Warm-up

- Differentiate $x^{17} - x^{-17}$ with respect to x.
  - The derivative of x^a is ax^{a-1} so the derivative of this expression is $17x^{16} + 17x^{-18}$.
- Differentiate $2\sqrt{x} + 3\sqrt[3]{x}$ with respect to x.
  - Remember that $\sqrt{x} = x^{1/2}$ and $\sqrt[3]{x} = x^{1/3}$, so the derivative of this expression is $x^{-1/2} + x^{-2/3}$, which we could write as $\frac{1}{\sqrt{x}} + \frac{1}{x^{2/3}}$ if we wanted to.
- Differentiate $1 - e^{3x}$ with respect to x.
  - Remember that the derivative of a constant is 0, so the derivative of this expression is just $-3e^{3x}$.
- Find the tangent to the curve $y = e^x + x^2$ at $x = 2$.
  - We need to find the value of the derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $x = 2$ because that's equal to the gradient of the tangent. We can differentiate to find $\frac{\mathrm{d}y}{\mathrm{d}x} = e^x + 2x$ so that gradient we wanted is $e^2 + 4$. We also want the tangent to have the same value at $x = 2$ as the curve; that's $e^x + x^2$ at $x = 2$, which is also $e^2 + 4$. So our tangent is $y = (e^2 + 4)(x - 1)$.
- Find the normal to the parabola $y = x^2$ at $x = 3$.
  - First find the derivative at $x = 3$, which is 6 for this parabola. That's the gradient of the tangent, and the normal is at right-angles to the tangent, so it has gradient $-\frac{1}{6}$. We have $y = -\frac{x}{6} + c$ and we want the normal to go through the point $(3, 9)$. So we want $c = \frac{19}{2}$.
- Find the turning points of the curve $y = x^4 - 2x^3 + x^2$. Identify whether the turning points are maxima or minima. For which values of x is $y = x^4 - 2x^3 + x^2$ increasing? For which values of x is it decreasing?
  - The turning points must have $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ so we must have $4x^3 - 6x^2 + 2x = 0$. That happens when $x = 0$ or when $2x^2 - 3x + 1 = 0$ which happens when $(2x - 1)(x - 1) = 0$, which is either $x = 1$ or $x = \frac{1}{2}$.
  - Now find the second derivative to check whether these are minima or maxima. $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 12x^2 - 12x + 2$, which is positive for $x = 0$, negative for $x = \frac{1}{2}$, and positive for $x = 1$. So we have a (local) minimum, then a (local) maximum, then a (local) minimum.
  - The function is decreasing for $x < 0$, then increasing for $0 < x < \frac{1}{2}$, then decreasing for $\frac{1}{2} < x < 1$ then increasing for $x > 1$.

- Two points A and B are on the curve $y = x^3 + x^2 + x + 1$. A is held fixed at $(1, 4)$. The point B is moved along the curve towards A. What happens to the line through A and B?
  - The line definitely goes through A, which doesn't move. The thing we learn from "differentiation from first principles" is that the gradient of the line gets closer and closer to the derivative of the function at A.
  - The derivative is $3x^2 + 2x + 1$ which is 6 at $x = 1$. The value is 4, so the tangent is $y = 6x - 2$. So if the line through A and B is $y = mx + c$ then m gets closer and closer to 6 and c gets closer and closer to -2.
- Suppose that the derivative of a polynomial $p(x)$ with respect to x is $q(x)$. Find an expression for $\int q(x)\,\mathrm{d}x$.
  - This must be $p(x) + c$ for some constant c.
- Find the area enclosed between $x = -1$ and $x = 1$ by the polynomial $y = x^4 - 4x^2 + 3$ and the x-axis.
  - First check if there are any points in that range where $y = 0$. This function is a quadratic in x^2, with roots $x^2 = 1$ or $x^2 = 3$. So $y = 0$ at $x = \pm 1$ but not in between. In between, we have $y > 0$ (I checked the value at $x = 0$).
  - So we want $\displaystyle\int_{-1}^{1} x^4 - 4x^2 + 3\,\mathrm{d}x = \left[\frac{x^5}{5} - \frac{4x^3}{3} + 3x\right]_{-1}^{1} = \frac{2}{5} - \frac{8}{3} + 6 = \frac{56}{15}$.
- Find $\displaystyle\int_{-1}^{1} 1 + x + x^2 + x^3 + x^4 + x^5 + x^6\,\mathrm{d}x$.
  - This is $\left[x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7}\right]_{-1}^{1} = 2 + \frac{2}{3} + \frac{2}{5} + \frac{2}{7} = \frac{352}{105}$.
- Find $\displaystyle\int_{-\pi}^{\pi} x^{2021}\,\mathrm{d}x$.
  - This is $\left[\frac{x^{2022}}{2022}\right]_{-\pi}^{\pi} = 0$. In general, any "odd function" with $f(-x) = -f(x)$ which we integrate from $-a$ to a for any real number a will give zero.
- Find

$$\int \frac{x+3}{x^3}\,\mathrm{d}x, \qquad \int \sqrt[3]{x}\,\mathrm{d}x, \qquad \int \left(\left(x^2\right)^3\right)^5\,\mathrm{d}x, \qquad \int \left(x^2+1\right)^3\,\mathrm{d}x$$

  - $\displaystyle\int \frac{x+3}{x^3}\,\mathrm{d}x = \int \frac{1}{x^2} + \frac{3}{x^3},\mathrm{d}x = -\frac{1}{x} - \frac{3}{2x^2} + c$ where c is a constant.

