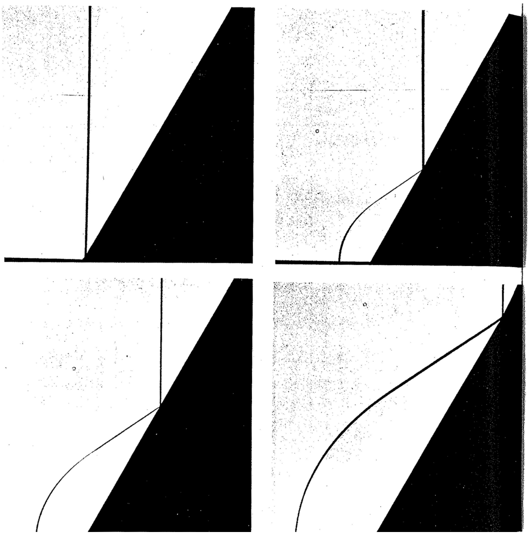
Shock reflection problem: existence and stability of global solutions

Mikhail Feldman, University of Wisconsin-Madison

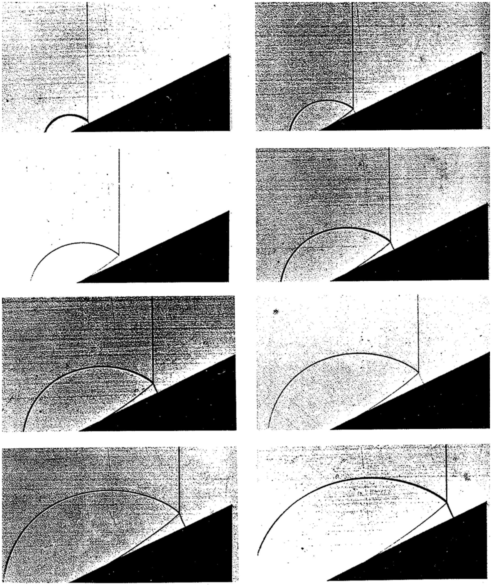
Based on works with
Gui-Qiang Chen (Oxford),
Wei Xiang(Hong Kong), and Myoungjean Bae(KAIST)

July 22, 2022

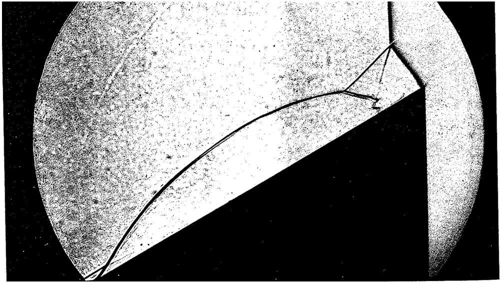
Shock reflection by a wedge: Regular reflection



Shock reflection by a wedge: Mach reflection



Shock reflection by a wedge: Irregular Mach reflection.



Self-similar flow: $(\vec{u}, p, \rho)(x,t) = (\vec{u}, p, \rho)(\frac{x}{t})$.

Shock reflection

First described by E. Mach 1878. Reflection patterns: Regular reflection, Mach reflection.

J. von Neumann, 1940s: on transition between patterns

Later works: experimental, computational. Asymptotic analysis: Lighthill, Keller, Blank, Hunter, Harabetian, Morawetz.
Reference: book by J. Glimm and A. Majda, survey by D. Serre.

Analysis: Special models (Transonic small disturbance eq., pressure-gradient system, nonlinear wave eq.): Gamba, Rosales, Tabak, Canic, Keyfitz, Kim, Lieberman, Y.Zheng, G.-Q. Chen-W. Xiang.

Local existence results: S.-X. Chen.

More recent results for potential flow:

Existence of global shock reflection solutions for potential flow: G.-Q.Chen-F., Elling

The complete up-to-date results on existence of regular reflection solutions and their proofs are presented in the monograph "The Mathematics of Shock Reflection-Diffraction and von Neumann conjectures" by G.-Q.Chen-F, 2018.

Other self-similar shock reflection problems:

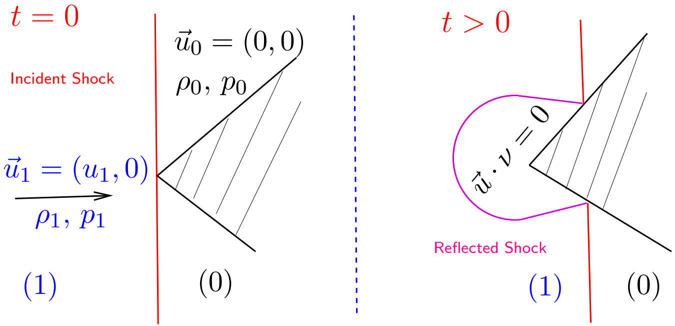
Prandtl Reflection: Elling-Liu, Bae-G.-Q.Chen-F

Shock interactions/reflection for Chaplygin gas: D. Serre

Properties of solutions of self-similar reflection problems: Bae-G.-Q.Chen-F, G.-Q. Chen-F.-W. Xiang, Elling.

Stability and uniqueness of regular reflection solutions. G.-Q. Chen-F.-W. Xiang.

Shock reflection as a Riemann problem in domain with boundary, with slip boundary conditions



Initial data: Constant (uniform) states (0) and (1):
State (0): velocity $\vec{u}_0 = (0,0)$, density ρ_0, pressure p_0.
State (1): velocity $\vec{u}_1 = (u_1, 0)$, density ρ_1, pressure p_1.

$t > 0$: Self-similar solution: $(\vec{u}, \rho, p) = (\vec{u}, \rho, p)(\vec{\xi})$, where
$\vec{\xi} = \dfrac{\vec{x}}{t}$.

Equations of gas dynamics

Isentropic Compressible Euler system:

$$\partial_t \rho + \mathbf{div}(\rho\, \vec{u}) = 0,$$
$$\partial_t (\rho \vec{u}) + \mathbf{div}(\rho\, \vec{u} \otimes \vec{u}) + \nabla p = 0,$$

where:

$\vec{u} = (u_1, u_2)$ – velocity

ρ – density

$p = \rho^\gamma$ – pressure

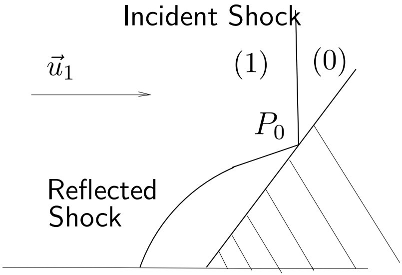
$\gamma > 1$ – adiabatic exponent (it is a given constant)

Potential flow: Conservation of mass, Bernoulli's law

$$\rho_t + \mathbf{div}(\rho\, \nabla \Phi) = 0,$$
$$\Phi_t + \frac{1}{2} |\nabla \Phi|^2 + \frac{\rho^{\gamma - 1} - 1}{\gamma - 1} = const$$

where Φ – velocity potential: $\vec{u} = \nabla_x \Phi$.

