$ we find that the first-order correction $\hat{p}_1^{(1)}(x)$ is governed by the differential equation

$$\frac{1}{2} \frac{\partial^2 \hat{p}_1^{(1)}(x)}{\partial x^2} + k \frac{\partial \hat{p}_1^{(1)}(x)}{\partial x} - \frac{1}{2} \gamma \hat{p}_1^{(1)} = - \frac{1}{2} \beta x \frac{\partial^2 \hat{p}_0^{(1)}(x)}{\partial x^2} - k x \frac{\partial \hat{p}_0^{(1)}(x)}{\partial x} + k \hat{p}_0^{(1)}(x).$$

(3.1.25)

The solution for $\hat{p}_1^{(1)}(x)$ is hence determined by an inhomogeneous differential equation which depends on the leading-order solution $\hat{p}_0^{(1)}(x)$. We will later see that also the differential equations to higher orders in $\epsilon$ have a similar structure. In particular, the dependency of the problem to lower-order solutions is retained to all orders. Therefore the solution to a given order cannot be obtained separately, but the problem can only be solved iteratively starting at the leading order in $\epsilon$.

We observe that the homogeneous part of the differential equation

$$\frac{1}{2} \frac{\partial^2 \hat{p}_1^{(1)}(x)}{\partial x^2} + k \frac{\partial \hat{p}_1^{(1)}(x)}{\partial x} - \frac{1}{2} \gamma^2 \hat{p}_1^{(1)}$$

(3.1.26)

is equal to that of the leading-order equation (3.1.17) and is consequently solved by $\hat{p}_0^{(1)}(x)$. To find a special solution of the full inhomogeneous equation we try an Ansatz with terms proportional to the leading-order solutions. Besides, due to the term

$$k \hat{p}_1^{(0)}(x)$$

(3.1.27)

an additional constant term $-2k^2/\gamma^2$ enters the equation for $x < 0$. To account for this term we add a term proportional to the Heaviside step function to our Ansatz, which is then given

$$\hat{p}_1^{(1)}(x) = f_1(x) A e^{(-k-\eta)x} + g_1(x) B e^{(-k+\eta)x} + \frac{4k^2}{\gamma^4} \mathcal{H}(-x),$$

(3.1.28)
where \( f_1(x) \) and \( g_1(x) \) are polynomial functions of \( x \). Inserting the Ansatz into eq. (3.1.25) we obtain a differential equation for \( f_1(x) \) and \( g_1(x) \)

\[
A e^{(-k-\eta)x} \left( \frac{1}{2} f_1''(x) - \eta f_1'(x) + \frac{1}{2} \beta x(k + \eta)^2 - kx(k + \eta) - k \right) + B e^{(-k+\eta)x} \left( \frac{1}{2} g_1''(x) + \eta g_1'(x) + \frac{1}{2} \beta x(k - \eta)^2 - kx(k - \eta) - k \right) = 0. \tag{3.1.29}
\]

Since the differential equation has to be fulfilled for all values of \( x > x_1 \), both terms in brackets have to be zero. This leaves us with two separate differential equations for \( f_1(x) \) and \( g_1(x) \).

Let us first consider the differential equation for \( f_1(x) \):

\[
\frac{1}{2} f_1''(x) - \eta f_1'(x) + \frac{1}{2} \beta x(k + \eta)^2 - kx(k + \eta) - k = 0. \tag{3.1.30}
\]

As an Ansatz for the solution of the differential equation, we use a quadratic polynomial function

\[
f_1(x) = ax^2 + bx \tag{3.1.31}
\]

with \( a \) and \( b \) real-valued constants. Since the differential equation (3.1.30) only contains derivatives of the function \( f_1(x) \) and not the function itself, we could add an arbitrary constant to our Ansatz for \( f_1(x) \). This is a manifestation of the fact that the leading-order solution \( \tilde{p}^{(1)}_0 \) solves the homogeneous part of the first-order differential equation (3.1.25).

Adding a constant to \( f_1(x) \) results in adding a constant times the leading-order solution to \( \tilde{p}^{(1)}_1(x) \). For now, we set the constant to zero and use the quadratic Ansatz above. Plugging the Ansatz into the differential equation for \( f_1(x) \) and solving for the constants \( a \) and \( b \) we obtain the following solution:

\[
f_1(x) = \frac{1}{4\eta}(k + \eta)(k\beta + \eta\beta - 2k)x^2 + \frac{1}{4\eta^2} \left( (k + \eta)^2 - 6k\eta - 2k^2 \right) x. \tag{3.1.32}
\]

By means of the same approach, we can derive a similar expression for the function \( g_1(x) \):

\[
g_1(x) = -\frac{1}{4\eta}(k - \eta)(k\beta - \eta\beta - 2k)x^2 + \frac{1}{4\eta^2} \left( (k - \eta)^2 + 6k\eta - 2k^2 \right) x. \tag{3.1.33}
\]

Note that we have not yet specified the \( x \) range for which the solution above is applicable. Taking the boundary conditions of the all-order problem into account we find

\[
\tilde{p}^{(1)}_1(x) = \begin{cases} 
C e^{(-k-\eta)x} f_1(x), & \text{for } x \geq 0 \\
A e^{(-k-\eta)x} f_1(x) + B e^{(-k+\eta)x} g_1(x) + \frac{4k^2}{\gamma^2}, & \text{for } x < 0
\end{cases} \tag{3.1.34}
\]

where \( A, B \) and \( C \) are the real-valued constants from the leading-order solution.

Considering also the boundary conditions at \( x_1 \) and adding \( \tilde{p}^{(1)}_1 \) to the leading-order solution, we obtain the full solution to first order in \( \epsilon \) in the form

\[
\tilde{p}^{(1)}_0(x) + \epsilon \tilde{p}^{(1)}_1(x) = \begin{cases} 
C e^{(-k-\eta)x} \left[ 1 + \epsilon f_1(x) \right], & \text{for } x \geq 0 \\
A e^{(-k-\eta)x} \left[ 1 + \epsilon f_1(x) \right] + B e^{(-k+\eta)x} \left[ 1 + \epsilon g_1(x) \right] - x - \frac{2k}{\gamma^2} + \frac{4k^2}{\gamma^2}, & \text{for } x_1 < x < 0 \\
-x, & \text{for } x \leq x_1
\end{cases} \tag{3.1.35}
\]

As mentioned before, instead of expanding the constants \( A, B \) and \( C \) as well as the exercise boundary \( x_1 \) in powers of \( \epsilon \), we will later numerically determine the constants by means
of the boundary conditions at $x_1$ and the matching conditions at $x = 0$. Before doing so we want to briefly sketch the derivation of the correction to second order in $\epsilon$.

The pricing problem for the second-order correction is determined by the terms which are proportional to $\epsilon^2$ in eq. (3.1.13). Collecting all these terms we obtain the following differential equation for $\hat{p}_2^{(1)}(x)$:

$$
\frac{1}{2} \frac{\partial^2 \hat{p}_2^{(1)}(x)}{\partial x^2} + k \frac{\partial \hat{p}_2^{(1)}(x)}{\partial x} - \frac{1}{2} \gamma^2 \hat{p}_2^{(1)}(x) = -\frac{1}{4} \beta(\beta - 1)x^2 \frac{\partial^2 \hat{p}_0^{(1)}(x)}{\partial x^2} - \frac{1}{2} \beta x \frac{\partial^2 \hat{p}_1^{(1)}(x)}{\partial x^2} - k x \frac{\partial \hat{p}_1^{(1)}(x)}{\partial x} + k p_1^{(1)}(x). \tag{3.1.36}
$$

Again we observe that the homogeneous part of the differential equation (l.h.s. of the equation) is solved by the leading-order solution. Besides, the inhomogeneous part on the r.h.s. of the equation depends on the lower-order contributions $\hat{p}_1^{(1)}(x)$ and $p_0^{(1)}(x)$.

The approach to solve the differential equation is hence similar to that for solving the first-order problem, i.e. our Ansatz is the leading-order solution with both terms multiplied by polynomial functions $g_2(x)$ and $f_2(x)$. Besides we add a term to cancel the contribution proportional to the Heaviside step function coming from the term $k p_1^{(1)}(x)$ on the r.h.s.:

$$
\hat{p}_2^{(1)}(x) = f_2(x)Ae^{(-k-\eta)x} + g_2(x)B e^{(-k+\eta)x} - \frac{8k^3}{\gamma^6} \mathcal{H}(-x). \tag{3.1.37}
$$

By plugging this Ansatz into eq. (3.1.36) we obtain separate differential equations for $f_2(x)$ and $g_2(x)$. In this case, the polynomial Ansatz for the functions $f_2(x)$ and $g_2(x)$ needs to be of fourth order in $x$ to solve the equation, i.e.

$$
f_2(x) = ax^4 + bx^3 + cx^2 + dx. \tag{3.1.38}
$$

Inserting the Ansatz into the differential equation and evaluating the pre-factors, we obtain the following expressions for the polynomial functions:

$$
f_2(x; \eta) = x^4 \left( \frac{(k + \eta)^2(-2k + k\beta + \beta\eta)^2}{32\eta^2} + x^3 \frac{k\beta(k + \eta)(k(\beta - 2) + \eta(\beta - 4))}{12\eta^2} \right)
+ \left( \frac{x}{32\eta^5} + \frac{x^2}{32\eta^4} + \frac{x^3}{48\eta^3} \right)(k + \eta)
\times \left( 20k^3 - 20k^3\beta + 5k^3\beta^2 + 28k^2\eta - 20k^2\beta\eta + 3k^2\beta^2\eta + 4k\beta\eta^2 
- k\beta^2\eta^2 - 4\beta\eta^3 + \beta^2\eta^3 \right), \tag{3.1.39}
$$

$$
g_2(x; \eta) = f_2(x; -\eta). \tag{3.1.40}
$$

which complete the derivation of the second-order correction to the value of the randomized American put. The function $g_2(x)$ can directly be obtained from the function $f_2(x)$ by replacing $\eta$ by $-\eta$.

Adding the second-order correction to the first-order solution our full solution for the
option’s value to second order in $\epsilon$ becomes

$$
\hat{p}_0^{(1)}(x) + \epsilon \hat{p}_1^{(1)}(x) + \epsilon^2 \hat{p}_2^{(1)}(x) = \begin{cases} 
C e^{(-k-\eta)x} \left[ 1 + \epsilon f_1(x) + \epsilon^2 f_2(x) \right], & \text{for } x \geq 0 \\
A e^{(-k-\eta)x} \left[ 1 + \epsilon f_1(x) + \epsilon^2 f_2(x) \right] + B e^{(-k+\eta)x} \left[ 1 + \epsilon g_1(x) + \epsilon^2 g_2(x) \right] - x - \frac{2k}{\gamma} + \epsilon \frac{4k^2}{\gamma^2} - \epsilon^2 \frac{4k^3}{\gamma^3}, & \text{for } x_1 < x < 0 \\
-x, & \text{for } x \leq x_1 
\end{cases}
$$

(3.1.41)

We note that going to even higher orders in the expansion in $\epsilon$ further terms proportional to $e^{(-k-\eta)x}$ and $e^{(-k+\eta)x}$ enter the expression for the option’s value. The corrections are given by additional terms in the square brackets of the form $\epsilon^n f_n(x)$ or $\epsilon^n g_n(x)$, where $f_n(x)$ and $g_n(x)$ are higher-order polynomial functions.

As already mentioned, if the exercise boundary $x_1$ is not expanded, the put value to any order in $\epsilon$ comprises four yet undetermined parameters: $A$, $B$, $C$ and the optimal exercise boundary $x_1$ itself. The values of these parameters are determined by the boundary conditions at $x = 0$ and $x = x_1$. We require that the option’s value and its first-order derivative are continuous at $x = 0$ and $x = x_1$ (i.e. the function is continuous and differential for all values of $x$) and hence we obtain four equations which determine the four parameters.

$$
\begin{align*}
\lim_{x \downarrow 0} \hat{p}_1^{(1)}(x) &= \lim_{x \uparrow 0} \hat{p}_1^{(1)}(x), \\
\lim_{x \downarrow 0} \frac{\partial \hat{p}_1^{(1)}(x)}{\partial x} &= \lim_{x \uparrow 0} \frac{\partial \hat{p}_1^{(1)}(x)}{\partial x}, \\
\lim_{x \uparrow x_1} \hat{p}_1^{(1)}(x) &= -x_1, \\
\lim_{x \downarrow x_1} \frac{\partial \hat{p}_1^{(1)}(x)}{\partial x} &= -1.
\end{align*}
$$

(3.1.42)

In the following we numerically solve the system of equations for the considered set of option parameters and thereby determine the value of the four constants.

### 3.1.1 Numerical Analysis

We examine the goodness of our first-order randomization results by comparing the obtained maturity-randomized American put value and optimal exercise boundary with the value and exercise boundary of its fixed-maturity equivalent. To numerically calculate the value and exercise boundary of the fixed-maturity option we implement a finite-differences scheme by discretizing the time and asset-price dimension. We thereby restrict the asset-price dimension to a finite domain $S \in [0, S_{\text{max}}]$ for a sufficiently large value of $S_{\text{max}}$ and introduce a uniform grid with $N+1$ grid points in the asset-price dimension and $M+1$ grid points in the time dimension.

To solve the Black-Scholes differential equation in the CEV model we apply an implicit Euler scheme, i.e. the time derivatives are approximated by forward differences and the resulting linear equations are solved backwards in time. To account for the free boundary condition of the LCP at any step of the calculation we implement a projected successive over-relaxation algorithm (PSOR) [8, 19]. The PSOR algorithm makes sure that at each time step of the backward calculation the inequalities of the LCP are fulfilled.

In the limit $N \to \infty$, $M \to \infty$ the finite-differences result converges to the true value of the option for continuous asset prices and time. However, the runtime of the numerical algorithm increases with the number of grid points in the asset-price and time dimension. For a sensible choice of the grid size it is important that the convergence of the numerical
Figure 3.1: Relative error of the numerical American put value in the CEV model between a finite-differences scheme with grid size $N = 1600$, $M = 400$ and a finite-differences scheme with grid size $N = 800$, $M = 200$. The relative error is shown for three different values of the parameter $\alpha$. The option parameters are $r = 0.02$, $\sigma = 0.3$, $\hat{S} = 100$, $T = 1$, $K = 100$.

The finite-differences results is good enough. The convergence for a given number of grid points can be assessed by looking at the relative error of the obtained option value by doubling the number of grid points in both the asset-price and time dimension:

$$
\Delta V(S) = \frac{V^{2N,2M}(S) - V^{N,M}(S)}{V^{N,M}(S)}
$$

In Fig. 3.1 the relative error of the fixed-maturity American put value in the CEV model from a finite differences scheme with $N = 1600$ and $M = 400$ grid points compared to a grid with $N = 800$ and $M = 200$ grid points is shown. The relative error is calculated for three different values of $\alpha$ and the following market and option parameters

$$
r = 0.02, \quad \sigma = 0.3, \quad \hat{S} = 100, \quad T = 1, \quad K = 100.
$$

The upper boundary of the asset-price grid is set to $S_{\text{max}} = 400$.

As can be seen from the figure, the relative error obtained from doubling the number of grid points in both dimensions is smaller than $10^{-3}$ in the relevant asset-price region around the option’s strike. This observation holds for all three considered values of the CEV parameter $\alpha$. Consequently, we can state that the accuracy of the finite-differences results for a grid size of $N = 1600$, $M = 400$ is of the order of $10^{-3}$. We will hence adapt the results for the fixed-maturity American put obtained from this finite-differences setup as a benchmark for the analysis of the goodness of the first-order maturity-randomized approximation.

In the upper plot of Figures 3.2 to 3.4 the value of an American put in the CEV model obtained from the finite-differences scheme is shown together with the value of the first-order maturity-randomized American put obtained from the asymptotic expansion to second order in $\epsilon$. The numerical values are calculated for three different values of $\alpha$ and the following market and option parameters

$$
r = 0.02, \quad \sigma = 0.3, \quad \hat{S} = 100, \quad T = 1, \quad K = 100.
$$
Figure 3.2: Comparison of the American put results in the CEV model for $\alpha = 0$ between a finite-differences scheme using the PSOR algorithm (numerical) and first-order randomization using an asymptotic expansion in $\epsilon$ (analytical). The parameters are $r = 0.02$, $\sigma = 0.3$, $\hat{S} = 100$, $T = 1$, $K = 100$. The upper plot shows the put value, the lower plot shows the relative error of the randomized value with respect to the finite-differences value.