  - $\displaystyle\int \sqrt[3]{x}\,\mathrm{d}x = \int x^{1/3}\,\mathrm{d}x = \frac{3}{4}x^{4/3} + c$ where c is a constant.
  - $\displaystyle\int \left(\left(x^2\right)^3\right)^5\,\mathrm{d}x = \int x^{30}\,\mathrm{d}x = \frac{x^{31}}{31} + c$ where c is a constant.
  - $\displaystyle\int \left(x^2+1\right)^3\,\mathrm{d}x = \int x^6 + 3x^4 + 3x^2 + 1\,\mathrm{d}x = \frac{x^7}{7} + \frac{3x^5}{5} + x^3 + x + c$ where c is a constant.

- By thinking about the area that the integral represents, explain why

$$\int_{-1}^{1} f(x)\,\mathrm{d}x = \int_{-1}^{1} f(-x)\,\mathrm{d}x.$$

  - The graph of $f(-x)$ is the graph of $f(x)$ reflected in y-axis. Also, note that if we reflect the interval $-1 \leqslant x \leqslant 1$ in the y-axis then we get the same interval back. On the left-hand side, we're finding the area under $f(x)$ (or maybe negative the area in any regions where f is negative). On the right-hand side, we're calculating exactly the same area, but with the shape of the graph reflected.

MAT questions

MAT 2014 Q1C

- First work out the derivative $3kx^2 - 2(k+1)x + (2-k)$ and then the second derivative $6kx - 2(k+1)$.
- At $x = 1$, the first derivative is $3k - 2(k+1) + (2-k)$, which is zero for any k, and the second derivative is $6k - 2(k+1) = 4k - 2$.
- For this to be a local minimum we would need zero derivative (which we have!) and we would need the second derivative to be positive, so we would need $k > \frac{1}{2}$.
- The answer is (c).

MAT 2017 Q1A

- Let's look for the stationary points by working out the derivative $6x^2 - 2kx + 2$.
- That's got two distinct roots if the discriminant $4k^2 - 48$ is positive, which happens if $k^2 > 12$.
- Technically we should probably check that the second derivative isn't zero at these points, but since the first derivative is a quadratic, it can only have zero derivative at a root if there's a single repeated root (which is not the case).
- The answer is (b).

MAT 2014 Q1J

- This question is tough. We're not going to work out what the function $f(x)$ is! We're just going to work out precisely the information $\int_{-1}^{1} f(x)\,\mathrm{d}x$. Let's give that number a name (it is a number, not a function of x). I'm going to call it A.
- Now the equation we've been given is

$$6 + f(x) = 2f(-x) + 3Ax^2$$

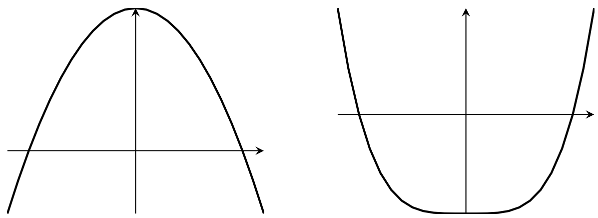
- I don't know much about $f(x)$, and frankly I don't care much about the details of $f(x)$, I just want to find A. I've noticed that if I integrate the left-hand side of the equation, I get $12 + A$.
- If I integrate the right-hand side I get

$$2\int_{-1}^{1} f(-x)\,\mathrm{d}x + 3A\int_{-1}^{1} x^2\,\mathrm{d}x.$$

  I've moved the A outside of the integral there because it's just a constant.
- Now note that $\int_{-1}^{1} f(-x)\,\mathrm{d}x = \int_{-1}^{1} f(x)\,\mathrm{d}x$; we talked about this in the warm-up. That's A.
- So we have $12 + A = 2A + 3A\int_{-1}^{1} x^2\,\mathrm{d}x$. We're almost done, I just need to calculate one integral and I'll get an equation for A.
- I get $12 + A = 2A + 2A$ so $A = 4$.
- The answer is (a).

MAT 2016 Q1H

- First, let's calculate the two areas in the question.