Regular reflection in self-similar coordinates $\vec{\xi} = \frac{\vec{x}}{t}$



Given:
State (0): velocity $\vec{u}_0 = (0,0)$, density ρ_0, pressure p_0.
State (1): velocity $\vec{u}_1 = (u_1, 0)$, density ρ_1, pressure p_1.

Problem: Find self-similar solution: $(\vec{u}, \rho, p) = (\vec{u}, \rho, p)(\vec{\xi})$, where $\vec{\xi} = \dfrac{\vec{x}}{t}$, with asymptotic conditions at infinity determined by states (0) and (1), and satisfying $u \cdot \nu = 0$ on the boundary.

Self-similar potential flow

$\Phi(\vec{x},t) = t\psi(\xi,\eta)$, $\rho(\vec{x},t) = \rho(\xi,\eta)$ with $(\xi,\eta) = \frac{\vec{x}}{t} \in \mathbb{R}^2$.

Pseudo-potential: $\varphi = \psi - \frac{1}{2}(\xi^2 + \eta^2)$.

Equation for φ:

$$\text{div}\,\big(\rho(|\nabla\varphi|^2,\varphi)\nabla\varphi\big) + 2\rho(|\nabla\varphi|^2,\varphi) = 0,$$

with $$\rho(|\nabla\varphi|^2,\varphi) = \big(K - (\gamma-1)(\varphi + \frac{1}{2}|\nabla\varphi|^2)\big)^{\frac{1}{\gamma-1}}.$$

Equation is of mixed type:

elliptic $|\nabla\varphi| < c(|\nabla\varphi|^2,\varphi,K)$,

hyperbolic $|\nabla\varphi| > c(|\nabla\varphi|^2,\varphi,K)$,

where **speed of sound** c is:

$$c^2 = \rho^{\gamma-1} = K - (\gamma-1)(\varphi + \frac{1}{2}|\nabla\varphi|^2).$$

Uniform states

Solutions with constant (physical) velocity (u, v):

$$\varphi(\xi, \eta) = -\frac{\xi^2 + \eta^2}{2} + u\xi + v\eta + const.$$

Any such function is a solution.
Also (from formula) density $\rho(\nabla\varphi, \varphi) = const$, thus sonic speed $c = \rho^{\frac{\gamma-1}{2}} = const$. Then ellipticity region

$$|\nabla\varphi(\xi, \eta)| = |(u, v) - (\xi, \eta)| < c$$

is circle, centered at (u, v), radius c.

Shocks, RH conditions, Entropy condition

Shocks are discontinuities in the pseudo-velocity $\nabla\varphi$:

if Ω^+ and $\Omega^- := \Omega \setminus \overline{\Omega^+}$ are nonempty and open, and $S := \partial\Omega^+ \cap \Omega$ is a C^1 curve where $\nabla\varphi$ has a jump, then $\varphi \in C^1(\Omega^\pm \cup S) \cap C^2(\Omega^\pm)$ is a global weak solution of

$$\text{div}\left(\rho(|\nabla\varphi|^2, \varphi)\nabla\varphi\right) + 2\rho(|\nabla\varphi|^2, \varphi) = 0,$$

with $\quad \rho(|\nabla\varphi|^2, \varphi) = \left(K - (\gamma - 1)(\varphi + \frac{1}{2}|\nabla\varphi|^2)\right)^{\frac{1}{\gamma-1}}.$

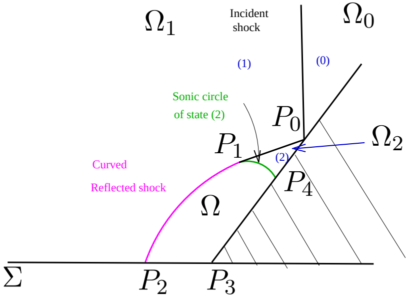
in Ω if and only if φ satisfies potential flow equation in $\Omega^\pm$ and the Rankine-Hugoniot (RH) condition on S:

$$[\varphi]_S = 0,$$
$$\left[\rho(|\nabla\varphi|^2, \varphi)\nabla\varphi \cdot \nu\right]_S = 0,$$

where $[\cdot]_S$ is jump across S.

Entropy Condition on S: density increases across S in the pseudo-flow direction.

Shock reflection as a free boundary problem



$$\text{div}\left(\rho(|\nabla\varphi|^2,\varphi)\nabla\varphi\right) + 2\rho(|\nabla\varphi|^2,\varphi) = 0 \quad \text{in } \Omega,$$

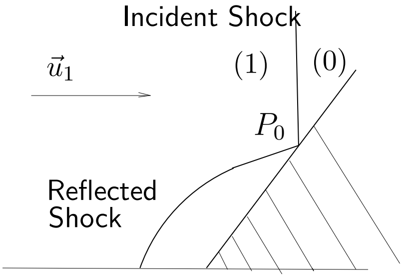
$$\left.\begin{array}{l} \rho(|\nabla\varphi|^2,\varphi)\nabla\varphi\cdot\nu = \rho(|\nabla\varphi_1|^2,\varphi_1)\nabla\varphi_1\cdot\nu \\ \varphi = \varphi_1 \end{array}\right\} \text{ on } P_1P_2$$

$\varphi = \varphi_2$ on P_1P_4 (and prove $D_\nu\varphi = D_\nu\varphi_2$ on P_1P_4)

$\varphi_\nu = 0$ on Wedge P_3P_4, Symmetry line P_2P_3,

Solve for: Free boundary P_1P_2 and function φ in Ω.
Expect equation elliptic in Ω.