For the finite-differences scheme we again set $S_{\text{max}} = 400$. From Fig. 3.2 we see that the value of the random-maturity option yields a fairly good approximation of the fixed-maturity American option value, however for at-the-money options the random-maturity value lies significantly below the finite-differences result. This undervaluation can heuristically be explained by the additional uncertainty introduced by the random maturity. Therefore, the agent is facing an additional source of risk when holding a randomized option compared to holding a fixed-maturity option. This additional risk is taken account of by adding a negative risk premium to the option’s value. The premium is negative, since due to the additional uncertainty the agent is willing to pay less for a randomized option than for its fixed-maturity equivalent.
Figure 3.3: Comparison of the American put results in the CEV model for $\alpha = -0.5$ between a finite-differences scheme using the PSOR algorithm (numerical) and first-order randomization using an asymptotic expansion in $\epsilon$ (analytical). The parameters are $r = 0.02$, $\sigma = 0.3$, $\hat{S} = 100$, $T = 1$, $K = 100$. The upper plot shows the put value, the lower plot shows the relative error of the randomized value with respect to the finite-differences value.

The lower plots in Figures 3.2 to 3.4 show the relative error of the first-order maturity randomized put value compared to the fixed-maturity finite-differences result for the three different values of $\alpha$. The relative error is shown for the asymptotic expansion truncated at the zeroth, first and second order in $\epsilon$. For large values of the spot price, i.e. for options far out-of-the-money, the randomized put value to first and second order in $\epsilon$ is larger than the put value obtained from the finite-differences scheme (although the absolute deviation is small, the relative error is very large due to the small absolute value of the out-of-the-money option). This deviation does not necessarily indicate a shortcoming of the randomization approach or the asymptotic expansion, instead the finite-differences scheme undervalues the American put for large spot prices due to the
Figure 3.4: Comparison of the American put results in the CEV model for $\alpha = -1$ between a finite-differences scheme using the PSOR algorithm (numerical) and first-order randomization using an asymptotic expansion in $\epsilon$ (analytical). The parameters are $r = 0.02$, $\sigma = 0.3$, $S = 100$, $T = 1$, $K = 100$. The upper plot shows the put value, the lower plot shows the relative error of the randomized value with respect to the finite-differences value.

fact that the boundary condition for $S \to \infty$ has to be applied at the boundary of the asset-price grid, $S_{\text{max}} = 400$, $P(S_{\text{max}}) = 0$. Besides, we observe that the solutions to first and second order in $\epsilon$ are in general very close to each other and almost coincide for the case $\alpha = -1$, i.e. the contribution proportional to $\epsilon^2$ to the option’s value in the asymptotic expansion is small.

In Table 3.1 we have collected the values of the optimal exercise boundary for the random-maturity American put option in the CEV model for the three different values of the parameter $\alpha$. As a benchmark we again use the optimal exercise boundary of the fixed-maturity American put at $t = 0$ obtained from the finite-differences scheme. We note that the accuracy of the optimal exercise boundary in the finite-differences scheme is
Table 3.1: Critical stock price for the American put in the CEV model for three different values of $\alpha$. The table shows the results from the finite-differences scheme and from first-order randomization with the asymptotic expansion truncated at the zeroth, first and second order in $\epsilon$.

<table>
<thead>
<tr>
<th>Critical stock price at $t = 0$</th>
<th>$\alpha = 0$</th>
<th>$\alpha = -0.5$</th>
<th>$\alpha = -1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Finite Differences (PSOR)</td>
<td>61.1</td>
<td>56.1</td>
<td>48.4</td>
</tr>
<tr>
<td>1st-order Randomization, $O(\epsilon^0)$</td>
<td>46.44</td>
<td>46.44</td>
<td>46.44</td>
</tr>
<tr>
<td>1st-order Randomization, $O(\epsilon^1)$</td>
<td>58.97</td>
<td>54.15</td>
<td>47.29</td>
</tr>
<tr>
<td>1st-order Randomization, $O(\epsilon^2)$</td>
<td>60.16</td>
<td>55.13</td>
<td>47.28</td>
</tr>
</tbody>
</table>

constrained by the step size of the asset-price dimension, $\Delta S = 0.25$. For the randomized put we find that for the solution to zeroth order in $\epsilon$ we obtain the same result for all values of $\epsilon$. The numerical value of the exercise boundary is equal to the analytic result in Eq. (3.1.24). The fact that the exercise boundary is the same for all values of $\alpha$ is due to our special choice of the parameter $\hat{S} = 100$, which is equal to the value of the strike $K = 100$. For this particular choice, the zeroth-order valuation problem in Eq. (3.1.17) is independent from $\alpha$ and therefore also the analytic and numerical result for the exercise boundary is independent from $\alpha$. Since the exercise boundary is constant as a function of $\alpha$ at zeroth order in $\epsilon$, its value of course only poorly approximates the true value of the critical stock price for arbitrary values of $\alpha$. Instead the dependence on the parameter $\alpha$ is only retained when higher-order corrections are considered. If we take into account those terms to first or second order in $\epsilon$, we obtain a good approximation of the optimal exercise boundary for all considered values of $\alpha$. Going from first to second order the accuracy of the approximation for the optimal exercise boundary slightly increases.

However, including even higher-order terms in $\epsilon$ into our derivation will not significantly improve the approximation of the American put value and its optimal exercise boundary. Instead, we have to improve our approximation of the American option contract by considering randomized American options with narrower maturity distributions. As mentioned above, the maturity-randomized American put value converges to the fixed-maturity value by letting the variance of the maturity distribution approach zero. In our first-order randomization approach where we assume that the maturity of the option is given by the first jump of a Poisson process the variance of the distribution is equal to $T^2$. For higher-order randomized puts with the maturity determined by the $n$th jump of a Poisson process the variance of the maturity distribution is instead given by $T^2/n$. In the following section, we will therefore derive analytic expressions for higher-order randomized American put values in the CEV model.

### 3.2 Higher-Order Randomization

In section 2.4 it was shown that for randomized American put options with the random maturity given by the $n$th jump of a Poisson process the option value is given by the Laplace-Carson transform of the value of a fixed-maturity barrier option

$$P^{(n)}(S) = \sup_{B > 0} \lambda \int_0^\infty e^{-\lambda M} \tilde{D}(0, S, t_M, B) dt_M,$$  

(3.2.1)

where $\lambda = n/T$ and $\tilde{D}(0, S, t_M, B)$ is the value of a down-and-out barrier put with maturity $t_M$, barrier $B$, and rebate $K - B$. The payoff of the barrier option at $t_M$ is equal to
$P^{(n-1)}(S)$, i.e. the value of a randomized American put maturing at the $(n-1)$th jump of a Poisson process with parameter $\lambda = n/T$.

The value of the fixed-maturity barrier option satisfies the Black-Scholes-like partial differential equation in the CEV model from Eq. (3.1.2). In analogy to the case of the first-order randomized put, the differential pricing equation for $P^{(n)}$ can be derived by applying the Laplace-Carson transform to both sides of Eq. (3.1.2). The resulting free-boundary pricing problem for the value of the randomized American put is

$$
\frac{1}{2} \sigma^2 S^2 \left( \frac{S}{S} \right)^{2\alpha} \frac{\partial^2 P^{(n)}(S)}{\partial S^2} + r S \frac{\partial P^{(n)}(S)}{\partial S} - r P^{(n)}(S) = \lambda \left( P^{(n)}(S) - P^{(n-1)}(S) \right), \quad S > S_n
$$

with boundary conditions

$$
\lim_{S \to \infty} P^{(n)}(S) = 0, \quad \lim_{S \downarrow S_n} P^{(n)}(S) = K - S_n,
$$

and $S_n$ the critical stock price. As already mentioned above, the value of the randomized American put maturing after $n$ jumps of the Poisson process depends on the value of the American put maturing after $n-1$ jumps of the Poisson process with $\lambda = n/T$. Therefore the pricing problem has to be solved iteratively.

Similar to the first-order randomization case, we will apply an asymptotic expansion to derive a solution in terms of a perturbative series in $\epsilon$. To this end, we adapt the definition of variables and the value function for the inner region

$$
s = \frac{S}{K}, \quad x = \frac{s-1}{\sigma \sqrt{T}}, \quad \epsilon = \sigma \sqrt{T}, \quad \hat{P}^{(n)}(x) = \frac{p^{(n)}(s)}{\epsilon} = \frac{P^{(n)}(S)}{\epsilon K}.
$$

Besides, to simplify the resulting differential equation we use the same definition of parameters

$$
\beta = 2\alpha + 2, \quad \hat{K} = \left( \frac{K}{S} \right)^{2\alpha}, \quad \gamma^2 = \frac{2\lambda T}{K}, \quad k = \frac{r \epsilon}{\sigma^2 K}.
$$

Plugging these definitions into the pricing problem in Eq. (3.2.2), we obtain the following differential equation in $\hat{p}^{(n)}(x)$:

$$
\frac{1}{2} (1 + \epsilon x) \beta \frac{\partial^2 \hat{p}^{(n)}(x)}{\partial x^2} + k (1 + \epsilon x) \frac{\partial \hat{p}^{(n)}(x)}{\partial x} - \epsilon k \hat{p}^{(n)}(x) = \frac{1}{2} \gamma^2 \left( \hat{p}^{(n)}(x) - \hat{p}^{(n-1)}(x) \right), \quad x > \bar{x}_n
$$

subject to the boundary conditions

$$
\lim_{x \to \infty} \hat{p}^{(n)}(x) = 0, \quad \lim_{x \downarrow \bar{x}_n} \hat{p}^{(n)}(x) = -\bar{x}_n, \quad \lim_{x \downarrow \bar{x}_n} \frac{\partial \hat{p}^{(n)}(x)}{\partial x} = -1.
$$

Since the problem has to be solved iteratively, we first have to consider the case of an American put maturing after the second jump of the Poisson process. The value of this put is given by $\hat{p}^{(2)}(x)$ and its optimal exercise boundary over the first period is $\bar{x}_2$. The differential equation for the option’s value then depends on $\hat{p}^{(1)}(x)$, the value of the
corresponding American put maturing at the first jump of a Poisson process with \( \lambda = 2/T \). We have determined this value in section 3.1 to second order in \( \epsilon \). We note that the optimal exercise boundary \( \bar{x}_1 \) for a randomized put maturing at the next jump is always larger than the optimal exercise boundary \( \bar{x}_2 \) of a randomized put maturing at the second jump of the same Poisson process (given that the first jump has not yet occurred, otherwise the optimal exercise boundary is of course the same). The reason for this is simply that the average time to maturity is larger in the latter case.

From the observation that \( \bar{x}_2 < \bar{x}_1 \) we know that the inhomogeneous part of the differential equation (3.2.6) is different for the three regimes of the first-order solution \( \hat{p}^{(1)}(x) \)

\[
\hat{p}^{(1)}(x) = \begin{cases} 
C^{(1)} e^{(-k-\eta)x} [1 + \epsilon f_1(x) + \ldots], & \text{for } x \geq 0 \\
A^{(1)} e^{(-k-\eta)x} [1 + \epsilon f_1(x) + \ldots] + B^{(1)} e^{(-k+\eta)x} [1 + \epsilon g_1(x) + \ldots] - x - \frac{2k}{\gamma^2} + \ldots, & \text{for } \bar{x}_1 < x < 0 \\
-x, & \text{for } x \leq \bar{x}_1,
\end{cases}
\]

where \( \eta \) is again defined as

\[
\eta = \sqrt{\gamma^2 + k^2}.
\]

In the interval \( \bar{x}_2 < x \leq \bar{x}_1 \), however, the differential equation for \( \hat{p}^{(2)}(x) \) is the same as for the first-order randomized American put. Hence, we can immediately state that the solution is the same as for \( \hat{p}^{(1)}(x) \) in this regime, however with numerically different constants \( A^{(2)} \) and \( B^{(2)} \) due to the different boundary conditions at \( \bar{x}_1 \) and \( \bar{x}_2 \).

For the other intervals \( x \geq 0 \) and \( \bar{x}_1 < x < 0 \) the solution has to be determined by solving the differential equation by means of a perturbative expansion in \( \epsilon \). To this end, we expand the second-order randomized put value in powers of the volatility parameter \( \epsilon = \sigma \sqrt{T} \):

\[
\hat{p}^{(2)}(x) = \hat{p}_0^{(2)}(x) + \epsilon \hat{p}_1^{(2)}(x) + \epsilon^2 \hat{p}_2^{(2)}(x) + \ldots
\]

and collect terms of equal powers in \( \epsilon \) in Eq. (3.2.6). We note that, similar to the analysis for the first-order randomized American put, we will not expand the exercise boundary \( \bar{x}_2 \) and the constants multiplying the homogeneous solution in powers of \( \epsilon \). Instead, we will later numerically solve for their values.

To zeroth order in \( \epsilon \) the resulting differential equation for \( \hat{p}_0^{(2)}(x) \) is given by

\[
\frac{1}{2} \frac{\partial^2 \hat{p}_0^{(2)}(x)}{\partial x^2} + k \frac{\partial \hat{p}_0^{(2)}(x)}{\partial x} - \frac{1}{2} \gamma^2 \hat{p}_0^{(2)}(x) = -\frac{1}{2} \gamma^2 \hat{p}_0^{(1)}(x).
\]

In analogy to the derivations in section 3.1, we first try to determine the general solution to the homogeneous part of the differential equation (l.h.s. of the equation) and then find a special solution which takes the inhomogeneous part of the equation into account.

In the interval \( \bar{x}_1 < x < 0 \), the general solution to the homogeneous part of the equation is the same as in the first-order randomization case, namely

\[
A_1^{(2)} e^{(-k-\eta)x} + B_1^{(2)} e^{(-k+\eta)x}
\]

with \( A_1^{(2)} \) and \( B_1^{(2)} \) real-valued constants. To find a special solution to the inhomogeneous equation we apply the following Ansatz:

\[
\hat{p}_0^{(2)}(x) = A_1^{(2)} e^{(-k-\eta)x} + B_1^{(2)} e^{(-k+\eta)x} + f_0^{(2)}(x) A^{(1)} e^{(-k-\eta)x} + g_0^{(2)}(x) B^{(1)} e^{(-k+\eta)x} + [-x]^+ - \frac{4k}{\gamma^2} \mathcal{H}(-x),
\]
where we added a term \([-x]^+ - \mathcal{H}(-x)4k/\gamma^2\) to account for the inhomogeneous term of the leading-order solution \(\hat{p}_0^{(1)}(x)\) from the previous section:

\[
\hat{p}_0^{(1)}(x) = A^{(1)}e^{(-k-\eta)x} + B^{(1)}e^{(-k+\eta)x} + [-x]^+ - \mathcal{H}(-x)\frac{2k}{\gamma^2}.
\]  (3.2.14)

Here \(A^{(1)}\) and \(B^{(1)}\) are real-valued constants which are determined by the boundary conditions of the first-order randomization problem, i.e. the values for \(A^{(1)}\) and \(B^{(1)}\) are obtained by solving the valuation problem for the first-order randomized American put as described in the previous section letting \(\lambda = 2/T\). Plugging the Ansatz into Eq. (3.2.11) we obtain two separate differential equations for the functions \(f_0^{(2)}(x)\) and \(g_0^{(2)}(x)\). Using a polynomial Ansatz for both functions the solutions are easily found to be

\[
f_0^{(2)}(x) = \frac{\gamma^2x}{2\eta},
\]

\[
g_0^{(2)}(x) = -\frac{\gamma^2x}{2\eta}.
\]  (3.2.15)

For the solution for \(x > 0\) the constants multiplying the terms proportional to \(\exp([-k+\eta]|x|)\) have to be zero due to the boundary condition \(\lim_{x\to\infty} \hat{p}^{(2)}(x) = 0\). Therefore the solution to zeroth order in \(\epsilon\) for the second-order randomized American put is given by

\[
\hat{p}_0^{(2)}(x) = \begin{cases} 
C^{(2)}e^{(-k-\eta)x} + f_0^{(2)}(x)C^{(1)}e^{(-k-\eta)x}, & \text{for } x \geq 0 \\
A^{(2)}e^{(-k-\eta)x} + B^{(2)}e^{(-k+\eta)x} + f_0^{(2)}(x)A^{(1)}e^{(-k-\eta)x} + g_0^{(2)}(x)B^{(1)}e^{(-k+\eta)x} - x - \frac{4k}{\gamma^2}, & \text{for } x_1 < x < 0 \\
A^{(2)}e^{(-k-\eta)x} + B^{(2)}e^{(-k+\eta)x} - x - \frac{2k}{\gamma^2}, & \text{for } x_2 < x \leq x_1 \\
-x, & \text{for } x \leq x_2,
\end{cases}
\]  (3.2.16)

where the real-valued constants \(A^{(1)}_1, A^{(2)}_1, B^{(2)}_1, B^{(2)}_2, C^{(2)}\) and the optimal exercise boundary \(x_2\) are determined by the boundary conditions at \(x_2\) and by assuming that the solution is continuous and differentiable at \(x = 0\) and \(x = x_1\). The values for \(A^{(1)}\), \(B^{(1)}\), \(C^{(1)}\) and \(x_1\) are given by the solution to the randomized American put maturing at the first jump of the Poisson process.

