

Revision Questions

1. $a_3 = 3^2 - 3 = 6$. $a_{10} = 90$.

$$a_{n+1} - a_n = ((n+1)^2 - (n+1)) - (n^2 - n) = 2n.$$

$a_{n+1} - 2a_n + a_{n-1} = (a_{n+1} - a_n) - (a_n - a_{n-1}) = 2n - 2(n-1)$ using the previous part, and this is 2.

2. This is the sum of the first 11 terms of an arithmetic sequence with first term $a = 1$ and common difference 3. So the sum is $\frac{11}{2}(2 + 10 \times 3) = 176$. Also, $a_{1000} = 3001$.

3. This is the sum of the first 11 terms of a geometric sequence with first term $a = 1$ and common ratio $\frac{1}{3}$. So the sum is $\frac{(1-3^{-11})}{1-3^{-1}} = \frac{3}{2}(1-3^{-11})$. Also, $a_{1000} = 3^{-1000}$.

The common ratio is between -1 and 1 so the sum to infinity does converge. The sum to infinity is $\frac{1}{1-3^{-1}} = \frac{3}{2}$.

4. In particular, we would need $a_0 = b_0$ so $1 = A + B$. Also, we would need $a_1 = b_1$ so $4 = 3A + B$. So $A = \frac{3}{2}$ and $B = -\frac{1}{2}$.

We should check that this works for all n , so let's check that if $a_{n-1} = b_{n-1}$ then $a_n = b_n$. We know that $a_n = 3a_{n-1} + 1$ and we can work out $3b_{n-1} + 1 = 3 \times \frac{3}{2}3^{n-1} - \frac{3}{2} + 1 = \frac{3}{2}3^n - \frac{1}{2}$ which is exactly b_n . So if $a_{n-1} = b_{n-1}$ then $a_n = b_n$. The sequences match for all n .

5. At first sight, this doesn't look like enough information; we haven't been told the values of any of the terms in the sequence! The key is that we're asked to give our answer in terms of the first three terms of the sequence without solving for what those are.

For example, if we substitute $n = 0$ then we find $a_0 = C$. So we've got an expression for C in terms of a_0 .

Now substitute $n = 1$ and $n = 2$ to get $a_1 = A + B + C$ and $a_2 = 4A + 2B + C$. We want A and B in terms of the variables a_0, a_1, a_2 , and we can use the fact that $C = a_0$ to eliminate C . These are simultaneous equations for A and B with solution

$$A = \frac{1}{2}(a_2 - 2a_1 + a_0), \quad B = \frac{1}{2}(-a_2 + 4a_1 - 3a_0)$$

6. This is a geometric sequence with common ratio x^3 , and the sum to infinity converges if $|x^3| < 1$, which is precisely $|x| < 1$.

In that case, it converges to $(1 - x^3)^{-1}$.

7. This is a geometric sequence with common ratio $-\frac{x}{2}$, and the sum to infinity converges if $|\frac{-x}{2}| < 1$, which is precisely $|x| < 2$.

In that case, it converges to $\frac{2}{1 + \frac{x}{2}} = \frac{4}{2 + x}$.

8. The 15th term will be equal to the first plus 14 times the common difference, so $5 + 14 \times 3 = 47$.
9. This is strange because the next k terms should each be kd more than the corresponding term k places before it in the sequence. For the sum to be the same, we would need $d = 0$ (a constant sequence) or $k = 0$ (no numbers in the statement!).
10. The n^{th} term is the same thing as the sum of the first n terms minus the sum of the first $(n - 1)$ terms, so we want $(3n^2 + 5n) - (3(n - 1)^2 + 5(n - 1)) = 6n + 2$.
Alternatively, just set $n = 1$ and $n = 2$ to find the first term is 8 and the sum of the first two terms is 22 so the second term is 14 and the common difference is therefore 6.
11. $2 + 4 + 6 + 8 + \dots + 200$ is the sum of an arithmetic progression with first term $a = 2$, common difference $d = 2$ and number of terms $n = 100$, so the sum is

$$\frac{n}{2}(2a + (n - 1)d) = 50 \times 202 = 10100.$$

12. What's the common ratio? We have $a = 3$ and $ar^2 = 27$ so either $r = 3$ or $r = -3$. The sum of the first five terms is

$$\frac{a(r^5 - 1)}{r - 1} = \frac{3((\pm 3)^5 - 1)}{(\pm 3) - 1} = \text{either } 363 \text{ or } 183$$