Regular reflection, state (2)



$\varphi =$ pseudo-potential between the reflected shock and the wall
$\varphi_1 =$ pseudo-potential of state (1)

Denote $\nabla\phi(P_0) = (u_2, v_2)$, where $\phi = \varphi + \dfrac{\xi^2 + \eta^2}{2}$. Since $\varphi_\nu = 0$ on wedge, then $v_2 = u_2 \tan\theta_w$. Here θ_w is wedge angle.

Rankine-Hugoniot conditions at reflection point P_0, for φ and φ_1: algebraic equations for u_2, $\varphi(P_0)$

Regular reflection, state (2), detachment angle

If solution exists: Let

$$\varphi_2(\xi,\eta) = -(\xi^2+\eta^2)/2 + u_2\xi + v_2\eta + C,$$

where C determined by $\varphi_2(P_0) = \varphi_1(P_0)$.

Existence of state (2) is necessary condition for existence of regular reflection

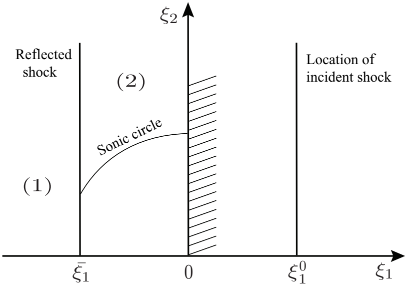
Given γ, ρ_0, ρ_1, there exists $\theta_{detach} \in (0, \frac{\pi}{2})$ such that:

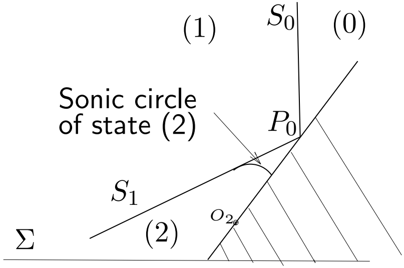
state (2) exists for $\theta_w \in (\theta_{detach}, \frac{\pi}{2})$,
state (2) does not exist for $\theta_w \in (0, \theta_{detach})$.

If φ_2 exist, then RH is satisfied along the line
$S_1 := \{\varphi_1 = \varphi_2\}$.

Weak and Strong State (2); Sonic angle

For each $\theta_w \in (\theta_{detach}, \frac{\pi}{2})$ there exists two possible States (2): weak and strong, with $\rho_2^{weak} < \rho_2^{strong}$. We always choose weak state (2). For strong state (2), existence of global regular reflection solution is not expected, Elling (2011) confirms that.





There exist $\theta_{sonic} \in (\theta_{detach}, \frac{\pi}{2})$ such that:
State 2 is supersonic at P_0 for $\theta_w \in (\theta_{sonic}, \frac{\pi}{2})$.
State 2 is subsonic at P_0 for $\theta_w \in (\theta_{detach}, \hat{\theta}_{sonic})$ where $\hat{\theta}_{sonic} \in (\theta_{detach}, \theta_{sonic}]$.

Von Neumann's conjectures on transition between different reflection patterns

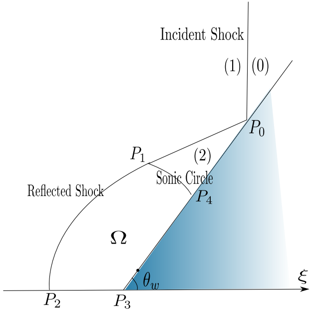
Recall: sonic angle θ_{sonic} and detachment angle θ_{detach} satisfy $0 < \theta_{detach} < \theta_{sonic} < \frac{\pi}{2}$.

Sonic conjecture:
Regular reflection for $\theta_w \in (\theta_{sonic}, \frac{\pi}{2})$, Mach reflection for $\theta_w < \theta_{sonic}$.

Von Neumann's detachment conjecture:
Regular reflection for $\theta_w \in (\theta_{detach}, \frac{\pi}{2})$, Mach reflection for $\theta_w < \theta_{detach}$.

G.-Q. Chen - F.(2018): existence of regular reflection for $\theta_w \in (\theta_{detach}, \frac{\pi}{2})$ for potential flow equation if ρ_0, ρ_1 satisfy $u_1 \leq c_1$ (weaker incident shocks), and up to a critical angle otherwise. Given $\rho_0 > 0$ there exists $\rho_1^* > \rho_0$ such that $u_1 < c_1$ for $\rho_1 \in (\rho_0, \rho_1^*)$ and $u_1 > c_1$ for $\rho_1 > \rho_1^*$.
Structure: supersonic and subsonic regular reflections.

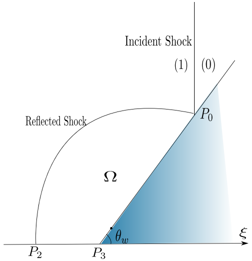
Supersonic regular reflection



Supersonic regular reflection: State (2) is supersonic at P_0.
Structure of solution φ:

- $\varphi = \varphi_i$ in Ω_i, i=0,1,2.
- $\varphi \in C^1(\overline{P_0 P_2 P_3})$, in particular C^1 across sonic arc $P_1 P_4$.
- Shock $P_0 P_2$ has flat part $P_0 P_1$, curved part $P_1 P_2$, and is C^1 across P_1.
- Equation is strictly elliptic in $\overline{\Omega} \setminus \overline{P_1 P_4}$.

Subsonic regular reflection

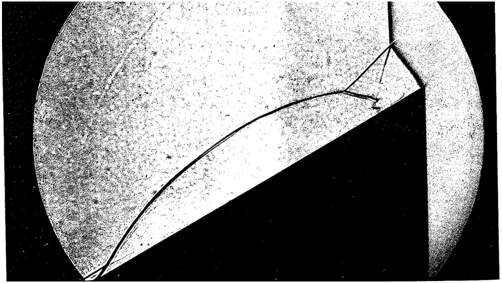


Subsonic regular reflection: State (2) is subsonic at P_0.
Structure of solution φ:

- $\varphi = \varphi_i$ in Ω_i, i=0,1.
- $\varphi \in C^1(\overline{P_0 P_2 P_3})$.
- $\varphi = \varphi_2$, $D\varphi = D\varphi_2$ at P_0.
- Shock $P_0 P_2$ is C^1.
- Equation is strictly elliptic in $\overline{\Omega} \setminus \{P_0\}$.

Attached shocks and cases $u_1 \leq c_1$, $u_1 > c_1$

Issue: experiments indicate that shock can hit the corner of wedge in certain cases:
For irregular Mach reflection see Fig. 238 (page 144) of M. Van Dyke, *An Album of Fluid Motion*, The Parabolic Press: Stanford, 1982.