The differential equation for the first-order correction in \(\epsilon\), \(\hat{p}_1^{(2)}(x)\), is given by collecting all terms proportional to \(\epsilon\) in Eq. (3.2.6). The resulting differential equation is

\[
\frac{1}{2} \frac{\partial^2 \hat{p}_1^{(2)}(x)}{\partial x^2} + k \frac{\partial \hat{p}_1^{(2)}(x)}{\partial x} - \frac{1}{2} \frac{\partial \hat{p}_0^{(2)}(x)}{\partial x} = -\frac{1}{2} \beta x \frac{\partial \hat{p}_0^{(2)}(x)}{\partial x} - kx \frac{\partial \hat{p}_0^{(2)}(x)}{\partial x} + k \hat{p}_0^{(2)}(x) - \frac{\gamma^2}{2} \hat{p}_1^{(1)}(x),
\]  (3.2.17)

which depends both on the leading-order solution \(\hat{p}_0^{(2)}(x)\) and the first-order correction to the first-order randomized American put value \(\hat{p}_1^{(1)}(x)\). The homogeneous part of this equation is again the same as in the case of first-order randomization. Hence, we can immediately state that the general part of the solution has the same structure as for the first-order randomized put. For the special solution we try the following Ansatz:

\[
\hat{p}_1^{(2)}(x) = f_1(x)A^{(2)}_1 e^{(-k-\eta)x} + g_1(x)B^{(2)}_1 e^{(-k+\eta)x} + f_1^{(2)}(x)A^{(1)}_1 e^{(-k-\eta)x} + g_1^{(2)}(x)B^{(1)}_1 e^{(-k+\eta)x} + \frac{12k^2}{\gamma^4} \mathcal{H}(-x),
\]  (3.2.18)

where \(f_1(x)\) and \(g_1(x)\) are given in Eqs. (3.1.32) and (3.1.33). Besides, we added a term proportional to the Heaviside step function to cancel the additional constant terms for
\[ x < 0 \text{ in Eq. (3.2.17)}. \]

Inserting this Ansatz into the differential equation for the first-order correction in Eq. (3.2.17) yields two separate differential equations for the yet unknown functions \( f_1^{(2)}(x) \) and \( g_1^{(2)}(x) \). The resulting differential equations can be solved by means of a polynomial Ansatz of third order in \( x \). The solution is then given by

\[
f_1^{(2)}(x; \eta) = - \left( \frac{x}{\eta} + x^2 \right) k(\eta^2 - k^2) \frac{(2 - \beta)k + (3 - \beta)\eta}{4\eta^3} \]
\[ - x^3(\eta^2 - k^2) \frac{(2 - \beta)k - \beta\eta}{8\eta^2}, \quad (3.2.19) \]

\[
g_1^{(2)}(x; \eta) = f_1^{(2)}(x; -\eta). \]

After having determined these functions we can write down the full solution to first order in \( \epsilon \) for the American put maturing at the second jump of the Poisson process:

\[
\tilde{p}_0^{(2)}(x) + \epsilon \tilde{v}_1^{(2)}(x) = \begin{cases} 
C^{(2)} e^{(-k-\eta)x} [1 + \epsilon f_1(x)] + C^{(1)} e^{(-k-\eta)x} \left[ f_0^{(2)}(x) + \epsilon f_1^{(2)}(x) \right], & \text{for } x \geq 0 \\
A_1^{(2)} e^{(-k-\eta)x} [1 + \epsilon f_1(x)] + B_1^{(2)} e^{(-k+\eta)x} [1 + \epsilon g_1(x)] + A_1^{(1)} e^{(-k-\eta)x} \left[ f_0^{(2)}(x) + \epsilon f_1^{(2)}(x) \right] \\
+ B_1^{(1)} e^{(-k+\eta)x} \left[ g_0^{(2)}(x) + \epsilon g_1^{(2)}(x) \right] - x - \frac{4k}{\gamma} + \epsilon \frac{12k^2}{\gamma}, & \text{for } x_1 < x < 0 \\
A_2^{(2)} e^{(-k-\eta)x} [1 + \epsilon f_1(x)] + B_2^{(2)} e^{(-k+\eta)x} [1 + \epsilon g_1(x)] - x - \frac{2k}{\gamma} + \epsilon \frac{4k^2}{\gamma}, & \text{for } x_2 < x \leq x_1 \\
-x, & \text{for } x \leq x_2. \quad (3.2.20) \end{cases}
\]

The structure of higher-order corrections in \( \epsilon \) is very similar to that of the leading- and first-order solutions. The corrections to the general solution proportional to the constants \( A_1^{(2)}, B_1^{(2)} \) and \( C^{(2)} \) are to any order in \( \epsilon \) the same as in the first-order randomization case. The reason for this is that the homogeneous part of the differential equation is the same as in the first-order randomization case. However, due to the different inhomogeneous terms additional corrections proportional to \( A^{(1)}, B^{(1)} \) and \( C^{(1)} \) enter the solution for the second-order randomized American put. In general, those corrections can be determined by means of the approach outlined above, i.e. by applying an Ansatz similar to that in Eq. (3.2.18) to derive differential equations for the polynomial functions \( f_n^{(2)} \) and \( g_n^{(2)} \). For this dissertation, we have derived the second-order corrections in \( \epsilon \) to the second-order randomized American put value. The lengthy expressions for the higher-order corrections can be found in the Appendix.

After having derived the expression for the second-order randomized American put value we have to make use of the boundary conditions to determine the five constants \( A_1^{(2)}, A_2^{(2)}, B_1^{(2)}, B_2^{(2)}, C^{(2)} \) and the critical stock price \( x_2 \). Beside the free boundary conditions at \( x = x_2 \) we get four additional boundary conditions by requiring that the put value is continuous and differentiable at \( x = 0 \) and \( x = x_1 \). All in all, we are faced with a
system of six equations to determine the six unknown real-valued constants

\[
\begin{align*}
\lim_{x \to 0} \hat{p}^{(2)}(x) &= \lim_{x \to 0} \hat{p}^{(2)}(x), \\
\lim_{x \to 0} \frac{\partial \hat{p}^{(2)}(x)}{\partial x} &= \lim_{x \to 0} \frac{\partial \hat{p}^{(2)}(x)}{\partial x}, \\
\lim_{x \downarrow x_1} \hat{p}^{(2)}(x) &= \lim_{x \uparrow x_1} \hat{p}^{(2)}(x), \\
\lim_{x \downarrow x_1} \frac{\partial \hat{p}^{(2)}(x)}{\partial x} &= \lim_{x \uparrow x_1} \frac{\partial \hat{p}^{(2)}(x)}{\partial x}, \\
\lim_{x \downarrow x_2} \hat{p}^{(2)}(x) &= -x_2, \\
\lim_{x \downarrow x_2} \frac{\partial \hat{p}^{(2)}(x)}{\partial x} &= -1.
\end{align*}
\]  

(3.2.21)

In our implementation we use a numerical nonlinear system solver to determine the numerical values of the six constants for a given set of option parameters.

Before comparing the results for the second-order randomized American put with the benchmark results from the finite-differences scheme, let us briefly outline the derivation and structure of the results for the third-order randomized American put, i.e. an American put which matures at the third jump of a Poisson process with parameter \( \lambda = 3/T \).

In this case the put value \( \hat{p}^{(3)}(x) \) is determined by the free boundary pricing problem

\[
\frac{1}{2} (1 + x \epsilon)^2 \frac{\partial^2 \hat{p}^{(3)}(x)}{\partial x^2} + k(1 + x \epsilon) \frac{\partial \hat{p}^{(3)}(x)}{\partial x} - k \epsilon \hat{p}^{(3)}(x) = \frac{1}{2} \gamma^2 \left( \hat{p}^{(3)}(x) - \hat{p}^{(2)}(x) \right), \quad x > x_3
\]

subject to the boundary conditions

\[
\begin{align*}
\lim_{x \to \infty} \hat{p}^{(3)}(x) &= 0, \\
\lim_{x \downarrow x_3} \hat{p}^{(3)}(x) &= -x_3, \\
\lim_{x \downarrow x_3} \frac{\partial \hat{p}^{(3)}(x)}{\partial x} &= -1,
\end{align*}
\]  

(3.2.23)

where \( x_3 \) is the optimal exercise boundary and \( \hat{p}^{(2)}(x) \) is the value of an American put maturing at the second jump of a Poisson process with \( \lambda = 3/T \). Similar to the previous cases, the problem can be solved by expanding the option’s value in powers of \( \epsilon \) and making use of the analytic results for \( \hat{p}^{(2)}(x) \).

The general structure of the solution is similar to that of the second-order randomized
Figure 3.5: Comparison of the American put results in the CEV model for \( \alpha = 0 \) between a finite-differences scheme using the PSOR algorithm (numerical) and maturity randomization using an asymptotic expansion in \( \epsilon \) (analytical). The parameters are \( r = 0.02, \sigma = 0.3, S = 100, T = 1, K = 100 \). The plot shows the relative error of the first-, second-, and third-order randomized value as well as the Richardson extrapolated randomized value with respect to the finite-differences value.

American put, but with one additional interval in \( x \)

\[
\tilde{p}^{(3)}(x) = \begin{cases} 
C^{(3)} e^{-(k-\eta)x} \left[ 1 + \epsilon f_1(x) + \ldots \right] \\
+ C^{(2)} e^{-(k-\eta)x} \left[ f_0^{(2)}(x) + \epsilon f_1^{(2)}(x) + \ldots \right] \\
+ C^{(1)} e^{-(k-\eta)x} \left[ f_0^{(3)}(x) + \epsilon f_1^{(3)}(x) + \ldots \right], \\
& \text{for } x \geq x_3 \\
A_1^{(3)} e^{-(k-\eta)x} \left[ 1 + \epsilon f_1(x) + \ldots \right] \\
+ B_1^{(3)} e^{-(k+\eta)x} \left[ 1 + \epsilon g_1(x) + \ldots \right] \\
+ A_1^{(2)} e^{-(k-\eta)x} \left[ f_0^{(2)}(x) + \epsilon f_1^{(2)}(x) + \ldots \right] \\
+ B_1^{(2)} e^{-(k+\eta)x} \left[ g_0^{(2)}(x) + \epsilon g_1^{(2)}(x) + \ldots \right] \\
+ A_1^{(1)} e^{-(k-\eta)x} \left[ f_0^{(3)}(x) + \epsilon f_1^{(3)}(x) + \ldots \right] \\
+ B_1^{(1)} e^{-(k+\eta)x} \left[ g_0^{(3)}(x) + \epsilon g_1^{(3)}(x) + \ldots \right] - x + \ldots, \\
& \text{for } x_1 < x < 0 \\
A_2^{(3)} e^{-(k-\eta)x} \left[ 1 + \epsilon f_1(x) + \ldots \right] \\
+ B_2^{(3)} e^{-(k+\eta)x} \left[ 1 + \epsilon g_1(x) + \ldots \right] \\
+ A_2^{(2)} e^{-(k-\eta)x} \left[ f_0^{(2)}(x) + \epsilon f_1^{(2)}(x) + \ldots \right] \\
+ B_2^{(2)} e^{-(k+\eta)x} \left[ g_0^{(2)}(x) + \epsilon g_1^{(2)}(x) + \ldots \right] - x + \ldots, \\
& \text{for } x_2 < x \leq x_3 \\
A_3^{(3)} e^{-(k-\eta)x} \left[ 1 + \epsilon f_1(x) + \ldots \right] \\
+ B_3^{(3)} e^{-(k+\eta)x} \left[ 1 + \epsilon g_1(x) + \ldots \right] - x + \ldots, \\
& \text{for } x_3 < x \leq x_2 \\
-x, \\
& \text{for } x \leq x_3,
\end{cases}
\]

where the optimal exercise boundary \( x_3 \) and the seven real-valued constants \( A_{1/2/3}^{(3)}, B_{1/2/3}^{(3)} \) and \( C^{(3)} \) are determined by the free boundary condition at \( x_3 \) and by requiring that the
Figure 3.6: Comparison of the American put results in the CEV model for $\alpha = -0.5$ between a finite-differences scheme using the PSOR algorithm (numerical) and maturity randomization using an asymptotic expansion in $\epsilon$ (analytical). The parameters are $r = 0.02$, $\sigma = 0.3$, $\hat{S} = 100$, $T = 1$, $K = 100$. The plot shows the relative error of the first-, second-, and third-order randomized value as well as the Richardson extrapolated randomized value with respect to the finite-differences value.

value function is continuous and differentiable at $x = 0$, $x = \underline{x}_1$ and $x = \underline{x}_2$. For the third-order randomized American put we have evaluated the corrections to the value function up to second order in $\epsilon$. The results are shown in the Appendix.

### 3.2.1 Numerical Analysis

Let us now examine the accuracy of the results obtained from higher-order randomization compared to the first-order randomization results from the previous section. As a benchmark we again use the numerically calculated value of a fixed-maturity American put using a finite-differences scheme and the PSOR algorithm. The considered option and market parameters are also the same as in the previous section,

$$r = 0.02, \quad \sigma = 0.3, \quad \hat{S} = 100, \quad T = 1, \quad K = 100. \quad (3.2.25)$$

Besides, we examine the results for three different values of the CEV model parameter $\alpha = 0$, -0.5 and -1.

In Figures 3.5 to 3.7 the relative errors of the first-, second- and third-order randomized American put values in the CEV model with respect to the fixed-maturity American put value obtained from the finite-differences scheme are shown. The relative errors are shown for the three different values of $\alpha$. The asymptotic expansion for the randomized put values includes terms up to second order in $\epsilon$. For the finite-differences scheme we use the same setup as in the previous section, i.e. $S_{\text{max}} = 400$, $N = 1600$ and $M = 400$. As can be seen, the error in the put value is significantly reduced by going from first-order to third-order randomization. In general, the randomized put value lies still below the fixed-maturity put value obtained from the finite-differences scheme. As discussed in the previous section this undervaluation is due to the uncertainty in the maturity of the option. The higher the order of the randomization (i.e. the higher the number of jumps
Figure 3.7: Comparison of the American put results in the CEV model for $\alpha = -1$ between a finite-differences scheme using the PSOR algorithm (numerical) and maturity randomization using an asymptotic expansion in $\epsilon$ (analytical). The parameters are $r = 0.02$, $\sigma = 0.3$, $\hat{S} = 100$, $T = 1$, $K = 100$. The plot shows the relative error of the first-, second-, and third-order randomized value as well as the Richardson extrapolated randomized value with respect to the finite-differences value.

of the Poisson process) the lower is the variance of the maturity distribution and hence the better is the agreement with the fixed-maturity results. To illustrate the convergence of the series, we show the maximum of the relative error between the randomization and finite-differences results for the $S$-range around the option’s strike (between $S = 60$ and $S = 130$) in Fig. 3.8. The maximum of the relative error is plotted against $1/n$, i.e. the inverse of the order of randomization. As can be seen, the maximum relative error is roughly proportional to the inverse of the order of randomization. Hence, the relative error is also roughly proportional to the variance of the maturity distribution (which is $T^2/n$). In the limit $n \to \infty$ the maturity distribution becomes a delta function and we therefore expect the value of the randomized American put to match the value of the equivalent fixed-maturity American put.

To speed up the convergence for higher-order randomization results, we use Richardson extrapolation to calculate extrapolated option values from the results for the first three orders of randomization. As shown in [5] an extrapolated American put pricing function can be calculated from fixed-order randomized pricing functions by means of the following series

$$P^{(1:N)}(x) = \sum_{n=0}^{N} \frac{(-1)^{N-n}n^N}{n!(N-n)!} P^{(n)}(x),$$  \hspace{1cm} (3.2.26)

where $P^{(n)}(x)$ is the $n$th-order randomized American put pricing function. Since we have calculated randomized American put values up to third order in randomization, we can calculate the extrapolated pricing function $P^{(1:N)}(x)$ with $N = 3$ by means of the above equation. In Figures 3.5 to 3.7 beside the relative errors for the fixed-order randomized American put values also the relative errors for the extrapolated American put values are shown. The relative error for the extrapolated American put values are significantly smaller than for the fixed-order results. In fact, in the relevant stock price range around the option’s strike the maximum of the relative error is only around 1% for all considered
Figure 3.8: Maximum of the relative errors from Fig. 3.7 in the stock price range between $S = 50$ and $S = 140$ for the first-order, second-order and third-order randomized American put values for $\alpha = -1$. The errors are plotted against the inverse of the randomization order $1/n$.