13. First note that $a_1 = 3$ (the only previous term is 3) and then $a_2 = 3 + 3 = 6$. After that, $a_3 = 6 + 3 + 3 = 12$ and it seems like the terms double each time.

That's true because $a_n = a_{n-1} + a_{n-2} + \dots + a_0$ and $a_{n-1} = a_{n-2} + \dots + a_0$ so $a_n = a_{n-1} + a_{n-1} = 2a_{n-1}$.

So after a_0 this is a geometric sequence with $a_n = 3 \times 2^{n-1}$ for $n \geq 1$.

14. For C_1 I'll put $n = 0$ into the equation, so that the left-hand side reads C_1 . The right-hand side is then

$$\sum_{i=0}^0 C_i C_{-i}.$$

That looks pretty weird written out like that; the sum from $i = 0$ to 0 is just the term with $i = 0$, which is C_0^2 . So $C_1 = C_0^2 = 1$.

Now for C_2 , I'll put $n = 1$ into the equation. The right-hand side is more interesting now; I get

$$\sum_{i=0}^1 C_i C_{1-i}$$

which is $C_0 C_1 + C_1 C_0$. So $C_2 = C_0 C_1 + C_1 C_0 = 2$.

Then $C_3 = C_2 C_0 + C_1 C_1 + C_0 C_2 = 5$ and $C_4 = C_3 C_0 + C_2 C_1 + C_1 C_2 + C_0 C_3 = 14$.

MAT 2008 Q2

(i) Perhaps $(x, y) = (1, 0)$ if you count zero as positive. Or $(x, y) = (3, 2)$ if you don't.

(ii) We would need

$$(3x_n + 4y_n)^2 - 2(ax_n + by_n)^2 = x_n^2 - 2y_n^2$$

Multiply out the left-hand side.

$$9x_n^2 + 24x_ny_n + 16y_n^2 - 2a^2x_n^2 - 4abx_ny_n - 2b^2y_n^2 = x_n^2 - 2y_n^2$$

This looks unlikely to be true, but one way to make it work would be to make the coefficients of x_n^2 , and x_ny_n , and y_n^2 agree between the two expressions. We would need

$$9 - 2a^2 = 1 \quad \text{and} \quad 24 - 4ab = 0 \quad \text{and} \quad 16 - 2b^2 = -2$$

Amazingly, we can satisfy all of these by setting $a = 2$ and $b = 3$.

(iii) Use the sequences x_n and y_n . The relationship we proved in the previous part of the question means that

$$x_{n+1}^2 - 2y_{n+1}^2 = x_n^2 - 2y_n^2 = x_{n-1}^2 - 2y_{n-1}^2 = \cdots = x_2^2 - 2y_2^2 = x_1^2 - 2y_1^2 = 1.$$

If we generate the first few terms with the rules in part (ii), we find

$$x_1 = 3, \quad y_1 = 2, \quad x_2 = 17, \quad y_2 = 12, \quad x_3 = 99, \quad y_3 = 70.$$

So $X = 99$, $Y = 70$ works (check; $99^2 = 9801$, $70^2 = 4900$, so $X^2 - 2Y^2 = 1$).

(iv) These values of x_n and y_n satisfy $(x_n)^2 - 2(y_n)^2 = 1$. If we divide this by y_n^2 then we get

$$\left(\frac{x_n}{y_n}\right)^2 - 2 = \frac{1}{(y_n)^2}$$

If y_n is very large, then the right-hand side of this equality is very small. So $(x_n/y_n)^2$ must be close to 2. Since x_n/y_n is positive, we can conclude that it is close to $\sqrt{2}$ for large n .