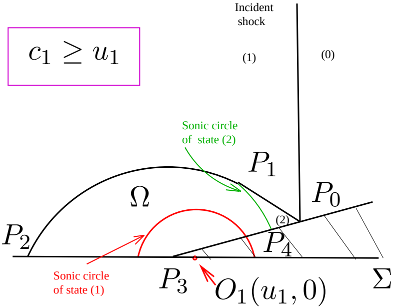


Attached shocks and cases $u_1 \leq c_1$, $u_1 > c_1$

Independent parameters are densities $\rho_1 > \rho_0$. Velocity $u_1 = u_1(\rho_0, \rho_1)$. Also, $c_1 = \rho_1^{\gamma-1}$.

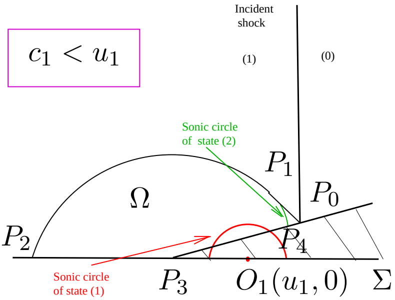
For each $\rho_0 > 0$ there exists $\rho^* > \rho_0$ such that
$u_1 < c_1$ if $\rho_1 \in (\rho_0, \rho^*)$ (weaker incident shock);
$u_1 > c_1$ if $\rho_1 > \rho^*$. (stronger incident shock)

We show: attached shocks do not occur for regular reflection if $u_1 \leq c_1$, since Sonic Circle of State (1) separates shock and P_3:



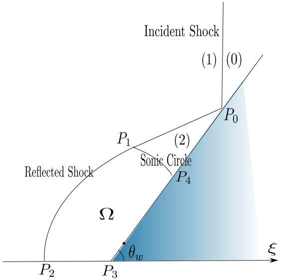
Attached shocks and cases $u_1 \le c_1$, $u_1 > c_1$

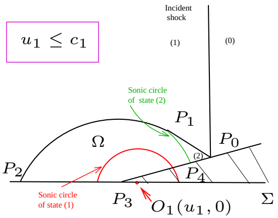
In the case $u_1 > c_1$, Sonic Circle of State (1) does not separate shock and P_3:



Then we cannot exclude possibility of "attached shocks" with $P_2 = P_3$. In case of Mach reflection, such configurations are obtained in experiments.

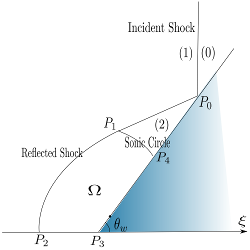
Existence of regular reflection solutions, case $u_1 \leq c_1$.





Theorem 1. (G.-Q. Chen-F.). If $\rho_1 > \rho_0 > 0$, $\gamma > 1$ satisfy $u_1 \leq c_1$, then a regular reflection solution φ exists for all wedge angles $\theta_w \in (\theta_{detach}, \frac{\pi}{2})$. The type of reflection (supersonic or subsonic) for each θ_w is determined by the type of State 2 at the reflection point P_0 for θ_w. Moreover, solution satisfies the following additional properties:

Properties of solution: supersonic case



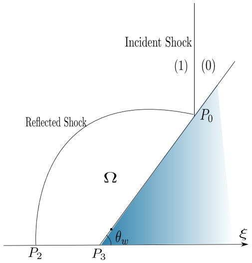
1) Equation is elliptic for φ in Ω, ellipticity degenerates near sonic arc P_1P_4.

2) φ is $C^{1,1}$ near and across the sonic arc P_1P_4;

3) Reflected shock is $C^{2,\beta}$, and a graph for a cone of directions $Con(\vec{e}_\eta, \vec{e}_{S_1})$ between $\vec{e}_\eta = (0,1)$ and $\vec{e}_{S_1} = P_0P_1$;

4) $\varphi_2 \leq \varphi \leq \varphi_1$ in Ω, and $\partial_e(\varphi_1 - \varphi) < 0$ if $e \in Con(\vec{e}_\eta, \vec{e}_{S_1})$.

Properties of solution: subsonic case



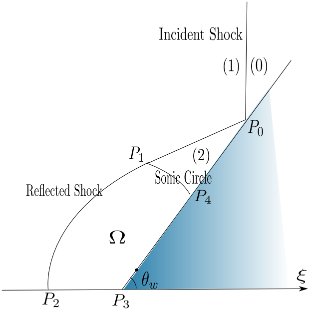
1) Equation is elliptic for φ in Ω, except for the sonic wedge angle (then ellipticity degenerates at P_0).

2) φ is $C^{2,\alpha}$ inside Ω, and $C^{1,\alpha}$ near and up to the reflection point P_0, and $\varphi = \varphi_2$, $D\varphi = D\varphi_2$ at P_0;

3) Reflected shock is $C^{2,\alpha}$ away from P_0 and $C^{1,\alpha}$ up to P_0, and a graph for a cone of directions $Con(\vec{e}_\eta, \vec{e}_{S_1})$;

4) $\varphi_2 \le \varphi \le \varphi_1$ in Ω, and $\partial_e(\varphi_1 - \varphi) < 0$ if $e \in Con(\vec{e}_\eta, \vec{e}_{S_1})$.

Stability of normal reflection as $\theta_w \to \pi/2$



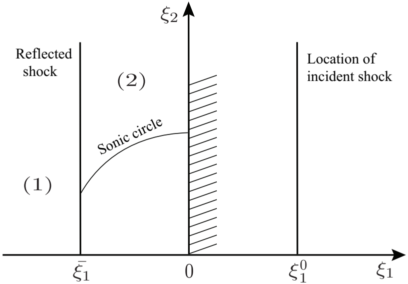
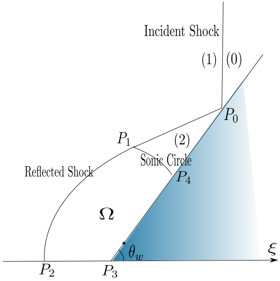
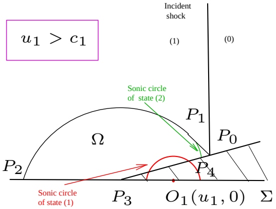


Figure: Normal reflection

Furthermore, the solutions $\varphi^{(\theta_w)}$ converge in $W^{1,1}_{loc}$ to the solution of the normal refection as $\theta_w \to \pi/2$.