Table 3.2: Critical stock price for the American put in the CEV model for three different values of $\alpha$. The table shows the results from the finite-differences scheme and from first-, second- and third-order randomization with the asymptotic expansion truncated at the second order in $\epsilon$. Results obtained from Richardson extrapolation of the randomization results from the first three orders are also shown.

<table>
<thead>
<tr>
<th>Critical stock price at $t = 0$</th>
<th>$\alpha = 0$</th>
<th>$\alpha = -0.5$</th>
<th>$\alpha = -1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Finite Differences (PSOR)</td>
<td>61.1</td>
<td>56.1</td>
<td>48.4</td>
</tr>
<tr>
<td>1st-order Randomization, $O(\epsilon^2)$</td>
<td>60.16</td>
<td>55.13</td>
<td>47.28</td>
</tr>
<tr>
<td>2nd-order Randomization, $O(\epsilon^2)$</td>
<td>60.53</td>
<td>55.55</td>
<td>47.93</td>
</tr>
<tr>
<td>3rd-order Randomization, $O(\epsilon^2)$</td>
<td>60.67</td>
<td>55.70</td>
<td>48.16</td>
</tr>
<tr>
<td>Richardson Extrapolation, $O(\epsilon^2)$</td>
<td>60.95</td>
<td>56.02</td>
<td>48.67</td>
</tr>
</tbody>
</table>

The values for the critical stock prices at $t = 0$ of the random-maturity and fixed-maturity American puts in the CEV model are shown in Table 3.2. The values for the first- to third-order randomized puts are obtained from our asymptotic expansion including terms up to the second order in $\epsilon$. Again, the results are shown for three different values of the parameter $\alpha$. The optimal exercise boundary for the first-order randomized American put already gives a good approximation for the critical stock price of the fixed-maturity put. As expected, for any value of $\alpha$ the accuracy of the approximation is further increased by considering higher-order randomized American put results. Still, we observe that the critical stock prices of the random-maturity American puts in the CEV model are generally smaller than the critical stock price of the fixed-maturity equivalent. Again, we have also calculated Richardson extrapolated critical stock prices taking into account the results from randomization up to third order. The extrapolated critical stock prices almost exactly agree with the results from the finite-differences scheme. The relative deviations are significantly below 1%. These results for the critical stock prices also confirm the accurate convergence of higher-order randomization results.
Chapter 4

American Options in a Fast Mean-Reverting Stochastic Volatility Model

In the second part of the thesis we study the maturity randomization approach for a fast mean-reverting stochastic volatility model. Thereby, we follow the approach discussed in [1], where maturity randomization is applied to the valuation of American puts on a non-dividend paying stock in a stochastic volatility model. As an extension to the analysis in [1], we assume that the underlying stock process includes a constant dividend yield. In this case, early execution is also optimal for American calls. Hence, we will also derive the value of maturity-randomized American calls in the fast mean-reverting stochastic volatility model.

Besides, we extend the analysis in [1] by considering higher-order maturity randomization. In [1] the American put value is only derived for an exponential maturity distribution, i.e. the option matures at the first jump of a Poisson process. In this dissertation we also derive the American put and call value for randomized options maturing at the second and third jump of a Poisson process. The resulting maturity distributions in those higher-order cases are much narrower than the exponential distribution, and hence better approximate the fixed maturity of a regular American option contract.

For the considered fast mean-reverting stochastic volatility model we apply a similar notation as in [1]; but as already mentioned we assume that the underlying asset price process $S_t$ includes a constant dividend yield $\delta$. Under the risk-neutral probability measure the stochastic differential equations for the stock price $S_t$ and volatility $Y_t$ are then given by the following equations:

\begin{align*}
\mathrm{d}S_t &= (r - \delta)S_t \mathrm{d}t + f(Y_t)S_t \mathrm{d}W^{(1)}_t, \quad S_0 > 0 \\
\mathrm{d}Y_t &= \frac{1}{\epsilon}(m - Y_t) + \nu \frac{2}{\epsilon} \mathrm{d}W^{(2)}_t, \quad Y_0 > 0, \quad (4.0.1)
\end{align*}

where $r$ denotes the risk-free interest rate, $m$ and $\nu$ are real-valued constants determining the mean and volatility of the stochastic volatility process, and $\epsilon$ is the intrinsic time-scale of the volatility process. For a fast mean-reverting volatility process the intrinsic time-scale $\epsilon$ is small. We note that in this context the parameter $\epsilon$ is defined differently than in the previous chapter on the CEV model. The correlation of the two one-dimensional Brownian motions $W^{(1)}_t$ and $W^{(2)}_t$ is denoted by $\rho$, i.e. $\mathrm{d}\{W^{(1)}, W^{(2)}\}_t = \rho \mathrm{d}t$.

We note that with the special choice of parameters the stochastic volatility process $Y$ is an Ornstein-Uhlenbeck process with a normal invariant distribution $\Phi$ with mean $m$ and variance $\nu^2$. The distribution does not depend on the intrinsic time-scale $\epsilon$. The function
which determines the value of the stock price volatility based on the dynamics of the stochastic volatility process \( Y \) is required to be continuous and differentiable. Besides, the function needs to be positive real-valued, \( f : \mathbb{R} \to \mathbb{R}_+ \setminus \{0\} \), and well-behaved with respect to the distribution \( \Phi \) of the stochastic volatility, \( \int f^2(y) \Phi(y) dy < \infty \).

### 4.1 Maturity-Randomized American Put

As discussed in section 2.2, to obtain a first-order maturity randomized option contract the fixed maturity of the American put is replaced by an exponentially distributed random maturity. Following the discussion in section 2.2 it is possible to derive a time-independent pricing equation for the randomized American put in the stochastic volatility model. For the first-order randomized American put we hereby obtain the following PDE:

\[
L^\epsilon P^{(1)}(x, y) + \lambda (K - x)^+ = 0, \quad x > x_1(y),
\]

subject to the following boundary conditions at the optimal exercise boundary \((x_1, y)\)

\[
P^{(1)}(x_1, y) = K - x_1, \\
\frac{\partial P^{(1)}}{\partial x}(x_1, y) = -1, \\
\frac{\partial P^{(1)}}{\partial y}(x_1, y) = 0,
\]

where \( x = S \) is the underlying stock price. Again, we have adapted the notation from [1] for the generator \( L^\epsilon \) of the differential equation. It is given by

\[
L^\epsilon = \frac{1}{\epsilon} L_0 + \frac{1}{\sqrt{\epsilon}} L_1 + L_2
\]

with the operators \( L_{0,1,2} \) defined as

\[
L_0 = \nu^2 \frac{\partial^2}{\partial y^2} + (m - y) \frac{\partial}{\partial y}, \\
L_1 = \sqrt{2} \mu y f(y) \frac{\partial^2}{\partial y \partial x}, \\
L_2 = \frac{1}{2} f(y) x^2 \frac{\partial^2}{\partial x^2} + (r - \delta) x \frac{\partial}{\partial x} - (r + \lambda).
\]

where \( \lambda \) is the decay parameter of the exponential distribution. Since there is no analytic solution to the free boundary problem for an arbitrary volatility function \( f(y) \), the objective is to perform an asymptotic expansion in the limit \( \epsilon \to 0 \) and solve the resulting problem in terms of a perturbative series.

Following the discussion in [1] we expand both the value of the American put, as well as, the optimal exercise boundary. Due to the special choice of the parameters in the stochastic volatility process, the expansion is performed in powers of \( \sqrt{\epsilon} \) instead of \( \epsilon \):

\[
P^{(1)}(x, y) = P_0(x, y) + \sqrt{\epsilon} P_1(x, y) + \epsilon P_2(x, y) + \ldots, \\
x_1(y) = x_0(y) + \sqrt{\epsilon} x_1(y) + \epsilon x_2(y) + \ldots,
\]

where we drop the superscript (1) in the functions of the expansions for brevity. Since we have expanded the optimal exercise boundary in powers of \( \sqrt{\epsilon} \), it is also required to
expand the boundary conditions at \( x_1 \) accordingly. The expansion in \( \sqrt{\epsilon} \) yields

\[
P_0(x_0, y) + \sqrt{\epsilon}\left( x_1(y) \frac{\partial P_0}{\partial x}(x_0, y) + P_1(x_0, y) \right) + \ldots = K - x_0 - \sqrt{\epsilon}x_1 + \ldots,
\]
\[
\frac{\partial P_0}{\partial x}(x_0, y) + \sqrt{\epsilon}\left( x_1(y) \frac{\partial^2 P_0}{\partial x^2}(x_0, y) + \frac{\partial P_1}{\partial x}(x_0, y) \right) + \ldots = -1,
\]
\[
\frac{\partial P_0}{\partial y}(x_0, y) + \sqrt{\epsilon}\left( x_1(y) \frac{\partial^2 P_0}{\partial x \partial y}(x_0, y) + \frac{\partial P_1}{\partial y}(x_0, y) \right) + \ldots = 0.
\]

(4.1.6)

After plugging the expansion in \( \sqrt{\epsilon} \) into the original pricing problem in Eq. (4.1.1) and multiplying the equation on both sides by \( \epsilon \), we can collect terms of equal powers in \( \sqrt{\epsilon} \) and iteratively solve the resulting PDEs. Starting at the zeroth order we face the following pricing problem for \( P_0(x, y) \):

\[
L_0 P_0(x, y) = 0, \quad x > x_0(y)
\]

(4.1.7)

with boundary conditions

\[
P_0(x_0, y) = K - x_0, \quad \frac{\partial P_0}{\partial x}(x_0, y) = -1.
\]

(4.1.8)

As discussed in [1], the differential equation and boundary condition at \( x_0 \) imply that \( P_0 \) is independent of \( y \) for all values of \( x \). This in turn means that also \( x_0 \) is independent of \( y \). Since the zeroth-order PDE does not include any derivatives in \( x \), we cannot yet make any statement regarding the \( x \)-dependence of the solution \( P_0 \). Instead, we have to consider higher orders in the expansion to examine the dynamics of the leading-order solution with respect to \( x \).

Making use of the zeroth-order result (namely that \( P_0 \) is independent from \( y \)) the PDE and boundary conditions to first order in the expansion in \( \sqrt{\epsilon} \) become

\[
L_0 P_1(x, y) = 0, \quad x > x_0,
\]

(4.1.9)

\[
P_1(x_0, y) = 0, \quad x_1(y) \frac{\partial^2 P_0}{\partial x^2}(x_0, y) + \frac{\partial P_1}{\partial x}(x_0, y) = 0.
\]

(4.1.10)

The PDE and the first boundary condition at \( x_0 \) are equivalent to the zeroth-order problem and hence we infer that also \( P_1 \) is independent of \( y \). Consequently, the second boundary condition implies that \( x_1 \) has no \( y \)-dependence as well. Again the PDE does not contain any information regarding the dynamics of \( P_1 \) with respect to the stock price \( x \).

To second order in the expansion in \( \sqrt{\epsilon} \) we obtain the following PDE for \( P_2 \) and \( P_0 \):

\[
L_0 P_2(x, y) + L_2 P_0(x) + \lambda(K - x)^+ = 0, \quad x > x_0
\]

(4.1.11)

with the boundary condition for \( P_2 \)

\[
P_2(x_0, y) = 0.
\]

(4.1.12)

Since the operator \( L_2 \) only contains partial derivatives with respect to \( x \) and none with respect to \( y \), Eq. (4.1.11) is a Poisson equation in \( y \). As discussed in [1], a well-behaved solution for the Poisson equation exists only if the source term \( L_2 P_0(x) + \lambda(K - x)^+ \) is centered with respect to the distribution of the stochastic volatility process \( Y \) (with infinitesimal generator \( L_0 \)). This means that the mean of the source term with respect to
the distribution of \( y \) needs to be equal to zero:

\[
\int_{-\infty}^{\infty} \left[ \mathcal{L}_2 P_0(x) + \lambda(K - x)^+ \right] \Phi(y) \, dy = 0, \tag{4.1.13}
\]

where \( \Phi \) is the probability density function of the stochastic process \( Y \). Since the whole dependence on \( y \) in the source term is contained in the volatility function \( f \), we define the mean of the squared volatility function as

\[
\bar{\sigma}^2 = \int_{-\infty}^{\infty} f^2(y) \Phi(y) \, dy, \tag{4.1.14}
\]

and rewrite the centering condition as an ordinary differential equation in \( x \):

\[
\frac{1}{2\bar{\sigma}^2 x^2} \partial^2 P_0 \partial x^2 + (r - \delta)x \partial P_0 \partial x - (r + \lambda)P_0 + \lambda(K - x)^+ = 0. \tag{4.1.15}
\]

Let us also recall the boundary conditions for \( P_0 \) at \( x_0 \):

\[
P_0(x_0) = K - x_0, \quad \frac{\partial P_0}{\partial x}(x_0) = -1. \tag{4.1.16}
\]

We note that the pricing problem for the zeroth-order solution \( P_0 \) given by the ordinary differential equation (4.1.15) and the boundary conditions at \( x_0 \) is equivalent to the pricing problem for a randomized American put (with exponential maturity distribution) in the Black-Scholes model. The value of a first-order randomized American put in the Black-Scholes model is determined by the same free-boundary problem with the mean of the squared stochastic volatility function, \( \bar{\sigma}^2 \), replaced by the square of the constant Black-Scholes volatility, \( \sigma^2 \). This problem has been solved in [5] for American puts on dividend-paying stocks. The solution is given by

\[
P_0(x) = \begin{cases} 
  a_1 \left( \frac{x}{K} \right)^{\beta_1} + a_2 \left( \frac{x}{x_0} \right)^{\beta_1}, & \text{for } x \geq K \\
  b_1 \left( \frac{x}{K} \right)^{\beta_2} + a_2 \left( \frac{x}{x_0} \right)^{\beta_1} + KR - xD, & \text{for } x_0 < x < K \\
  K - x, & \text{for } x \leq x_0,
\end{cases} \tag{4.1.17}
\]

where

\[
\beta_1 = \gamma - \Delta, \quad \beta_2 = \gamma + \Delta, \quad \gamma = \frac{1}{2} - \frac{r - \delta}{\sigma^2}, \\
\Delta = \sqrt{\gamma^2 + \frac{2\lambda}{R\sigma^2}}, \quad R = \frac{\lambda}{\lambda + r}, \quad D = \frac{\lambda}{\lambda + \delta}, \\
a_1 = qKR - \hat{q}KD, \quad a_2 = qK(1 - R) - \hat{q}x_0(1 - D), \quad b_1 = \hat{p}KD - pKR, \\
p = \frac{\Delta - \gamma}{2\Delta}, \quad q = 1 - p, \quad \hat{p} = \frac{\Delta - \gamma + 1}{2\Delta}, \quad \hat{q} = 1 - \hat{p}. \tag{4.1.18}
\]

By means of the boundary conditions at \( x_0 \), it can be shown that the exercise boundary \( x_0 \) to leading order is implicitly given by the following equation:

\[
\left( \frac{x_0}{K} \right)^{\beta_2} = pK(1 - R) - qx_0(1 - D) \frac{\hat{p}KD - pKR}{\hat{p}KD - pKR}. \tag{4.1.19}
\]

For arbitrary values of \( \beta_2 \) this equation can only analytically be solved for \( x_0 \), if the dividend yield \( \delta \) is equal to zero and hence \( D = 1 \). In this case the optimal exercise
boundary is
\[ x_0 = K \left( \frac{pRr}{\lambda(\hat{p} - Rp)} \right)^{1/\beta_2}, \]  
(4.1.20)
where we made use of the following relation
\[ 1 - R = 1 - \frac{\lambda}{\lambda + r} = R^r. \]  
(4.1.21)

By means of the centering condition in Eq. (4.1.13) it is also possible to derive an expression for the second-order correction \( P_2 \). The differential equation (4.1.11) can be rewritten as
\[ L_0 P_2 = -L_2 P_0 - \lambda(K - x) + \int_{-\infty}^{\infty} L_2 P_0(x) \Phi(y) dy. \]  
(4.1.22)
Since, as shown above, \( P_0 \) is independent from \( y \) and the only term depending on \( y \) in the infinitesimal generator \( L_2 \) is the term proportional to the second derivative with respect to \( x \), the equation reduces to
\[ L_0 P_2 = -\frac{1}{2} \left( f^2(y) - \overline{\sigma^2} \right) x^2 \frac{\partial^2 P_0}{\partial x^2}. \]  
(4.1.23)
Let us assume that we have found a solution \( \psi(y) \) to the differential equation
\[ L_0 \psi(y) = f^2(y) - \overline{\sigma^2}, \]  
(4.1.24)
then the general solution to the differential equation (4.1.23) is given by
\[ P_2(x, y) = -\frac{1}{2} \psi(y) x^2 \frac{\partial^2 P_0}{\partial x^2} + c(x), \]  
(4.1.25)
where \( c(x) \) is an arbitrary function independent from \( y \).