Extension

- Consider $x_n^2 - 3y_n^2 = 1$. Let $x_1 = 1$ and $y_1 = 0$. Suppose $x_{n+1} = ax_n + by_n$ and $y_{n+1} = cx_n + dy_n$. It would be nice to have $(ax_n + by_n)^2 - 3(cx_n + dy_n)^2 = x_n^2 - 3y_n^2$. Comparing terms, we want $a^2 - 3c^2 = 1$ and $ab - 3cd = 0$ and $b^2 - 3d^2 = -3$. We can achieve all of these with $a = 2$, $b = 3$, $c = 1$, and $d = 2$. Then generate some more solutions for (x, y) ; $(1, 0) \rightarrow (2, 1) \rightarrow (7, 4) \rightarrow (26, 15) \rightarrow (97, 56)$. So $\sqrt{3} \approx \frac{97}{56}$.

MAT 2012 Q5

(i) $P_2 = P_1 \mathbf{L} P_1 \mathbf{R} = \mathbf{FLFR L FLFR R}$

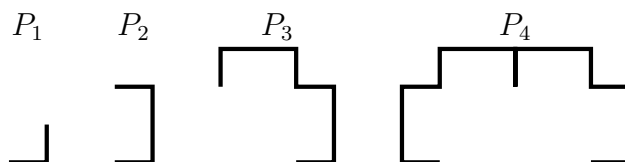
- (ii) There's one **F** in P_0 , and then P_{n+1} contains two copies of P_n (and no other **F** commands) so it has twice as many **F** commands as P_n . Therefore P_n has 2^n **F** commands.
- (iii) P_{n+1} contains twice as many commands as P_n , plus two extra commands (the **L** added in between and the **R** at the end). So $l_{n+1} = 2l_n + 2$.

Consider $l_n + 2$. The right-hand side of our equation is $2(l_n + 2) - 2$ so we can write

$$(l_{n+1} + 2) = 2(l_n + 2)$$

This means that $l_n + 2$ just doubles when n is increased by 1. So $l_n = A \times 2^n$ for some n . Note that $l_0 = 1$ so $l_0 + 2 = 3$. Compare this with $A \times 2^0$ and we see that $A = 3$. Therefore $l_n = 3 \times 2^n - 2$. Check that $l_0 = 1$, and $l_1 = 4$, and $l_2 = 10$. Check that $2 \times (3 \times 2^n - 2) + 2 = 3 \times 2^{n+1} - 2$.

- (iv) The robot turns left just as often as it turns right, so after performing the program P_n , the robot is facing along the positive x -axis again.
- (v) First draw the paths for P_1 and P_2 and P_3 .



- (vi) If the robot starts at (a, b) facing along the positive x -axis and performs program P_n , it moves to $(a + x_n, b + y_n)$. But if it first turns left and then performs program P_n , then what would have been a motion in the positive x -direction becomes a motion in the positive y -direction, and what would have been a motion in the positive y -direction becomes a motion in the *negative* x -direction. So if the robot starts at (a, b) facing along the positive x -axis and performs program $\mathbf{L}P_n$, then it is now at $(a - y_n, b + x_n)$.

For program P_{n+1} the robot first performs P_n , moving to (x_n, y_n) , and then performs $\mathbf{L}P_n$, moving to $(x_n - y_n, y_n + x_n)$ by the argument above.

Note that (x_4, y_4) is $(-4, 0)$, so the overall effect of the operation P_4 is a bit like **F** but four times as much displacement, and in the opposite direction (and taking a complicated route to get there, of course). That means that as we generate P_5 and P_6 and P_7 and P_8 , the overall displacements are the same as for P_1 and P_2 and P_3 and P_4 , except 4 times as large and with a minus sign. In particular, the displacement for P_8 is $((-4)^2, 0) = (16, 0)$.

That's the same as **F** except 16 times as much displacement. This pattern continues; from here it's like starting again but with **F** being 16 times as large. The displacement for P_{8k} is therefore 16^k .