Existence of regular reflection solutions, case $u_1 > c_1$.



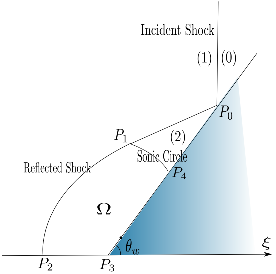


Theorem 1'. (G.-Q. Chen-F.). If $\rho_1 > \rho_0 > 0$, $\gamma > 1$ satisfy $u_1 > c_1$, then a regular reflection solution φ described in Th. 1 exists for all wedge angles $\theta_w \in (\theta_c, \frac{\pi}{2})$, where

-either $\theta_c = \theta_{detach}$,

-or $\theta_c > \theta_{detach}$ and for $\theta_w = \theta_c$ there exists an attached weak solution of regular reflection problem, i.e. with $P_2 = P_3$.

Regularity in Ω near sonic arc (supersonic case)



Theorem 3 (M. Bae-G.-Q. Chen-F.). For any admissible solution φ of supersonic reflection structure:

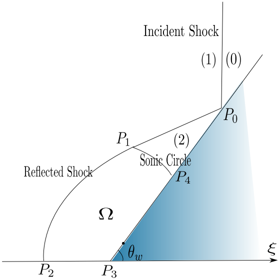
1) For every P in sonic arc $(P_1P_4]$ (i.e. excluding P_1)
$\varphi \in C^{2,\alpha}(\overline{\Omega} \cap B_R(P))$, for some small $R > 0$, any $\alpha \in (0,1)$.

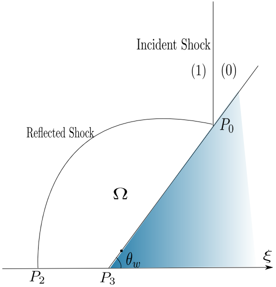
2) $D^2\varphi$ has a jump across sonic arc P_1P_4:

$D_{rr}\varphi_{|\Omega} - D_{rr}\varphi_2 = \frac{1}{\gamma+1}$ on arc$(P_1P_4]$
Thus φ is $C^{1,1}$ but *not* C^2 across sonic arc,

3) $D^2\varphi$ in Ω does *not* have a limit at P_1.

Shock reflection: free boundary problem





$$\text{div}\left(\rho(|\nabla\varphi|^2,\varphi)\nabla\varphi\right) + 2\rho(|\nabla\varphi|^2,\varphi) = 0 \quad \text{in } \Omega,$$

$$\left.\begin{array}{l} \rho(|\nabla\varphi|^2,\varphi)\nabla\varphi\cdot\nu = \rho(|\nabla\varphi_1|^2,\varphi_1)\nabla\varphi_1\cdot\nu \\ \varphi = \varphi_1 \end{array}\right\} \text{ on } P_1P_2$$

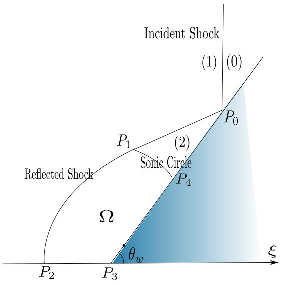
$\varphi = \varphi_2$ on P_1P_4 (and prove $D_\nu\varphi = D_\nu\varphi_2$ on P_1P_4)

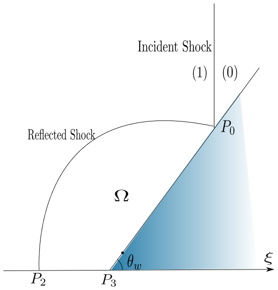
$\varphi_\nu = 0$ on Wedge P_3P_4, Symmetry line P_2P_3,

For subsonic reflection: $\varphi = \varphi_2$ and $D\varphi = D\varphi_2$ at P_0.

Solve for: Free boundary P_1P_2 (resp. P_0P_2 for subsonic case) and function φ in Ω. Expect equation elliptic in Ω.

Proof of Th. 1, 1' is obtained by solving free boundary problem using method of continuity/degree theory in the set of "admissible solutions"





Admissible solutions:

(a) Have structure supersonic or subsonic reflections depending on θ_w. Recall: this includes ellipticity in Ω and some regularity of P_0P_2 and of φ in $\overline{P_0P_2P_3}$;

(b) $\varphi_2 \leq \varphi \leq \varphi_1$ in Ω;

(c) satisfy nonstrict monotonicity $\partial_e(\varphi_1 - \varphi) \leq 0$ in Ω for any $e \in Con(e_\eta, e_{S_1})$.

Solving FBP

- Prove strict monotonicity of $\varphi_1 - \varphi$ for each direction $e \in \text{Cone}(e_\eta, e_{S_1}) \Longrightarrow \Gamma_{shock}$ is a graph, Lip$[\Gamma_{shock}] \leq C$.
- Derive basic uniform apriori estimates for admissible solutions: $\|\varphi\|_{C^{0,1}(\Omega)} \leq C$, diam$(\Omega) \leq C$, $0 < \rho_{min} \leq \rho(\nabla\varphi, \varphi) \leq \rho_{\max}$.
- Prove geometric properties of the free boundary Γ_{shock}: Uniform estimates on separation of shock with wedge and the symmetry line, uniform lower bound dist$(\Gamma_{shock}, B_{c_1}(O_1)) \geq \frac{1}{C}$.
- Prove "ellipticity"$(\xi, \eta) \geq \frac{1}{C}$dist$((\xi, \eta), \Gamma_{sonic})$.
- Derive apriori estimates for φ in weighted/scaled $C^{2,\alpha}$ in $\overline{\Omega}$, including for degenerate elliptic region near sonic arc.
- Use method of continuity/degree theory to prove existence of admissible solutions for each wedge angle up to the detachment angle θ_{detach} if $u_1 \leq c_1$, or up to the critical angle θ_c if $u_1 > c_1$, with "attached solution" for θ_c if $\theta_c \geq \theta_{detach}$.

Convexity of shock, uniqueness

Theorem 3. (Chen-F.-W. Xiang) For admissible solutions, shock is strictly convex in its relative interior.
Moreover, regular reflection solution satisfying (a)-(b) of the definition of admissible solutions, have cone of monotonicity (c) if and only if the shock is (strictly) convex.