We will make use of this result (even if it does not completely determine the second-order correction due to the arbitrary \( x \)-dependent function \( c(x) \)) in the derivation of the first-order correction \( P_1 \). To this end, we have to consider the third-order contribution to the pricing problem in Eq. (4.1.1) after expansion and multiplication by \( \epsilon \), i.e. the terms proportional to \( \epsilon^{3/2} \):
\[ L_0 P_3 + L_1 P_2 + L_2 P_1 = 0, \quad x > x_0. \]  
(4.1.26)
Similar to the second-order PDE, this is a Poisson equation in \( y \) and hence a well-behaved solution exists only, if the source term is centered with respect to the distribution \( \Phi(y) \), see [1]. In this case, this leads to the following solvability condition:
\[ \int_{-\infty}^{\infty} L_2 \Phi(y) dy P_1(x) = -\int_{-\infty}^{\infty} L_1 P_2(x, y) \Phi(y) dy, \]  
(4.1.27)
where we made use of the fact that \( P_1 \) is independent of \( y \). With the result for \( P_2 \) obtained above, the partial differential equation can be simplified to an ordinary differential equation in \( x \):
\[ \int_{-\infty}^{\infty} L_2 \Phi(y) dy P_1(x) = V \left( 2x^2 \frac{\partial^2 P_0}{\partial x^2} + x^3 \frac{\partial^3 P_0}{\partial x^3} \right), \]  
(4.1.28)
where \( V \) is defined as
\[ V = \sqrt{\frac{1}{2} \rho \nu} \int_{-\infty}^{\infty} f(y) \frac{\partial \psi(y)}{\partial y} \Phi(y) dy. \]  
(4.1.29)
We note that the homogeneous part of the differential equation (4.1.28) on the r.h.s. is
equal to the homogeneous part of the pricing problem for the zeroth-order solution \( P_0 \) in Eq. (4.1.15). Hence, the general solution to the homogeneous part is the same as for the zeroth-order contribution and given by

\[
\tilde{A} x^{\beta_1} + \tilde{B} x^{\beta_2}
\]  

(4.1.30)

with real-valued constants \( \tilde{A} \) and \( \tilde{B} \). The full solution to the differential equation is given by the sum of the general solution and a special solution taking into account the inhomogeneous part of the differential equation, which still has to be determined.

Plugging the solution for \( P_0 \) into the r.h.s. of Eq. (4.1.28) we find that the r.h.s. of the differential equation is given by a term proportional to \( x^{\beta_1} \) plus a term proportional to \( x^{\beta_2} \). This means that the inhomogeneous part of the differential equation is of the same powers in \( x \) as the solution to the homogeneous part of the equation. Therefore, the equation is solved by a special solution involving polynomials in \( x \) of powers \( \beta_1 \) and \( \beta_2 \) for which the powers of the polynomials do not change, if the differential operator \( x^i \partial^i / \partial x^i \), for \( i = 0, 1, 2 \), is applied. As also shown in [1] this is the case for a function of the form \( \ln(x)x^{\beta_1} \), and hence we try the following Ansatz for the special solution:

\[
\ln(x)(A_1 x^{\beta_1} + B_1 x^{\beta_2})
\]  

(4.1.31)

with real-valued constants \( A_1 \) and \( B_1 \). After inserting this Ansatz into Eq. (4.1.28) we can solve for the unknown constants. Due to the term proportional to the step function \( (K-x)^+ \) we have to separately solve the equation for \( x > K \) and \( x < K \). The calculation yields

\[
A_1 = -\frac{V \beta_1^2 (\beta_1 - 1)}{\Delta \sigma^2} \left( \frac{a_1}{K^{\beta_1}} + \frac{a_2}{x_0^{\beta_1}} \right), \quad B_1 = 0, \quad \text{for} \ x > K,
\]

\[
A_1 = -\frac{V \beta_1^2 (\beta_1 - 1) a_2}{\Delta \sigma^2 x_0^{\beta_1}}, \quad B_1 = \frac{V \beta_2^2 (\beta_2 - 1) b_1}{\Delta \sigma^2 K^{\beta_2}}, \quad \text{for} \ x \leq K.
\]  

(4.1.32)

Adding the solution for the homogeneous part of the ODE to the special solution we obtain the following general solution for the first-order correction to the American Put value

\[
P_1 = \begin{cases} 
\tilde{A}_1 x^{\beta_1} - \frac{V B_1 \ln(x)}{\Delta \sigma^2} \left( a_1 \left( \frac{x}{K} \right)^{\beta_1} + a_2 \left( \frac{x}{x_0} \right)^{\beta_1} \right), & \text{for} \ x \geq K, \\
\tilde{A}_2 x^{\beta_1} + \tilde{B}_1 x^{\beta_2} + \frac{V \ln(x)}{\Delta \sigma^2} \left( B_2 b_1 \left( x/K \right)^{\beta_2} - B_1 a_2 \left( x/x_0 \right)^{\beta_1} \right), & \text{for} \ x_0 < x < K, \\
0, & \text{for} \ x \leq x_0,
\end{cases}
\]  

(4.1.33)

where for brevity we have introduced the following parameters:

\[
B_1 = \beta_1^2 (\beta_1 - 1), \quad B_2 = \beta_2^2 (\beta_2 - 1).
\]  

(4.1.34)

The values for the constants \( \tilde{A}_1, \tilde{A}_2, \tilde{B}, \) and the first-order correction to the critical stock price \( x_1 \) have yet to be evaluated. They are determined by the boundary condition for the first-order correction at \( x_0 \),

\[
P_1(x_0) = 0, \quad \frac{\partial P_1}{\partial x}(x_0) = -\frac{\partial^2 P_0}{\partial x^2}(x_0),
\]  

(4.1.35)
and by demanding that the pricing function is continuous and differentiable at \( x = K \):

\[
\lim_{x \downarrow K} P_1(x) = \lim_{x \uparrow K} P_1(x), \quad \lim_{x \downarrow K} \frac{\partial P_1(x)}{\partial x} = \lim_{x \uparrow K} \frac{\partial P_1(x)}{\partial x}. \tag{4.1.36}
\]

Solving this system of equations algebraically, it can be found that the three constants are given by the following lengthy expressions:

\[
\begin{align*}
\bar{A}_1 &= \frac{V}{2\Delta^2 \sigma^2 x_0^\beta_1} \left[ (a_1 B_1 + b_1 B_2) \left( x_0^\beta_2 - \frac{x_0}{K}^\beta_1 \right) 
+ 2\Delta \ln(K) \left( a_1 B_1 \left( \frac{x_0}{K} \right)^\beta_1 + b_1 B_2 \left( \frac{x_0}{K} \right)^\beta_2 \right) 
+ 2\Delta \ln(x_0) \left( a_2 B_1 - b_1 B_2 \left( \frac{x_0}{K} \right)^\beta_2 \right) \right], \\
\bar{A}_2 &= \frac{V}{2\Delta^2 \sigma^2 x_0^\beta_1} \left[ (a_1 B_1 + b_1 B_2) \left( 1 + 2\Delta \ln(K) \right) \left( \frac{x_0}{K} \right)^\beta_2 
+ 2\Delta \ln(x_0) \left( a_2 B_1 - b_1 B_2 \left( \frac{x_0}{K} \right)^\beta_2 \right) \right], \\
\bar{B} &= -\frac{V}{2\Delta^2 \sigma^2 K^\beta_2} \left[ (a_1 B_1 + b_1 B_2) \left( 1 + 2\Delta \ln(K) \right) \right]. \tag{4.1.37}
\end{align*}
\]

To determine the value of the first-order correction to the exercise boundary \( x_1 \) the boundary condition

\[
\frac{\partial P_1(x)}{\partial x}(x_0) = -x_1 \frac{\partial^2 P_0(x_0)}{\partial x^2}(x_0) \tag{4.1.38}
\]

has to be solved for \( x_1 \). Hence, the correction is given by

\[
x_1 = -\frac{\partial P_1(x_0)}{\partial x} \frac{\partial^2 P_0(x_0)}{\partial x^2}(x_0). \tag{4.1.39}
\]

As mentioned above, an analytic expression for the zeroth-order approximation of the exercise boundary \( x_0 \) can only be obtained in case the dividend yield of the underlying stock is zero, otherwise the value for \( x_0 \) has to be determined numerically. Since the first-order correction \( P_1 \) depends on the value of \( x_0 \) the function has to be evaluated numerically at the value of \( x_0 \). We note that for \( \delta = 0 \) the above expressions reduce to the results obtained in [1] for a randomized American put on a non-dividend paying stock in the fast mean-reverting stochastic volatility model.

### 4.2 Maturity-Randomized American Call

After having derived the value and optimal exercise boundary for a first-order randomized American put to order \( \sqrt{\epsilon} \), let us now consider the valuation problem for a randomized American call in the fast mean-reverting stochastic volatility model. Similar to the put, exponential maturity valuation can be applied to derive a time-independent pricing equation for the randomized American call by replacing the fixed maturity by an exponential random variable. The value of the first-order randomized American call \( C^{(1)} \) is then given as the solution to the following time-independent free-boundary pricing problem:

\[
\mathcal{L} C^{(1)}(x, y) + \lambda (x - K)^+ = 0, \quad x < x_1(y), \tag{4.2.1}
\]

As having derived the value and optimal exercise boundary for a first-order randomized American put to order \( \sqrt{\epsilon} \), let us now consider the valuation problem for a randomized American call in the fast mean-reverting stochastic volatility model. Similar to the put, exponential maturity valuation can be applied to derive a time-independent pricing equation for the randomized American call by replacing the fixed maturity by an exponential random variable. The value of the first-order randomized American call \( C^{(1)} \) is then given as the solution to the following time-independent free-boundary pricing problem:

\[
\mathcal{L} C^{(1)}(x, y) + \lambda (x - K)^+ = 0, \quad x < x_1(y), \tag{4.2.1}
\]
where \((x_1, y)\) is the optimal exercise boundary. The boundary conditions at \((x_1, y)\) are:

\[
\begin{align*}
C^{(1)}(x_1, y) &= x_1 - K, \\
\frac{\partial C^{(1)}}{\partial x}(x_1, y) &= 1, \\
\frac{\partial C^{(1)}}{\partial y}(x_1, y) &= 0.
\end{align*}
\]

(4.2.2)

The generator \(\mathcal{L}^\epsilon\) of the differential equation is the same as for the American put and is given by Eq. (4.1.4).

By expanding the price of the American call and the optimal exercise boundary in powers of \(\sqrt{\epsilon}\), i.e.

\[
\begin{align*}
C^{(1)}(x, y) &= C_0(x, y) + \sqrt{\epsilon} C_1(x, y) + \epsilon C_2(x, y) + \ldots, \\
x_1(y) &= x_0(y) + \sqrt{\epsilon} x_1(y) + \epsilon x_2(y) + \ldots,
\end{align*}
\]

(4.2.3)

it is possible to derive analytic pricing equations iteratively for the different terms of the expansion. Similar to the case of the American put, it can be shown that the zeroth-order approximation of the American call price \(C_0\) is independent from \(y\) and is determined by the following differential equation:

\[
\frac{1}{2} \sigma^2 x \frac{\partial^2 C_0}{\partial x^2} + (r - \delta)x \frac{\partial C_0}{\partial x} - (r + \lambda)C_0 + \lambda(x - K)^+ = 0
\]

(4.2.4)

with boundary conditions

\[
\begin{align*}
C_0(x_0) &= x_0 - K, & \frac{\partial C_0}{\partial x}(x_0) &= 1,
\end{align*}
\]

(4.2.5)

where the mean of the squared volatility function \(\overline{\sigma^2}\) is given by Eq. (4.1.14).

By means of a polynomial Ansatz in \(x\) it is straightforward to solve the differential equation. The solution is then given by

\[
C_0(x) = \begin{cases} 
  b_1 \left( \frac{x}{R} \right)^{\beta_2} + b_2 \left( \frac{x}{x_0} \right)^{\beta_2}, & \text{for } x \leq K \\
  a_1 \left( \frac{x}{R} \right)^{\beta_1} + b_2 \left( \frac{x}{x_0} \right)^{\beta_2} + xD - KR, & \text{for } K < x < x_0 \\
  x - K, & \text{for } x \geq x_0,
\end{cases}
\]

(4.2.6)

where

\[
a_1 = qKR - \hat{q}KD, \quad b_1 = \hat{p}KD - pKR, \quad b_2 = -pK(1 - R) + \hat{p}x_0(1 - D).
\]

(4.2.7)

The other parameters are defined as in the case of the American put in the previous section. We note that the solution to the zeroth-order problem in \(\epsilon\) is the same as for a first-order randomized American call in the Black-Scholes model with \(\overline{\sigma^2}\) replaced by the square of the Black-Scholes volatility \(\sigma^2\). Maturity randomization for the American call in the Black-Scholes model has already been studied in [5]. The result obtained above can hence also be found in [5]. The optimal exercise boundary \(x_0\) is implicitly determined by the following equation:

\[
\left( \frac{x_0}{K} \right)^{\beta_1} = \frac{\hat{q}x_0(1 - D) - \hat{p}K(1 - R)}{qKR - \hat{q}KD}.
\]

(4.2.8)

Since there is no analytic solution to this equation, the optimal exercise boundary has to
be determined numerically.

To derive the correction to $O(\sqrt{\delta})$ to the pricing function of the randomized American call we follow the same approach as for the American put. Similar to the case of the put also the first-order correction to the American call can be shown to be independent from $y$. The correction $C_1$ is then determined by the following differential equation in $x$:

$$
\frac{1}{2} \sigma^2 x^2 \frac{\partial^2 C_1}{\partial x^2} + (r - \delta)x \frac{\partial C_1}{\partial x} - (r + \lambda)C_1 = V \left( 2x^2 \frac{\partial^2 C_0}{\partial x^2} + x^3 \frac{\partial^3 C_0}{\partial x^3} \right),
$$

(4.2.9)

where $V$ is given by Eq. (4.1.29) and the boundary conditions at $x_0$ are

$$
C_1(x_0) = 0, \quad \frac{\partial C_1}{\partial x}(x_0) = -x_1 \frac{\partial^2 C_0}{\partial x^2}(x_0).
$$

(4.2.10)

The problem can be solved by observing that the homogeneous part of the differential equation is the same as for the leading-order problem. It hence amounts to finding a special solution that satisfies the inhomogeneous part of the equation and then determining the constants multiplying the solution to the homogeneous part by means of the boundary conditions. Following the discussion in section 4.1 the general solution is found to be

$$
C_1(x) = \begin{cases} 
B_1 x^{\beta_2} + \frac{V B_1 \ln(x)}{\Delta \sigma^2} \left( b_1 \left( \frac{x}{K} \right)^{\beta_2} + b_2 \left( \frac{x}{x_0} \right)^{\beta_2} \right), & \text{for } x \leq K \\
A x^{\beta_1} + B_2 x^{\beta_2} + \frac{V \ln(x)}{\Delta \sigma^2} \left( B_2 b_2 \left( \frac{x}{x_0} \right)^{\beta_2} - B_1 a_1 \left( \frac{x}{K} \right)^{\beta_1} \right), & \text{for } K < x < x_0 \\
0, & \text{for } x \geq x_0,
\end{cases}
$$

(4.2.11)

where the constants $B_1$, $A$ and $B_2$ are given by

$$
B_1 = -\frac{V}{2 \Delta^2 \sigma^2 x_0^{\beta_2}} \left[ (a_1 B_1 + b_1 B_2) \left( \left( \frac{x_0}{K} \right)^{\beta_1} - \left( \frac{x_0}{K} \right)^{\beta_1} \right) \right. \\
+ 2 \Delta \ln(K) \left( a_1 B_1 \left( \frac{x_0}{K} \right)^{\beta_1} + b_1 B_2 \left( \frac{x_0}{K} \right)^{\beta_2} \right) \\
+ 2 \Delta \ln(x_0) \left( b_2 B_2 - a_1 B_1 \left( \frac{x_0}{K} \right)^{\beta_1} \right),
$$

$$
A = -\frac{V}{2 \Delta^2 \sigma^2 K^{\beta_1}} \left( a_1 B_1 (1 - 2 \Delta \ln(K)) + b_1 B_2 \right),
$$

$$
B_2 = \frac{V}{2 \Delta^2 \sigma^2 x_0^{\beta_2}} \left[ (a_1 B_1 (1 - 2 \Delta \ln(K)) + b_1 B_2) \left( \frac{x_0}{K} \right)^{\beta_1} \\
- 2 \Delta \ln(x_0) \left( b_2 B_2 - a_1 B_1 \left( \frac{x_0}{K} \right)^{\beta_1} \right) \right].
$$

(4.2.12)

4.3 Higher-Order Maturity-Randomized American Options

In the previous two sections we have derived the value of a randomized American put and call in the fast mean-reverting stochastic volatility model. In the derivation we have assumed that the options mature at the first jump of a Poisson process with exponential decay rate $\lambda$. As shown in chapter 3 for the CEV model the first-order randomized option contract can be a fairly good approximation of the fixed-maturity American option contract, however the accuracy strongly increases when higher-order randomization, i.e. options maturing at a larger number of jumps of the Poisson process, are considered. By considering more jumps in the Poisson process, the probability distribution of the random maturity gets narrower and therefore better approximates the fixed maturity of the usual
American option contract. Hence, let us now consider higher-order randomized option contracts in the fast mean-reverting stochastic volatility model.