- The x-axis and $f(x)$ enclose a region with $-\sqrt{a} < x < \sqrt{a}$ where $f(x)$ is positive. The area is
$$\int_{-\sqrt{a}}^{\sqrt{a}} a - x^2 \, dx = 2a^{3/2} - \frac{2}{3}a^{3/2} = \frac{4}{3}a^{3/2}.$$

- The x-axis and $g(x)$ enclose a region with $-a^{1/4} < x < a^{1/4}$ where $g(x)$ is negative. The area is
$$-\int_{-a^{1/4}}^{a^{1/4}} x^4 - a \, dx = -\frac{2}{5}a^{5/4} + 2a^{5/4} = \frac{8}{5}a^{5/4}$$
.

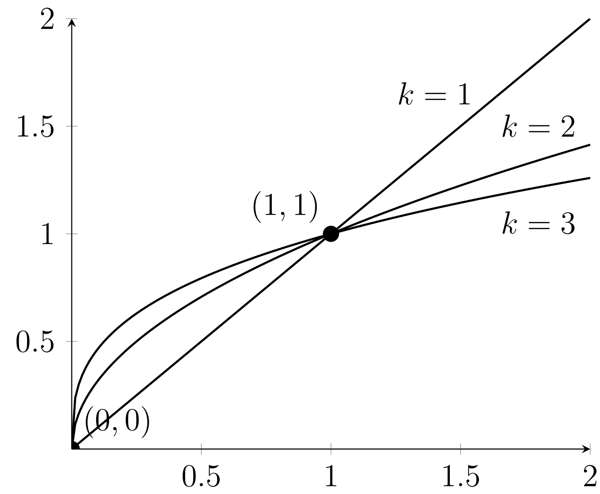
- We want to find out when the first area is bigger. We need
$$\frac{4}{3}a^{3/2} > \frac{8}{5}a^{5/4}.$$

- This rearranges to $a > \left(\frac{6}{5}\right)^4$.

- The answer is (e).

MAT 2017 Q3

(i) Here's a sketch



Note that all the curves pass through the point $(0, 0)$ and through $(1, 1)$. For $0 < x < 1$ the curve with the highest value of k is the largest, but for $x > 1$ the curve with the lowest value of k is the largest.

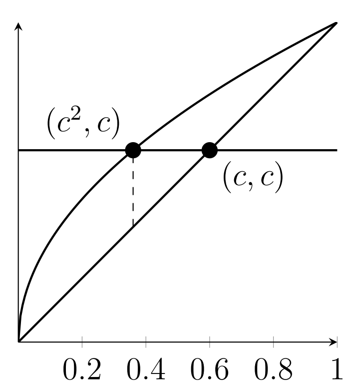
(ii) For $0 < x < 1$, the curve $y = f_{k+1}(x)$ has a larger value than the curve $y = f_k(x)$. The area we want is

$$\int_0^1 f_{k+1}(x) - f_k(x)\,\mathrm{d}x = \int_0^1 x^{1/(k+1)} - x^{1/k}\,\mathrm{d}x = \left[\frac{x^{1+1/(k+1)}}{1 + 1/(k+1)} - \frac{x^{1+1/k}}{1 + 1/k}\right]_0^1 = \frac{k+1}{k+2} - \frac{k}{k+1}.$$

We can check that if $k = 1$ then this is $\frac{2}{3} - \frac{1}{2} = \frac{1}{6}$.

(iii) Remember that $y = f_1(x)$ is just $y = x$. This intersects the line $y = c$ for $x = c$, so the point of intersection is (c, c). The curve $y = f_2(x)$ is just $y = x^{1/2}$ so the point of intersection with the line $y = c$ is (c^2, c).

(iv) Here's a sketch



We can write the area between the curves and under $y = c$ in two parts as

$$A = \int_0^{c^2} x^{1/2} - x \, \mathrm{d}x + \int_{c^2}^{c} c - x \, \mathrm{d}x.$$

We can calculate these integrals to get (after simplifying)

$$A = \frac{c^2}{2} - \frac{c^3}{3}.$$

Now for the two regions to have equal area, they must each have area $\frac{1}{12}$, because we found in part (ii) that the overall area is $\frac{1}{6}$. This gives the equation $\frac{1}{12} = \frac{c^2}{2} - \frac{c^3}{3}$ which we can rearrange for $4c^3 - 6c^2 + 1 = 0$.

We know that $0 < c < 1$, so let's try guessing simple fractions as roots of this cubic. It turns out that $c = \frac{1}{2}$ works. Perhaps we could factorise the cubic to check that this is the only root in that range, but it's easier to argue geometrically; as c increases from zero to one, the area under the line $y = c$ only increases, so we can't have two different values of c which split the region in half.

Reflection

- Splitting the area up into multiple parts is a good tactic. In the long question, I split the area up into a region to the left of a point of intersection and a separate region to the right. I could have spotted that the region to the right was just a right-angled isosceles triangle, which might have been a faster way to get the area.
- If you've never seen something like $\int_{-1}^{1} f(t) \, \mathrm{d}t$ before, it's worth thinking about what this means and why it's the same thing as $\int_{-1}^{1} f(x) \, \mathrm{d}x$.
- Drawing sketches is still important, even if the question doesn't ask you to!