Based on Theorem 2, we prove:

Theorem 4. (Chen-F.-Xiang) Admissible solutions are unique (and exist, by Thms. 1, 2).

Corollary. (Chen-F.-Xiang) Regular reflections solutions with convex shocks are unique (and exist by Thms. 1, 2).

To put these results in a wider context, compare them with the known results on uniqueness/nonuniqueness for 2-D Riemann problems in whole space: Similar to that case, we show uniqueness of self-similar solutions of the prescribed structure (regular reflections; convex shocks).

Outline of proof of uniqueness

We prove uniqueness of admissible solutions (thus with convex shock).

By Th. 1, when $\theta_w \to \frac{\pi}{2}-$, admissible solutions converge to normal reflection. Also we have uniform estimates for admissible solutions. Then use the method of continuity:

Suppose φ, $\hat{\varphi}$ are two non-equal admissible solutions for some $\theta_w^* \in (\theta_w^d, \frac{\pi}{2})$. Then it is sufficient to:

1. Construct continuous in C^1 families $\theta_w \mapsto \varphi^{(\theta_w)}$, $\theta_w \mapsto \hat{\varphi}^{(\theta_w)}$ for $\theta_w \in [\theta_w^*, \frac{\pi}{2})$, with $\varphi^{(\theta_w^*)} = \varphi$, $\hat{\varphi}^{(\theta_w^*)} = \hat{\varphi}$,
2. Show "local uniqueness": if two admissible solutions for same θ_w are close in C^1 in the intersection of their subsonic regions, and if their shocks are close to each other, then the solutions are equal.

Since $\varphi^{(\frac{\pi}{2})} = \hat{\varphi}^{(\frac{\pi}{2})}$ are the unique normal reflection, the two properties above lead to a contradiction for non-equal φ, $\hat{\varphi}$.

Proof of uniqueness: Role of convexity (heuristic)

When formally linearize FBP, variations of shock locations introduce an additional zero-order term in the oblique boundary condition derived from RH condition $\rho D\varphi \cdot \nu = \rho_1 D\varphi_1 \cdot \nu$. This term has the "correct" sign if shock is convex:

Formal linerization of RH conditions: shock is $\eta = f(\xi)$ with $\Omega \subset \{\eta < f(\xi)\}$ after rotating coordinates. Then RH:

$\varphi^\varepsilon(\xi, f^\varepsilon(\xi)) = \varphi_1(\xi, f^\varepsilon(\xi))$;

$\Big((\rho(|D\varphi^\varepsilon|^2, \varphi^\varepsilon)D\varphi^\varepsilon - \rho_1 D\varphi_1) \cdot (D\varphi_1 - D\varphi^\varepsilon)\Big)(\xi, f^\varepsilon(\xi)) = 0,$

where we use that $\nu = \frac{D\varphi_1 - D\varphi^\varepsilon}{|D\varphi_1 - D\varphi^\varepsilon|}$. Here $\varphi^\varepsilon = \varphi + \varepsilon\delta\varphi + \ldots,$ same for f^ε. Taking $\frac{d}{d\varepsilon}$ at $\varepsilon = 0$ in 1st condition and using $\partial_\nu(\varphi_1 - \varphi) > 0$ and on shock, so $\partial_\eta(\varphi_1 - \varphi) > 0$:

$$\delta f = \frac{1}{\partial_\eta(\varphi_1 - \varphi)}\delta\varphi.$$

Now take $\frac{d}{d\varepsilon}$ at $\varepsilon = 0$ in 2nd RH condition

$$\Big((\rho(|D\varphi^\varepsilon|^2, \varphi^\varepsilon)D\varphi^\varepsilon - \rho_1 D\varphi_1) \cdot (D\varphi_1 - D\varphi^\varepsilon)\Big)(\xi, f^\varepsilon(\xi)) = 0,$$

Get two terms. First, linearization of oblique condition:

$$\frac{d}{d\varepsilon}\Big[\Big((\rho(|D\varphi^\varepsilon|^2, \varphi^\varepsilon)D\varphi^\varepsilon - \rho_1 D\varphi_1) \cdot (D\varphi_1 - D\varphi^\varepsilon)\Big)\Big]_{\varepsilon=0}(\xi, f(\xi))$$

$= a\partial_\nu \delta\varphi + b\partial_\tau \delta\varphi + c\delta\varphi$, where $a(\xi) \geq \lambda > 0$, $c(\xi) \leq -\lambda < 0$

Second term comes from the perturbation of shock location:

$$\partial_\eta\Big[\Big((\rho(|D\varphi|^2, \varphi)D\varphi - \rho_1 D\varphi_1) \cdot (D\varphi_1 - D\varphi)\Big)\Big]\delta f$$
$$= A(\varphi_1 - \varphi)_{\tau\tau}\delta f = \frac{A}{(\varphi_1 - \varphi)_\eta}(\varphi_1 - \varphi)_{\tau\tau}\delta\varphi,$$

where $A > 0$. Convexity of shock equivalent to $(\varphi_1 - \varphi)_{\tau\tau} < 0$, and then the coefficient of $\delta\varphi$ has "correct" sign.

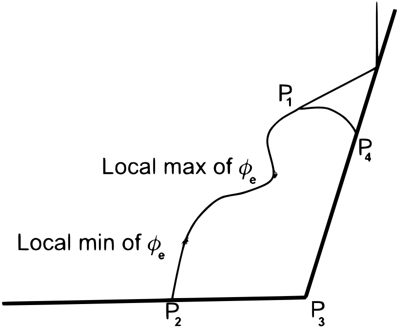
Outline of proof of convexity

Function $\phi = \varphi - \varphi_1$ satisfies equation

$$(c^2 - \varphi_\xi^2)\phi_{\xi\xi} - 2\varphi_\xi\varphi_\eta\phi_{\xi\eta} + (c^2 - \varphi_\eta^2)\phi_{\eta\eta} = 0,$$

where $c = c(|D\varphi|^2, \varphi)$ is the speed of sound, $c^2 = \rho^{\gamma-1}$. Equation is elliptic in Ω. $\phi = 0$ on $\Gamma_{shock} = P_1P_2$ (resp. on P_0P_2 for subsonic reflections). Also, $\phi < 0$ in Ω, which means $\phi_{\tau\tau} > 0$ on "strictly convex" parts of shock, and $\phi_{\tau\tau} < 0$ on parts of shock which are strictly convex in opposite direction.