For the stochastic volatility model the same approach as outlined in section 2.4 can be applied to derive iterative pricing equations for higher-order randomized option contracts. The approach leads to the following free boundary pricing problem for the value of a randomized American put maturing at the $n$th jump of a Poisson process with parameter $\lambda = n/T$

$$\mathcal{L}^\epsilon P^{(n)}(x, y) + \lambda P^{(n-1)}(x, y) = 0, \quad \text{for} \ x \geq x_n(y),$$

subject to the boundary conditions

$$P^{(n)}(x_n, y) = K - x_n,$$
$$\frac{\partial P^{(n)}}{\partial x}(x_n, y) = -1,$$
$$\frac{\partial P^{(n)}}{\partial y}(x_n, y) = 0,$$

(4.3.2)

where $P^{(n-1)}(x, y)$ is the value of a randomized American put maturing at the $(n-1)$th jump of the Poisson process with rate $\lambda = n/T$.

The value of a second-order randomized American put is hence given by the following pricing problem:

$$\mathcal{L}^\epsilon P^{(2)}(x, y) + \lambda P^{(1)}(x, y) = 0, \quad \text{for} \ x \geq x_2(y),$$

(4.3.3)

with boundary conditions

$$P^{(2)}(x_2, y) = K - x_2,$$
$$\frac{\partial P^{(2)}}{\partial x}(x_2, y) = -1,$$
$$\frac{\partial P^{(2)}}{\partial y}(x_2, y) = 0.$$

(4.3.4)

In analogy to the first-order case, the problem can be solved by means of an asymptotic expansion in $\epsilon$. The expansion of the pricing function in powers of $\sqrt{\epsilon}$ reads

$$P^{(2)}(x, y) = P_0^{(2)}(x, y) + \sqrt{\epsilon}P_1^{(2)}(x, y) + \epsilon P_2^{(2)}(x, y) + \ldots.$$  

(4.3.5)

For a fully consistent expansion it would also be required to expand the exercise boundary in powers of $\epsilon$. However, instead of expanding the exercise boundary we want to apply the hybrid method which we also applied to the American put in the CEV model in chapter 3, namely solving for the exercise boundary numerically after having derived the structure of the pricing functions.

By the same argument as in the first-order randomization case, it can be shown that the zeroth-order approximation of the pricing function $P_0^{(2)}$ is independent from $y$ and is determined by the following differential equation in $x$:

$$\frac{1}{2} \sigma^2 x^2 \frac{\partial^2 P_0^{(2)}}{\partial x^2} + (r - \delta) x \frac{\partial P_0^{(2)}}{\partial x} - (r + \lambda) P_0^{(2)} + \lambda P_0^{(1)} = 0.$$  

(4.3.6)

This is the centering condition for the source term of the Poisson equation which is obtained by collection terms of $O(\epsilon^0)$ in Eq. (4.3.3). $P_0^{(1)}$ is the zeroth-order solution to the first-order randomization pricing problem. Since we do not want to expand the exercise boundary and the constants multiplying the general solution to the differential equation, the zeroth-order and first-order contributions in $\sqrt{\epsilon}$ for the first-order randomized Amer-
ican put are more generally given by
\[ P_0^{(1)}(x) + \sqrt{\epsilon} P_1^{(1)}(x) = \begin{cases} c^{(1)} \left( \frac{x}{K} \right)^{\beta_1} [1 + \sqrt{\epsilon} f_1(x)], & \text{for } x \geq K \\ a^{(1)} \left( \frac{x}{K} \right)^{\beta_2} [1 + \sqrt{\epsilon} g_1(x)] + b^{(1)} \left( \frac{x}{K} \right)^{\beta_1} [1 + \sqrt{\epsilon} f_1(x)] + KR - xD, & \text{for } x_1 < x < K \\ K - x, & \text{for } x \leq x_1, \end{cases} \] (4.3.7)

where \( a^{(1)} \), \( b^{(1)} \) and \( c^{(1)} \) are real-valued constants determined by the boundary conditions.

Here, the functions \( g_1(x) \) and \( f_1(x) \) are defined as
\[ f_1(x) = \frac{-V(\beta_1 - 1)\beta_1^2 \ln(x)}{\Delta \sigma^2}, \quad g_1(x) = \frac{V(\beta_2 - 1)\beta_2^2 \ln(x)}{\Delta \sigma^2}. \] (4.3.8)

Again we observe that the homogeneous part of the differential equation (4.3.6) is the same as in the first-order randomization case. In fact, for \( x \leq x_1 \) the pricing equation is the same as for the first-order randomization case with \( \lambda = 2/T \). For \( x > x_1 \), we try the following Ansatz to find a special solution to the inhomogeneous differential equation:
\[ P_0^{(2)} = b^{(1)} f_0^{(2)} \left( \frac{x}{K} \right)^{\beta_1} + a^{(1)} g_0^{(2)} \left( \frac{x}{K} \right)^{\beta_2} + KR^2 - xD^2, \] (4.3.9)

where \( f_0^{(2)} \) and \( g_0^{(2)} \) are real-valued functions of \( x \).

Plugging this Ansatz into Eq. (4.3.6) and collecting terms proportional to \( (x/K)^{\beta_1} \) and \( (x/K)^{\beta_2} \), it is possible to derive separate differential equations for both functions \( f_0^{(2)} \) and \( g_0^{(2)} \):
\[ x^2 f_0^{(2)''}(x) + (x - 2x\Delta)f_0^{(2)'}(x) - R\beta_1\beta_2 = 0, \]
\[ x^2 g_0^{(2)''}(x) + (x + 2x\Delta)g_0^{(2)'}(x) - R\beta_1\beta_2 = 0. \] (4.3.10)

The solutions to these equations can readily be obtained by means of standard techniques for solving differential equations:
\[ f_0^{(2)}(x) = -\frac{R\beta_1\beta_2 \ln(x)}{2\Delta}, \]
\[ g_0^{(2)}(x) = \frac{R\beta_1\beta_2 \ln(x)}{2\Delta}. \] (4.3.11)

Hence, the zeroth-order approximation of the second-order randomized American put value is given by
\[ P_0^{(2)}(x) = \begin{cases} c^{(2)} \left( \frac{x}{K} \right)^{\beta_1} + f_0^{(2)}(x)c^{(1)} \left( \frac{x}{K} \right)^{\beta_1}, & \text{for } x \geq K \\ a_1^{(2)} \left( \frac{x}{K} \right)^{\beta_2} + b_1^{(1)} \left( \frac{x}{K} \right)^{\beta_1} + g_0^{(2)}(x)a^{(1)} \left( \frac{x}{K} \right)^{\beta_2} + f_0^{(2)}(x)b^{(1)} \left( \frac{x}{K} \right)^{\beta_1} + KR^2 - xD^2, & \text{for } x_1 < x < K \\ a_2^{(1)} \left( \frac{x}{K} \right)^{\beta_2} + b_2^{(2)} \left( \frac{x}{K} \right)^{\beta_1} + KR - xD, & \text{for } x_2 < x \leq x_1 \\ K - x, & \text{for } x \leq x_2, \end{cases} \] (4.3.12)

where \( c^{(2)}, a_1^{(2)}, b_1^{(1)}, a_2^{(2)} \) and \( b_2^{(2)} \) are real-valued constants. The values of the constants and the exercise boundary \( x_2 \) have to be determined by the boundary conditions at \( x_2 \) and by demanding that the solution is continuous and differentiable at \( x = K \) and \( x = x_1 \).

By following the same approach as for the first-order randomized American put, it is
also possible to derive a pricing equation for the correction to \( O(\sqrt{\tau}) \). Collecting terms proportional to \( e^{1/2} \) in Eq. (4.3.3) we obtain a Poisson equation in \( y \) for \( P_3^{(2)} \). The first-order correction \( P_1^{(2)} \) is then given by evaluating the solvability condition of the Poisson equation, which is

\[
\int_{-\infty}^{\infty} L_2 \Phi(y) dy P_1^{(2)}(x) = V \left( 2x^2 \frac{\partial^2 P_0^{(2)}}{\partial x^2} + x^3 \frac{\partial^3 P_0^{(2)}}{\partial x^3} \right) - \lambda P_1^{(1)}(x),
\]

where \( P_1^{(1)}(x) \) is the \( O(\sqrt{\tau}) \) correction to the first-order randomized American put with \( \lambda = 2/T \). This differential equation has a very similar structure to the zeroth-order case, in particular the homogeneous part on the r.h.s. of the equation is the same as in the zeroth-order case. Hence, we can use a similar Ansatz to find a special solution for \( P_1^{(2)}(x) \) in the different regimes confined by \( K \) and \( x_0 \). The Ansatz is

\[
P_1^{(2)} = b_1^{(1)} f_1^{(2)}(x) \left( \frac{x}{K} \right)^{\beta_1} + a_1^{(1)} g_1^{(2)} \left( \frac{x}{K} \right)^{\beta_2}.
\]

Plugging the Ansatz into Eq. (4.3.13) and collecting terms proportional to \( (x/K)^{\beta_1} \) and \( (x/K)^{\beta_2} \) separate differential equations can be derived for both functions \( f_1^{(2)} \) and \( g_1^{(2)} \), which can be solved by means of standard ODE solving techniques.

Adding the first-order correction to the leading-order approximation we obtain the following solution to \( O(\sqrt{\tau}) \):

\[
P_0^{(2)}(x) + \sqrt{\tau} P_1^{(2)}(x) = \begin{cases} 
   c^{(2)} \left( \frac{x}{K} \right)^{\beta_1} (1 + \sqrt{\tau} f_1(x)) + c^{(1)} \left( \frac{x}{K} \right)^{\beta_1} (f_0^{(2)}(x) + \sqrt{\tau} f_1^{(2)}(x)), & \text{for } x \geq K \\
   a^{(2)} \left( \frac{x}{K} \right)^{\beta_2} (1 + \sqrt{\tau} g_1(x)) + a^{(1)} \left( \frac{x}{K} \right)^{\beta_2} (g_0^{(2)}(x)), & \text{for } x_1 < x < K \\
   +b^{(1)} \left( \frac{x}{K} \right)^{\beta_1} (f_0^{(2)}(x) + \sqrt{\tau} f_1^{(2)}(x)) + K R^2 - x D^2, & \text{for } x_1 < x < K \\
   a^{(2)} \left( \frac{x}{K} \right)^{\beta_2} (1 + \sqrt{\tau} g_1(x)) + b^{(2)} \left( \frac{x}{x_0} \right)^{\beta_1} (1 + \sqrt{\tau} f_1(x)) + K R - x D, & \text{for } x_2 < x \leq x_1 \]

\]

(4.3.15)

where

\[
f_1^{(2)}(x) = - RV \beta_1^2 \beta_2 \ln(x) \left[ \beta_1^2 + 2 \beta_2 - 3 \beta_1 \beta_2 + \beta_1 (\beta_1 - 1) \beta_2 \ln(x) \right] \\
g_1^{(2)}(x) = RV \beta_2^2 \beta_1 \ln(x) \left[ \beta_2^2 + 2 \beta_1 - 3 \beta_1 \beta_2 + (\beta_2 - 1) \beta_2 (\beta_2 - 1) \ln(x) \right].
\]

(4.3.16)

In analogy to the zeroth-order approximation the values of the constants \( c^{(2)}, a^{(2)}, b^{(2)}, a_0^{(2)}, b_0^{(2)} \), and the exercise boundary are given by the boundary conditions at \( x_2 \) and by the continuity and differentiability conditions at \( x = K \) and \( x = x_1 \). Hence, their numerical values implicitly contain corrections to \( O(\sqrt{\tau}) \).

By means of this iterative approach it is possible to derive and solve pricing equations for even higher-order randomized American puts in the stochastic volatility model. For this thesis, we have also derived third-order randomized American put values (i.e. the value of an American put maturing at the third jump of the Poisson process) up to \( O(\sqrt{\tau}) \) in the asymptotic expansion. The results are collected in the Appendix.

Besides, the same approach also allows for the calculation of higher-order randomized American call values. In addition to the results for the first-order randomized American
call in the fast mean-reverting stochastic volatility model obtained in section 4.2, we have derived expressions for the value of randomized American calls maturing at the second and third jump of the Poisson process up to $O(\sqrt{\epsilon})$. Again for brevity we have summarized the results in the Appendix.

4.4 Numerical Analysis

In this section, we examine the results which we obtained for the randomized American put and call in the fast mean-reverting stochastic volatility model. For comparability we use the same volatility model as in [1] in our analysis. Hence we choose the volatility function $f$ to be

$$f(y) = e^{\nu y}. \quad (4.4.1)$$

There are two parameters in our results for the randomized American put and call which depend on the function $f$: the mean of the squared volatility $\sigma^2$, and the parameter $V$ in the $O(\sqrt{\epsilon})$ correction. For the particular choice of the volatility function, the integrals over the probability distribution can be solved analytically and the two parameters can be evaluated to:

$$\sigma^2 = e^{2m+2\nu^2},$$

$$V = -\frac{\rho}{\nu^2\sqrt{2}} e^{3m} \left( e^{9\nu^2/2} - e^{5\nu^2/2} \right), \quad (4.4.2)$$

where $m, \nu$ and $\rho$ are the parameters of the stochastic volatility model. For the numerical values of the model parameters we also apply the same choice as in [1]:

$$m = -2, \quad \nu = 1, \quad \rho = -0.3, \quad \epsilon = 0.01. \quad (4.4.3)$$

For the randomized American put we choose the following market and option contract parameters:

$$K = 100, \quad T = 1.0, \quad r = 0.1, \quad \delta = 0. \quad (4.4.4)$$

For the randomized American call we apply the following choice:

$$K = 100, \quad T = 1.0, \quad r = 0.03, \quad \delta = 0.1. \quad (4.4.5)$$

As mentioned in the previous section, for higher-order randomization we do not expand the exercise boundary and constants in the solutions to the pricing equations in powers of $\epsilon$. Instead, we assume that the constants implicitly contain contributions from higher orders and solve for their values numerically by means of the boundary conditions at the exercise boundary and by demanding that the pricing function is continuous and differentiable everywhere.

In Fig. 4.1 we show the value of the American put obtained from our results for first-order, second-order and third-order randomization up to $O(\sqrt{\epsilon})$ in the asymptotic expansion. Beside the absolute value of the put, we also show the relative difference between the second-order and first-order results as well as the relative difference between the third-order and second-order results. The relative difference between the $(n+1)$th order and the $n$th order gives us an indication of the speed of the convergence of the series. In fact, as can be seen, the relative difference as a function of the underlying stock price $S$ strongly decreases when going from second order to third order in the randomization: the maximum of the relative difference between the second-order randomized put value and the first-order randomized put value in the relevant range of $S$ is roughly 10%, whereas the maximum of the relative difference between the third-order and second-order randomized
Figure 4.1: Results for the randomized American put in the fast mean-reverting stochastic volatility model with parameters from Eq. (4.4.3) and Eq. (4.4.4). The upper plot shows the put value, the lower plot shows the relative difference between the American put values obtained from the $(n+1)$th order and the $n$th order in the maturity randomization.

put value in the same range is less than $4\%$. This result indicates a fast convergence of the series.