Let $\boldsymbol{e} \in \mathbb{R}^2$, $\boldsymbol{e} \neq 0$. Then $v = \phi_{\boldsymbol{e}}$ satisfies equation $Lv = 0$ in Ω, where L is a linear elliptic 2nd order operator without zero order terms. From this and Rankine-Hugoniot conditions obtain, using maximum principles and Hopf's lemma:



Property 1

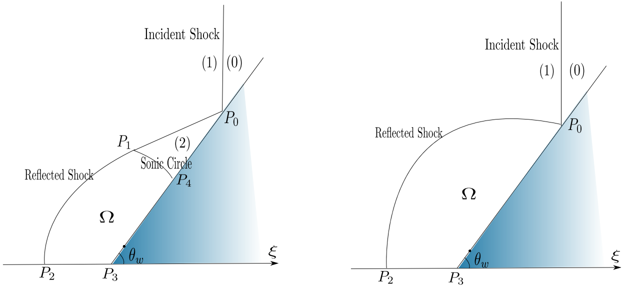
For $\boldsymbol{e} \in \mathbb{R}^2$ such that $\boldsymbol{e} \cdot \boldsymbol{\nu}_{sh} < 0$ on Γ_{shock}, where $\boldsymbol{\nu}_{sh}$ is interior unit normal:

If $\phi_{\boldsymbol{e}}$ has a local minimum relative Ω at $P \in \Gamma_{shock}$, then $\phi_{\tau\tau}(P) > 0$.

If $\phi_{\boldsymbol{e}}$ has a local maximum relative Ω at $P \in \Gamma_{shock}$, then $\phi_{\tau\tau}(P) < 0$.

Condition $\boldsymbol{e} \cdot \boldsymbol{\nu}_{sh} < 0$ on Γ_{shock} holds for any $\boldsymbol{e} \in Con(\vec{e}_{\eta}, \vec{e}_{S_1})$

We choose and fix $\boldsymbol{e} = \boldsymbol{\nu}_w$, where $\boldsymbol{\nu}_w$ is the interior unit normal on $\Gamma_{wedge} = P_3P_4$ (resp. $\Gamma_{wedge} = P_0P_3$ in the subsonic case). It satisfies: $\boldsymbol{\nu}_w \in Con(\vec{e}_\eta, \vec{e}_{S_1})$.



Then $v = \phi_{\boldsymbol{\nu}_w}$ satisfies oblique derivative condition on $\Gamma_{sym} = P_2P_3$; $\partial_{\boldsymbol{\nu}}(\varphi - \varphi_2) = 0$ on Γ_{wedge}, and $D(\varphi - \varphi_2) = 0$ on $\Gamma_{sonic} = P_1P_4$ (resp. at P_0 in the subsonic case). Also $\phi_{\boldsymbol{e}}$ is not constant in Ω.

From this, using that $\partial_{\boldsymbol{\nu}_w}(\varphi - \varphi_2) \leq 0$ in Ω, obtain: $\phi_{\boldsymbol{\nu}_w}$ cannot attain its local minimum (relative to Ω) on $\partial\Omega \setminus (\Gamma^0_{shock} \cup \{P_2\})$.

Convexity of Γ_{shock} is proved by a non-local argument.

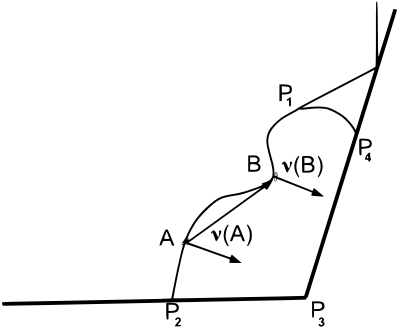
Technical tool: minimal (resp. maximal) chains.

Minimal chain $\{B_r(C^i)\}_{i=0}^k$ of (small) radius $r > 0$ is:

1) $C_0 \in \overline{\Omega}$
2) $C^{i+1} \in \overline{B_r(C^i) \cap \Omega}$ with $\phi_{\boldsymbol{\nu}_w}(C^{i+1}) = \min_{\overline{B_r(C^i) \cap \Omega}} \phi_{\boldsymbol{\nu}_w}$ for $i = 1, \dots, k$.
3) Endpoint: $\phi_{\boldsymbol{\nu}_w}(C^k) = \min_{\overline{B_r(C^k) \cap \Omega}} \phi_{\boldsymbol{\nu}_w}$.

For any $C_0 \in \overline{\Omega}$ which is not a local minimum (resp. maximum) and small $r > 0$, minimal (resp. maximal) chain exists (for some finite $k \geq 1$), and $\cup_{i=0}^k B_r(C^i) \cap \Omega$ is connected using that angles are $< \pi$ at corners of Ω. Also, for sufficiently small r depending on various parameters, minimal/maximal chains do not intersect, using regularity $\|\phi\|_{C^{1,\alpha}(\overline{\Omega})} \leq C$.

Endpoint C^k is a local minimum (resp. maximum) of $\phi_{\boldsymbol{\nu}_w}$, and $\phi_{\boldsymbol{\nu}_w}$ is non-constant, thus $C^k \in \partial\Omega$ by strong maximum principle. From properties $\phi_{\boldsymbol{\nu}_w}$ above: for any minimal chain: $C^k \in \Gamma^0_{shock} \cup \{P_2\}$.

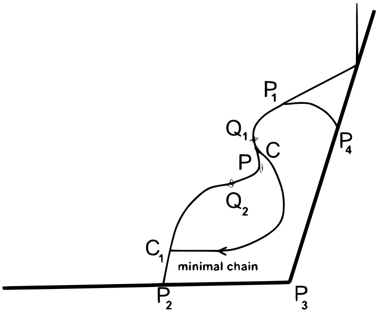


Property 2.

If $A, B \in \Gamma_{shock}$ and $\boldsymbol{\nu}_{sh}(A) = \boldsymbol{\nu}_{sh}(B)$, with $AB \cdot \boldsymbol{\nu}(A) > 0$, then $\phi_{\boldsymbol{\nu}_w}(A) > \phi_{\boldsymbol{\nu}_w}(B)$

Note: on picture, A lies on "convex" part of Γ_{shock}, and B lies on "non-convex" part of Γ_{shock}. We use this in the argument: minimal chain ends in A, and we further reduce $\phi_{\boldsymbol{\nu}_w}$ by finding B on a "non-convex" part of Γ_{shock}, then B is not a point of local minimum of $\phi_{\boldsymbol{\nu}_w}$, can start a minimal chain at B. After several steps there is no place for endpoint of chain, a contradiction.

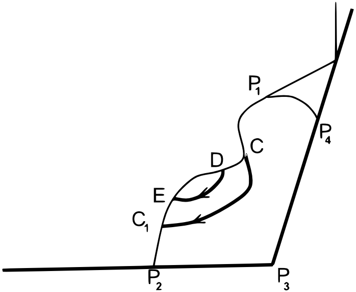
Steps of proof of convexity of Γ_{shock}



Suppose there exists $P \in \Gamma^0_{shock}$ with $\varphi_{\tau\tau}(P) < 0$ ("wrong direction of convexity"). Recall $\phi = \varphi - \varphi_1$.

Let Q_1Q_2 be the maximal interval on Γ_{shock} with $\phi_{\tau\tau} < 0$ and $P \in Q_1Q_2$. Then $Q_1Q_2 \subset (\Gamma_{shock})^0$, by monotonicity cone of ϕ. Let $C \in Q_1Q_2$ be such that $\phi_{\boldsymbol{\nu}_w}(C) = \min_{Q_1Q_2} \phi_{\boldsymbol{\nu}_w}$.

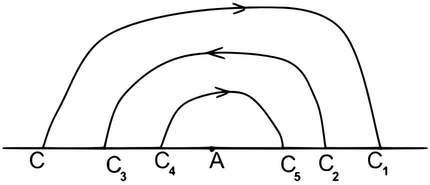
By Property 1, C is not a point of local minimum of $\phi_{\boldsymbol{\nu}_w}$ relative to $\overline{\Omega}$. Then there exists a minimal chain (with r small) starting at C, with endpoint at C_1.



Then $\phi_{\boldsymbol{\nu}_w}(C_1) < \phi_{\boldsymbol{\nu}_w}(C)$, and $\phi_{\boldsymbol{\nu}_w}$ has a local minimum at C_1. By Property 1, $\phi_{\tau\tau}(C_1) > 0$, i.e. C_1 is on "convex" part of shock.

A contradiction would be obtained, if we show, by Property 2, existence D on CC_1 with $\phi_{\boldsymbol{\nu}_w}(D) = \min_{CC_1} \phi_{\boldsymbol{\nu}_w} < \phi_{\boldsymbol{\nu}_w}(C_1)$ and $\phi_{\tau\tau}(D) \leq 0$. Then there exists a minimal chain from D, it must end at $E \in CC_1$ and $\phi_{\boldsymbol{\nu}_w}(E) < \phi_{\boldsymbol{\nu}_w}(D)$, which contradicts the definition of D.

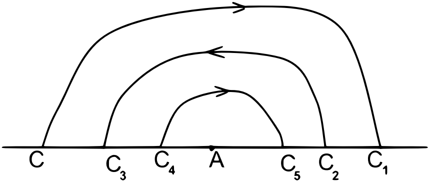
However, to use Property 2, we have to control the directions of $\boldsymbol{\nu}$ on Γ_{shock}. This requires extra steps.



We show $\max_{CC_1} \phi_{\boldsymbol{\nu}_w} > \phi_{\boldsymbol{\nu}_w}(C)$. Then there exists $A \in (CC_1)^0$ such that $\phi_{\boldsymbol{\nu}_w}(A) = \max_{CC_1} \phi_{\boldsymbol{\nu}_w}$. We show: A is a local maximum of $\phi_{\boldsymbol{\nu}_w}$ relative to $\overline{\Omega}$, and $\boldsymbol{\nu}(A) \neq \boldsymbol{\nu}(P)$ for all P in $CC_1 \setminus A$. We can control directions of $\boldsymbol{\nu}$ on subintervals AC and AC_1.

We show, using Property 2, that there exists C_2 on AC_1 with $\phi_{\boldsymbol{\nu}_w}(C_2) = \min_{AC_1} \phi_{\boldsymbol{\nu}_w} < \phi_{\boldsymbol{\nu}_w}(C_1)$ and $\phi_{\tau\tau}(C_2) \leq 0$.

Then there exists a minimal chain from C_2; its endpoint C_3 must be on CC_1 and $\phi_{\boldsymbol{\nu}_w}(C_3) < \phi_{\boldsymbol{\nu}_w}(C_2)$. It follows that $C_3 \in AC$.



Now we show, using Property 2, that there exists C_4 on AC_3 with $\phi_{\boldsymbol{\nu}_w}(C_4) = \min_{AC_3} \phi_{\boldsymbol{\nu}_w} < \phi_{\boldsymbol{\nu}_w}(C_3)$ and $\phi_{\tau\tau}(C_4) \leq 0$.

Then there exists a minimal chain from C_4; its endpoint C_5 must be on C_2C_3 and $\phi_{\boldsymbol{\nu}_w}(C_5) < \phi_{\boldsymbol{\nu}_w}(C_4)$. It follows that $C_5 \in AC_2$. But then

$$\phi_{\boldsymbol{\nu}_w}(C_5) < \phi_{\boldsymbol{\nu}_w}(C_4) < \phi_{\boldsymbol{\nu}_w}(C_3) < \phi_{\boldsymbol{\nu}_w}(C_2) = \min_{AC_1} \phi_{\boldsymbol{\nu}_w},$$

a contradiction.

Open problems

1) Prove existence of regular reflection solutions for Euler system. One of difficulties is in vorticity estimates, noticed by D. Serre for isentropic Euler system: vorticity is not in $L^2(\Omega)$. Singularities are expected near the tip of wedge. Thus one has to work in the low regularity framework: velocity is discontinuous (but subsonic) near tip of wedge. On the positive side, this may improve stability of solutions: For potential flow, regular reflection solution does not exist for non-symmetric perturbations of the incoming flow (J. Hu, 2018) because velocity cannot be subsonic and discontinuous in case of potential flow. For Euler system, existence for non-symmetric perturbations can be expected.

2) Uniqueness/nonuniqueness in various classes of solutions. For example, for reflection of oblique shock by a flat wall, in the class of self-similar solutions for Euler system, etc.

3) Mach reflection: develop apriori estimates.