We also observe that for the relevant $S$ range around the strike of the option, the relative difference is larger than zero for both considered orders of the randomization. This observation is in line with the result from chapter 3 for the CEV model, namely that the randomized American put value lies slightly below the true value of its fixed-maturity equivalent. The higher the order of the randomization, i.e. the more jumps in the Poisson process are considered and the narrower the distribution of the random maturity, the smaller is the difference between the randomized American put value and the value of the corresponding fixed-maturity American put.

The numerical values for the exercise boundaries or critical stock prices for the first-, second- and third-order randomized American puts are shown in Table 4.1. The value of
Table 4.1: Critical stock price for the randomized American put in the fast mean-reverting stochastic volatility model with parameters from Eq. (4.4.3) and Eq. (4.4.4). The table shows the results for first-order, second-order and third-order randomization to $O(\sqrt{\epsilon})$ in the asymptotic expansion.

<table>
<thead>
<tr>
<th>Critical stock price at $t = 0$</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1st-order Randomization, $O(\sqrt{\epsilon})$</td>
<td>71.76</td>
</tr>
<tr>
<td>2nd-order Randomization, $O(\sqrt{\epsilon})$</td>
<td>71.10</td>
</tr>
<tr>
<td>3rd-order Randomization, $O(\sqrt{\epsilon})$</td>
<td>70.80</td>
</tr>
</tbody>
</table>

Table 4.2: Critical stock price for the randomized American call in the fast mean-reverting stochastic volatility model with parameters from Eq. (4.4.3) and Eq. (4.4.5). The table shows the results for first-order, second-order and third-order randomization to $O(\sqrt{\epsilon})$ in the asymptotic expansion.

<table>
<thead>
<tr>
<th>Critical stock price at $t = 0$</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1st-order Randomization, $O(\sqrt{\epsilon})$</td>
<td>146.37</td>
</tr>
<tr>
<td>2nd-order Randomization, $O(\sqrt{\epsilon})$</td>
<td>147.99</td>
</tr>
<tr>
<td>3rd-order Randomization, $O(\sqrt{\epsilon})$</td>
<td>148.76</td>
</tr>
</tbody>
</table>

the exercise boundary slightly decreases with the order of the randomization. However, the difference between the numerical results is very small indicating that already the first-order result for the exercise boundary yields a good approximation of the exercise boundary to $O(\sqrt{\epsilon})$.

In Fig. 4.2 the results for the randomized American call in the fast mean-reverting stochastic volatility model are shown. The plots show the value of the first-order, second-order and third-order randomized American call to $O(\sqrt{\epsilon})$ in the asymptotic expansion. The call price is shown as a function of the underlying stock price $S$. Besides, also the relative difference between adjacent orders in the randomization are shown. Similar to the results for the American put, the relative difference between the third-order and second-order randomized call values is significantly smaller than the difference between the second-order and first-order values. Again, in the stock price regime around the option’s strike the relative difference is larger than zero. Hence, also for the randomized American call the expectation is that the true value of the fixed-maturity American call is larger than the value of the randomized American call. With increasing order in the randomization (considering more and more jumps in the Poisson process) the value of the randomized American call is expected to converge to the value of the equivalent fixed-maturity contract.

In Table 4.2 we have collected the values for the exercise boundaries obtained from our results for the randomized American calls to $O(\sqrt{\epsilon})$. We observe a slight increase of the exercise boundary with the order of the randomization.

To estimate the goodness of the approximation for the American option values obtained from the randomization approach, we compare our results for the American put with those from [1]. In [1] the approximation for the exercise boundary of the first-order randomized American put to $O(\sqrt{\epsilon})$ was used to calculate put values by means of a Monte-Carlo simulation. In contrast to our asymptotic expansion of the put value, the Monte-Carlo approach implicitly takes into account all orders in $\epsilon$ for the generation of the sample paths. Only the optimal exercise is approximated by maturity randomization and the
Figure 4.2: Results for the randomized American call in the fast mean-reverting stochastic volatility model with parameters from Eq. (4.4.3) and Eq. (4.4.5). The upper plot shows the randomized American call value, the lower plot shows the relative difference between the American call values obtained from the \((n + 1)\)th order and the \(n\)th order in the maturity randomization.

asymptotic expansion in \(\epsilon\). As a benchmark for the true value of the American put in [1] the “interleaving estimator” from [17] is used. In Table 4.3 we compare the American put values obtained from first-order, second-order and third-order randomization with the results obtained from the Monte-Carlo simulation and the interleaving estimator in [1]. The results from maturity randomization underestimate the true value of the American put, as expected from the observations above for the CEV model and stochastic volatility model. The higher the order of the randomization the smaller is the difference to the value from the Monte-Carlo simulation and the interleaving estimator. To extrapolate the results from maturity randomization we again apply Richardson extrapolation. By means of the formula in Eq. (3.2.26) we calculate extrapolated values for the randomized American put from the first three orders of randomization. The resulting put values are
Table 4.3: American put values in the fast mean-reverting stochastic volatility model with parameters from Eq. (4.4.3) and Eq. (4.4.4). The table shows the results for first-order, second-order and third-order randomization to $O(\sqrt{\epsilon})$ in the asymptotic expansion, as well as values obtained from Richardson extrapolation of the randomization results to the first three orders. As benchmarks the results obtained in [1] from the Monte-Carlo simulation using control variates (Agarwal et al. CV) and the Longstaff-Schwartz interleaving estimator are shown.

<table>
<thead>
<tr>
<th>American Put value</th>
<th>S = 90</th>
<th>S = 100</th>
<th>S = 110</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st-order Randomization, $O(\sqrt{\epsilon})$</td>
<td>13.966</td>
<td>9.140</td>
<td>6.177</td>
</tr>
<tr>
<td>2nd-order Randomization, $O(\sqrt{\epsilon})$</td>
<td>14.453</td>
<td>9.870</td>
<td>6.812</td>
</tr>
<tr>
<td>3rd-order Randomization, $O(\sqrt{\epsilon})$</td>
<td>14.660</td>
<td>10.144</td>
<td>7.065</td>
</tr>
<tr>
<td>Richardson Extrapolation, $O(\sqrt{\epsilon})$</td>
<td>15.142</td>
<td>10.740</td>
<td>7.634</td>
</tr>
<tr>
<td>Agarwal et al. CV</td>
<td>15.220 (0.035)</td>
<td>11.047 (0.021)</td>
<td>8.103 (0.021)</td>
</tr>
<tr>
<td>LS interleaving estimator</td>
<td>15.409 (0.021)</td>
<td>11.190 (0.031)</td>
<td>8.228 (0.031)</td>
</tr>
</tbody>
</table>

Also shown in Table 4.3. For the extrapolated put values the difference to the benchmark results is significantly smaller than for the fixed-order randomized values. However, the agreement is not as good as in the case of the CEV model. The reason for the larger size of the deviations is that in the asymptotic expansion only terms up to $O(\sqrt{\epsilon})$ are considered and hence the dynamics of the stochastic volatility are approximated less accurately than in our expansion for the CEV model. Still the obtained results show that the analytic approach of maturity randomization yields a good approximation of American option values given that only minimal numerical computation time is required.
Chapter 5

Summary and Conclusion

In this dissertation maturity randomization as an alternative approach for the valuation of American options was studied. We outlined the general concept of maturity randomization and derived valuation formulas for randomized American options, i.e. American options with the fixed maturity replaced by a random variable. The approach was applied to two different volatility models, namely the CEV model and a fast mean-reverting stochastic volatility model.

For the CEV model the resulting free-boundary pricing problem cannot analytically be solved. Therefore we performed an asymptotic expansion of the pricing function in a suitably chosen small parameter $\epsilon$ and derived the pricing function for a randomized American put as a perturbative series up to second order in the parameter $\epsilon$. We considered randomized American puts maturing at the first, second and third jump of a Poisson process. In the comparison with American put results obtained from a finite-differences scheme we found that the obtained randomized American put values and optimal exercise boundaries yield very good approximations for the equivalent fixed-maturity American put values. As expected, the higher the order of the randomization the better is the agreement with the fixed-maturity results. By means of Richardson extrapolation we calculated an extrapolated American put pricing function from the randomization results up to third order. The extrapolated American put value almost exactly agrees with the benchmark result from the finite-differences scheme. In contrast to the finite differences scheme there is hardly any numerical computation time required for the calculation of the randomized American put values and exercise boundary, since the pricing function is given in an analytic form.

In the second part of the dissertation we studied a fast mean-reverting stochastic volatility model. In this case we considered an underlying stock process including a constant dividend yield, hence also for American calls the optimal exercise is non-trivial. Consequently, we derived pricing functions for both randomized American puts and calls up to third order in the randomization. Again, we used an asymptotic expansion for the derivation of the pricing functions in terms of a perturbative series. As benchmark for our obtained option values we used the results from [1], where the optimal exercise boundary from first-order randomization was used as input for a Monte-Carlo simulation. The comparison showed that our results for the randomized American put yield a fairly good approximation of the benchmark. However, the agreement is not as good as in the case of the CEV model. The reason for the deviations is that in the asymptotic expansion we only calculated terms up to $O(\sqrt{\epsilon})$ and hence only roughly approximated the dynamics of the stochastic volatility process. To obtain an even better approximation higher order terms in the expansion should be considered.

All in all, we found that maturity randomization is a powerful tool for the valuation of American-styled option contracts. When a large enough number of jumps in the Poisson
process is considered and Richardson extrapolation is used to speed up the convergence, maturity randomization yields results in very good agreement with usual numerical methods for the valuation of American options. Since the pricing functions for randomized options can be derived in analytic form, the numerical computational effort is very small. In our approach only the optimal exercise boundary and the constants multiplying the general solutions to the differential equations have to be solved for numerically by means of the boundary conditions. This amounts to solving a small system of non-linear equation and hardly consumes any computation time. In contrast, most popular numerical approaches for pricing American options require much more computation time. In particular pricing of American options in stochastic volatility models usually requires Monte-Carlo simulations and is hence numerically very involved. Therefore, we conclude that maturity randomization can be an efficient and fast alternative to the popular numerical approaches for pricing American-styled options.
Appendices
Appendix A

Pricing Functions

A.1 CEV Model

In this appendix we collect our results obtained for the maturity-randomized American put in the CEV model up to second order in the asymptotic expansion in $\epsilon$. In the CEV model we use dimensionless pricing functions defined as

$$\hat{p}^{(i)}(x) = \frac{P^{(i)}(S)}{\epsilon K} \tag{A.1.1}$$

with the dimensionless variable

$$x = \frac{S/K - 1}{\sigma \sqrt{T}}. \tag{A.1.2}$$

The parameters used in the following equations are defined in section 3.1.

For the first-order randomized American put the pricing function is given by

$$\hat{p}^{(1)}(0) + \epsilon \hat{p}^{(1)}(1) + \epsilon^2 \hat{p}^{(1)}(2) =
\begin{cases}
    C^{(1)} e^{-(k-\eta)x} \left[ 1 + \epsilon f_1(x) + \epsilon^2 f_2(x) \right], & \text{for } x \geq 0 \\
    A^{(1)} e^{-(k-\eta)x} \left[ 1 + \epsilon f_1(x) + \epsilon^2 f_2(x) \right] \\
    + B^{(1)} e^{-(k+\eta)x} \left[ 1 + \epsilon g_1(x) + \epsilon^2 g_2(x) \right] - x - \frac{2k}{\eta} + \epsilon \frac{4k^2}{\eta} - \epsilon^2 \frac{8k^3}{\eta^3}, & \text{for } x_1 < x < 0 \\
    -x, & \text{for } x \leq x_1, \tag{A.1.3}
\end{cases}$$

where

$$f_1(x; \eta) = \frac{1}{4\eta} (k + \eta)(k + \eta - 2k)x^2 + \frac{1}{4\eta^2} \left[ (k + \eta)^2 - 6k \eta + 2k^2 \right] x,$$

$$g_1(x; \eta) = f_1(x; -\eta),$$

$$f_2(x; \eta) = x^3 \frac{(k + \eta)^2 (-2k + k + \beta \eta)}{32\eta^2} + x^3 \frac{k \beta (k + \eta)(k \beta - 2) + \eta(\beta - 4)}{12\eta^2}$$

$$+ \left( \frac{x}{32\eta^2} + \frac{x^2}{32\eta^4} + \frac{x^3}{48\eta^6} \right) (k + \eta)$$

$$\times \left( 20k^3 - 20k^3 \beta + 5k^3 \beta^2 + 28k^2 \eta - 20k^2 \beta \eta + 3k^2 \beta^2 \eta + 4k \beta \eta^2 \\ - k \beta^2 \eta^2 - 4 \beta \eta^3 + \beta^2 \eta^3 \right),$$

$$g_2(x; \eta) = f_2(x; -\eta). \tag{A.1.4}$$

The constants $A^{(1)}$, $B^{(1)}$, $C^{(1)}$ and the optimal exercise boundary $x_1$ are determined by the boundary conditions at $x = x_1$ and by demanding that the pricing function is continuous.
and differentiable at \( x = 0 \).

The pricing function for the second-order randomized American put is

\[
p_0^{(2)}(x) + \epsilon p_1^{(2)}(x) + \epsilon^2 p_2^{(2)}(x) = \begin{cases} 
C^{(2)}e^{(-k-\eta)x} \left[ 1 + \epsilon f_1(x) + \epsilon^2 f_2(x) \right] \\
+ C^{(1)}e^{(-k-\eta)x} \left[ f_0^{(2)}(x) + \epsilon f_1^{(2)}(x) + \epsilon^2 f_2^{(2)}(x) \right] , & \text{for } x \geq 0 \\
A_1^{(2)}e^{(-k-\eta)x} \left[ 1 + \epsilon f_1(x) + \epsilon^2 f_2(x) \right] \\
+ B_1^{(2)}e^{(-k+\eta)x} \left[ 1 + \epsilon g_1(x) + \epsilon^2 g_2(x) \right] \\
+ A^{(1)}e^{(-k-\eta)x} \left[ f_0^{(2)}(x) + \epsilon f_1^{(2)}(x) + \epsilon^2 f_2^{(2)}(x) \right] \\
+ B^{(1)}e^{(-k+\eta)x} \left[ g_0^{(2)}(x) + \epsilon g_1^{(2)}(x) + \epsilon^2 g_2^{(2)}(x) \right] \\
-x - \frac{4k}{\gamma} + \epsilon \frac{12k^2}{\gamma^2} - \epsilon^2 \frac{2k^3}{\gamma^2} , & \text{for } \underline{x}_1 < x < 0 \\
A_2^{(2)}e^{(-k-\eta)x} \left[ 1 + \epsilon f_1(x) + \epsilon^2 f_2(x) \right] \\
+ B_2^{(2)}e^{(-k+\eta)x} \left[ 1 + \epsilon g_1(x) + \epsilon^2 g_2(x) \right] - x - \frac{2k}{\gamma} + \epsilon \frac{4k^2}{\gamma^2} - \epsilon^2 \frac{8k^3}{\gamma^2} , & \text{for } \underline{x}_3 < x \leq \underline{x}_1 \\
-x , & \text{for } x \leq \underline{x}_2 ,
\end{cases}
\]

where

\[
\begin{align*}
f_0^{(2)}(x; \eta) &= \frac{\gamma^2 x}{2\eta} , \\
g_0^{(2)}(x; \eta) &= f_0^{(2)}(x; -\eta) , \\
f_1^{(2)}(x; \eta) &= - \left( \frac{x}{\eta} + x^2 \right) k(\eta^2 - k^2) \frac{(2 - \beta)k + (3 - \beta)\eta}{4\eta^3} \\
&\quad - x^3(k + \eta)(\eta^2 - k^2) \frac{(2 - \beta)k - \beta\eta}{8\eta^2} , \\
g_1^{(2)}(x; \eta) &= f_1^{(2)}(x; -\eta) , \\
f_2^{(2)}(x; \eta) &= \frac{\gamma^2}{192\eta^3} \left\{ 3x^3\eta^4(k + \eta)^2 [k(\beta - 2) + \beta\eta]^2 \\
&\quad - 4x^4\eta^3(k + \eta) \left[ -4k^3(\beta - 2)^2 + k^2(17\beta - 5\beta^2 - 14)\eta + 3k\beta\eta^2 + \beta(2 + \beta)\eta^3 \right] \\
&\quad + (3x + 3x^2\eta) \left[ 25k^4(\beta - 2)^2 + 32k^3(6 - 5\beta + \beta^2)\eta + 6k^2(14 - 8\beta + \beta^2)\eta^2 \\
&\quad + (\beta - 4)\beta\eta^4 \right] + x^3\eta^2 \left[ 45k^4(\beta - 2)^2 + 24k^3(14 - 13\beta + 3\beta^2)\eta \\
&\quad + 2k^2(70 - 64\beta + 13\beta^2)\eta^2 + (\beta - 4)\beta\eta^4 \right] \right\} , \\
g_2^{(2)}(x; \eta) &= f_2^{(2)}(x; -\eta) .
\end{align*}
\]

The constants \( A_1^{(2)} , A_2^{(2)} , B_1^{(2)} , B_2^{(2)} , C^{(2)} \) and the optimal exercise boundary \( \underline{x}_2 \) are determined by the boundary conditions at \( x = \underline{x}_2 \) and by demanding that the pricing function is continuous and differentiable at \( x = 0 \) and \( x = \underline{x}_1 \).
For the third-order randomized American put the pricing function is given by the following expression:

\[
\begin{align*}
 p_0^{(3)}(x) + \epsilon p_1^{(3)}(x) + \epsilon^2 p_2^{(3)}(x) &= \\
 &\begin{cases}
 C^{(3)}e^{(-k-\eta)x} [1 + \epsilon f_1(x) + \epsilon^2 f_2(x)] \\
 + C^{(2)}e^{(-k-\eta)x} \left[ f_0^{(2)}(x) + \epsilon f_1^{(2)}(x) + \epsilon^2 f_2^{(2)}(x) \right] \\
 + C^{(1)}e^{(-k-\eta)x} \left[ f_0^{(3)}(x) + \epsilon f_1^{(3)}(x) + \epsilon^2 f_2^{(3)}(x) \right], \\
 &\text{for } x \geq 0 \\
 A_1^{(3)}e^{(-k-\eta)x} [1 + \epsilon g_1(x) + \epsilon^2 g_2(x)] \\
 + B_1^{(3)}e^{(-k-\eta)x} \left[ g_0^{(3)}(x) + \epsilon g_1^{(3)}(x) + \epsilon^2 g_2^{(3)}(x) \right]
 &\text{for } x_1 < x < 0 \\
 A_2^{(3)}e^{(-k-\eta)x} [1 + \epsilon f_1(x) + \epsilon^2 f_2(x)] \\
 + B_2^{(3)}e^{(-k-\eta)x} \left[ g_0^{(2)}(x) + \epsilon g_1^{(2)}(x) + \epsilon^2 g_2^{(2)}(x) \right]
 &\text{for } x_2 < x \leq x_1 \\
 A_3^{(3)}e^{(-k-\eta)x} [1 + \epsilon g_1(x) + \epsilon^2 g_2(x)] \\
 + B_3^{(3)}e^{(-k-\eta)x} \left[ g_0^{(3)}(x) + \epsilon g_1^{(3)}(x) + \epsilon^2 g_2^{(3)}(x) \right]
 &\text{for } x_3 < x \leq x_2 \\
 -x, \\
 &\text{for } x \leq x_3,
 \end{cases}
\end{align*}
\]

where

\[
\begin{align*}
 f_0^{(3)}(x; \eta) &= \frac{x^4\gamma + x^2\gamma^4}{8\eta^2} + \frac{x^2\gamma^4}{8\eta^2}, \\
 g_0^{(3)}(x; \eta) &= f_0^{(3)}(x; -\eta), \\
 f_1^{(3)}(x; \eta) &= \frac{\gamma^4}{32\eta^4} \left[ 4x^3\eta^2(\beta - 2)(k^2 + k\eta) + x^4\eta^3(k + \eta)(-2k + k\beta + \beta\eta) \right] + (2kx + 2kx^2\eta)(-8k + 4k\beta - 9\eta + 3\beta\eta), \\
 g_1^{(3)}(x; \eta) &= f_1^{(3)}(x; -\eta), \\
 f_2^{(3)}(x; \eta) &= \frac{\gamma^4}{768\eta^9} \left\{ 3x^5\eta^5(kn + \eta^2)[k(\beta - 2) + \beta\eta^2] - x^5\eta^4(k + \eta) \left[ -25k^3(\beta - 2)^2 
 + k^2(-68 + 104\beta - 35\beta^2)\eta - 3k(\beta - 4)\beta\eta^2 + \beta(8 + 7\beta)\eta^3 \right] 
 + 4k^3\eta^2 \left[ 75k^4(\beta - 2)^2 + 48k^3(10 - 9\beta + 2\beta^2)\eta + 12k^2(14 - 11\beta + 2\beta^2)\eta^2 \right] 
 + (\beta - 4)\beta\eta^4 + x^4\eta^3 \left[ 109k^4(\beta - 2)^2 + 12k^3(58 - 59\beta + 15\beta^2)\eta \right. 
 + 6k^2(42 - 48\beta + 11\beta^2)\eta^2 - 4k(5 + \beta)\eta^3 + (\beta - 4)\beta^2\eta^4 \right. 
 + (3x + 3x^2\eta)(175k^4(\beta - 2)^2 + 192k^3(6 - 5\beta + \beta^2)\eta + 30k^2(14 - 8\beta + \beta^2)\eta^2 
 + 3(\beta - 4)\beta^3\eta^4) \right\}, \\
 g_2^{(3)}(x; \eta) &= f_2^{(3)}(x; -\eta).
\end{align*}
\]
A.2 Fast Mean-Reverting Stochastic Volatility Model

The results obtained for the maturity-randomized American put and call options in the fast mean-reverting stochastic volatility model are collected in this section. We have calculated the pricing functions for the first-, second- and third-order randomized American options up to $O(\sqrt{\epsilon})$ in the asymptotic expansion.

For the American put, the full results for first-order and second-order randomization have been presented in section 4.3. Hence in this appendix only the results for the third-order randomized put are shown.

The value of the third-order randomized American put under fast mean-reverting stochastic volatility as a function of the underlying stock price $x$ is given by

\[
P_0^{(3)}(x) + \sqrt{\epsilon}F_1^{(3)}(x) = \begin{cases} 
  c^{(3)}(\frac{x}{K})^{\beta_1} [1 + \sqrt{\epsilon}f_1(x)] + c^{(2)}(\frac{x}{K})^{\beta_1} \left[ f_0^{(2)}(x) + \sqrt{\epsilon}f_1^{(2)}(x) \right] 
  + f_0^{(3)}(x) + \sqrt{\epsilon}f_1^{(3)}(x), & \text{for } x \geq K \\
  a_1^{(3)}(\frac{x}{K})^{\beta_2} [1 + \sqrt{\epsilon}g_1(x)] + b_1^{(3)}(\frac{x}{K})^{\beta_1} [1 + \sqrt{\epsilon}f_1(x)] 
  + g_0^{(3)}(x) + \sqrt{\epsilon}g_1^{(3)}(x), & \text{for } \underline{x}_1 < x < K \\
  a_2^{(3)}(\frac{x}{K})^{\beta_2} [1 + \sqrt{\epsilon}g_1(x)] + b_2^{(3)}(\frac{x}{K})^{\beta_1} [1 + \sqrt{\epsilon}f_1(x)] 
  + b_0^{(3)}(\frac{x}{K})^{\beta_1} \left[ f_0^{(2)}(x) + \sqrt{\epsilon}f_1^{(2)}(x) \right] + KR^3 - xD^3, & \text{for } \underline{x}_2 < x \leq \underline{x}_1 \\
  a_3^{(3)}(\frac{x}{K})^{\beta_2} [1 + \sqrt{\epsilon}g_1(x)] + b_3^{(3)}(\frac{x}{K})^{\beta_1} [1 + \sqrt{\epsilon}f_1(x)] + KR - xD, & \text{for } x \leq \underline{x}_3 \\
  K - x, & \text{for } x \leq \underline{x}_3,
\end{cases}
\]

where

\[
f_0^{(3)}(x) = \frac{R^2\beta_1^2\beta_2^2}{8\Delta^3}(\ln(x) + \Delta \ln(x)^2),
\]
\[
g_0^{(3)}(x) = -\frac{R^2\beta_1^2\beta_2^2}{8\Delta^3}(\ln(x) - \Delta \ln(x)^2),
\]
\[
f_1^{(3)}(x) = \frac{R^2V\beta_1^2\beta_2^2}{8\Delta^5\sigma^2}(3\gamma^2 - 3\gamma^3 - \Delta^2 + 3\gamma\Delta^2) \ln(x) + \Delta \beta_1(3\gamma - 3\gamma^2 + \Delta + 3\Delta^2) \ln(x)^2 + \beta_1^2\Delta(1 - \beta_1) \ln(x)^3,
\]
\[
g_1^{(3)}(x) = -\frac{R^2V\beta_1^2\beta_2^2}{8\Delta^5\sigma^2}(3\gamma^2 - 3\gamma^3 - \Delta^2 + 3\gamma\Delta^2) \ln(x) - \Delta \beta_2(3\gamma - 3\gamma^2 - \Delta + 3\Delta^2) \ln(x)^2 + \beta_2^2\Delta(1 - \beta_2) \ln(x)^3.
\]

The other functions are defined in section 4.3. The constants $a_1^{(3)}$, $a_2^{(3)}$, $a_3^{(3)}$, $b_1^{(3)}$, $b_2^{(3)}$, $b_3^{(3)}$, $c^{(3)}$, $d^{(3)}$, $f_1^{(3)}$, $g_1^{(3)}$, $f_0^{(3)}$, $g_0^{(3)}$, $f_1^{(2)}$, $g_1^{(2)}$, $f_0^{(2)}$, $g_0^{(2)}$, $f_0^{(1)}$, $g_0^{(1)}$, $f_1^{(1)}$, $g_1^{(1)}$, $f_0^{(0)}$, $g_0^{(0)}$, $f_1^{(0)}$, $g_1^{(0)}$, $f_0^{(-1)}$, $g_0^{(-1)}$, $f_1^{(-1)}$, $g_1^{(-1)}$, $f_0^{(-2)}$, $g_0^{(-2)}$, $f_1^{(-2)}$, $g_1^{(-2)}$, $f_0^{(-3)}$, $g_0^{(-3)}$, $f_1^{(-3)}$, $g_1^{(-3)}$, $f_0^{(-4)}$, $g_0^{(-4)}$, $f_1^{(-4)}$, $g_1^{(-4)}$, $f_0^{(-5)}$, $g_0^{(-5)}$, $f_1^{(-5)}$, $g_1^{(-5)}$
The optimal exercise boundary \( x_3 \) are determined by the boundary conditions at \( x = x_3 \) and by demanding that the pricing function is continuous and differentiable at \( x = K, x = x_1 \) and \( x = x_2 \).

For the American call, we obtain the following pricing function from first-order randomization to \( O(\sqrt{\epsilon}) \):

\[
C_0^{(1)}(x) + \sqrt{\epsilon} C_1^{(1)}(x) = \\
\begin{cases}
    c^{(1)}(x_K) \beta_2 [1 + \sqrt{\epsilon} g_1(x)], & \text{for } x \leq K \\
    a^{(1)}(x_K) \beta_2 [1 + \sqrt{\epsilon} g_1(x)] + b^{(1)}(x_K) \beta_1 [1 + \sqrt{\epsilon} f_1(x)] + xD - KR, & \text{for } K < x < x_1 \\
    x - K, & \text{for } x \geq x_1
\end{cases}
\]

(A.2.3)

where the functions \( g_1 \) and \( f_1 \) are the same as for the American put. The constants \( c^{(1)}, b^{(1)}, c^{(1)} \) as well as the optimal exercise boundary \( x_1 \) have to be evaluated by means of the boundary conditions at \( x = x_1 \) and by the continuity and differentiability conditions at \( x = K \).

The value function for the second-order randomized American call is given by

\[
C_0^{(2)}(x) + \sqrt{\epsilon} C_1^{(2)}(x) = \\
\begin{cases}
    c^{(2)}(x_K) \beta_2 [1 + \sqrt{\epsilon} g_1(x)] + c^{(1)}(x_K) \beta_2 \left[ g_0^{(2)}(x) + \sqrt{\epsilon} g_1^{(2)}(x) \right], & \text{for } x \leq K \\
    a_1^{(2)}(x_K) \beta_2 \left[ g_0^{(2)}(x) + \sqrt{\epsilon} g_1^{(2)}(x) \right] + a^{(1)}(x_K) \beta_2 \left[ g_0^{(2)}(x) + \sqrt{\epsilon} g_1^{(2)}(x) \right] + xD - KR^2, & \text{for } K < x \leq x_1 \\
    + b^{(2)}(x_K) \beta_1 \left[ f_0^{(2)}(x) + \sqrt{\epsilon} f_1^{(2)}(x) \right] + xD - KR, & \text{for } x_1 < x < x_2 \\
    a_2^{(2)}(x_K) \beta_2 [1 + \sqrt{\epsilon} g_1(x)] + b_2^{(2)}(x_K) \beta_1 \left[ 1 + \sqrt{\epsilon} f_1(x) \right] + xD - KR, & \text{for } x \geq x_2
\end{cases}
\]

(A.2.4)

Again, the functions \( f_1^{(2)} \) and \( g_1^{(2)} \) are the same as for the American put, and the constants \( c^{(2)}, a_1^{(2)}, a_2^{(2)}, b_1^{(2)}, b_2^{(2)} \) and the second-order optimal exercise boundary \( x_2 \) are determined by the boundary conditions at \( x = x_2 \) and by demanding the functions to be continuous and differentiable at \( x = K \) and \( x = x_1 \).
For the third-order randomized American call, we obtain the following pricing function:

\[ C_0^{(3)}(x) + \sqrt{\epsilon} C_1^{(3)}(x) = \]

\[
\begin{cases}
  c^{(3)}(x/K) \beta_2 [1 + \sqrt{\epsilon} g_1(x)] + c^{(2)}(x/K) \beta_2 [g_0^{(2)}(x) + \sqrt{\epsilon} g_1^{(2)}(x)] \\
  + c^{(1)}(x/K) \beta_2 [g_0^{(3)}(x) + \sqrt{\epsilon} g_1^{(3)}(x)], & \text{for } x \leq K \\
  a_1^{(3)}(x/K) \beta_2 [1 + \sqrt{\epsilon} g_1(x)] + b_1^{(3)}(x/K) \beta_1 [1 + \sqrt{\epsilon} f_1(x)] \\
  + a_1^{(2)}(x/K) \beta_2 [g_0^{(2)}(x) + \sqrt{\epsilon} g_1^{(2)}(x)] \\
  + b_1^{(2)}(x/K) \beta_1 f_0^{(2)}(x) + \sqrt{\epsilon} f_1^{(2)}(x)], & \text{for } K < x \leq \bar{x}_1 \\
  a_2^{(3)}(x/K) \beta_2 [1 + \sqrt{\epsilon} g_1(x)] + b_2^{(3)}(x/K) \beta_1 [1 + \sqrt{\epsilon} f_1(x)] \\
  + a_2^{(2)}(x/K) \beta_2 [g_0^{(2)}(x) + \sqrt{\epsilon} g_1^{(2)}(x)] \\
  + b_2^{(2)}(x/K) \beta_1 f_0^{(2)}(x) + \sqrt{\epsilon} f_1^{(2)}(x)], & \text{for } \bar{x}_1 < x \leq \bar{x}_2 \\
  a_3^{(3)}(x/K) \beta_2 [1 + \sqrt{\epsilon} g_1(x)] + b_3^{(3)}(x/K) \beta_1 [1 + \sqrt{\epsilon} f_1(x)] + xD - KR, & \text{for } \bar{x}_2 < x < \bar{x}_3 \\
  x - K, & \text{for } x \geq \bar{x}_3
\end{cases}
\]

(A.2.5)

with the functions \( f^{(3)} \) and \( g^{(3)} \) from the third-order randomized American put in Eq. (A.2.2). The constants \( a_1^{(3)}, a_2^{(3)}, a_3^{(3)}, b_1^{(3)}, b_2^{(3)}, b_3^{(3)}, c^{(3)} \) and the optimal exercise boundary \( \bar{x}_3 \) are determined by the boundary conditions at \( x = \bar{x}_3 \) and by demanding that the pricing function is continuous and differentiable at \( x = K, x = \bar{x}_1 \) and \( x = \bar{x}_2 \).
Bibliography


