Plateau flow
alias
The heat flow for half-harmonic maps

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of a map $u \in H^{1/2}(S^1; N)$, where

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Numerous results on regularity, “bubbling”, and quantization have been obtained by Da Lio-Rivière, Da Lio-Martinazzi-Rivière, R. Moser, Schikorra, and others, in particular, when $N = S^k$.

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of the half-Laplacian as the Dirichlet-to-Neumann map. Thus, the representation of the operators $(-\Delta)^{1/2}$ and $(-\Delta)^{1/4}$ in Fourier space with symbols $|\xi|$, $\sqrt{|\xi|}$, and Parceval's identity give

$$\begin{aligned}\int_B |\nabla u|^2 dz &= \int_{S^1} u \partial_r u \, d\phi \\ &= \int_{S^1} u(-\Delta)^{1/2} u \, d\phi = \int_{S^1} |(-\Delta)^{1/4} u|^2 d\phi.\end{aligned}$$

Connection with minimal surfaces

The characterization $(-\Delta)^{1/2}u = \partial_r u$ of the half-Laplacian shows that a half-harmonic map $u \in H^{1/2}(S^1; \mathbb{R}^n)$ induces a minimal surface with free boundary on N (Da Lio-Rivière (2011)).

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Indeed, the harmonic extension $u \in H^1(B; \mathbb{R}^n)$ of u satisfies

$$d\pi(u)\partial_r u = d\pi(u)\big((-\Delta)^{1/2}u\big) = 0.$$

Thus, $u_r \cdot u_\phi = 0$ on $\partial B = S^1$, and the (analytic) Hopf differential

$$\Phi = z^2 u_z^2 = r^2|u_r|^2 - |u_\phi|^2 - 2iru_r \cdot u_\phi$$

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Moreover, from the condition $d\pi(u)\partial_r u = 0$ we infer that u meets N vertically.

The heat flow for half-harmonic maps

Wettstein (2021) studied the heat flow

$$d\pi(u)\big(u_t + (-\Delta)^{1/2}u\big) = 0 \text{ on } S^1 \times [0,\infty[, \tag{2}$$

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Note that with $(-\Delta)^{1/2}u = \partial_r u$, equation (2) is equivalent to the evolution equation

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for a family of maps $u = u(t) \in H^{1/2}(S^1; N)$ with harmonic extension $u \in H^1(B; \mathbb{R}^n)$. Observing that

$$\begin{aligned}\frac{d}{dt}E(u) = \int_B \nabla u \nabla u_t \, dx &= \int_{\partial B} u_r \cdot u_t \, d\phi \\ &= -\int_{\partial B} |d\pi(u)u_r|^2 d\phi = -\int_{\partial B} |u_t|^2 d\phi,\end{aligned}$$

we view (3) as heat flow for E with a free boundary constraint.

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Theorem (S.(2021)). Let $N \subset \mathbb{R}^n$ be a closed, smooth sub-manifold of $\mathbb{R}^n$, and suppose that the normal bundle $T^{\perp}N$ is parallelizable. For any $u_0 \in H^{1/2}(S^1; N)$ there exists a unique global weak solution $u = u(t)$ of (3) (hence (2)) with $u(0) = u_0$, whose energy is non-increasing and which is smooth away from finitely many points in space-time where non-trivial half-harmonic maps "bubble off".

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Remarks

• We exploit the surprising regularity properties of the normal component $d\pi_N^{\perp}(u)\partial_r u$ of the harmonic extension of u, likely related to the commutator estimates for the normal projection in the work of Da Lio-Rivière and others.

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• A simple identity, and the use of the Dirichlet-to-Neumann map for the harmonic extension $u \colon B \to \mathbb{R}^n$ of u instead of the half-Laplacian $(-\Delta)^{1/2}u$ allow to perform the analysis completely avoiding fractional calculus.

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• Even though (3) appears to be degenerate, within our framework we are able to obtain similar smoothing properties as in the case of the harmonic map heat flow of surfaces.

• When N is a smoothly embedded, oriented closed curve $\Gamma \subset \mathbb{R}^n$ the flow (3) may be viewed as an alternative gradient flow for the Plateau problem of disc-type minimal surfaces, the Plateau flow.

Remarks on the Plateau flow

• It is not clear whether the classical Plateau boundary condition (requiring monotonicity) is preserved along the flow (3), even when Γ is a strictly convex planar curve. But also without monotonicity energy quantization holds (see below).

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$$u_t + d\pi(u)\partial_\nu u = 0$$

instead of (2), where ν is the outward unit normal along $\partial S = S^1$ and where for each time we harmonically extend $u(t)$ to S; see Da Lio-Pigati (2020) in the time-independent case.

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• In these cases, in order to obtain minimal surfaces one would also need to flow the conformal type of the domain, similar to Rupflin-Topping for harmonic map flow.

Notation

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$$dist_N(q) = \eta(h(q)) \text{ in } N_\rho,\ \ dist_N(q) = 0 \text{ else.}$$

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Then for any $u \in H^{1/2}(S^1; N)$ with harmonic extension u we have

$$\nu(u)\partial_r\big(dist_N(u)\big) = \nu(u)\nu(u)\cdot u_r =: d\pi_N^\perp(u)u_r$$

on $\partial B = S^1$,where $d\pi_N^\perp(p) = 1 - d\pi_N(p)$ for each $p \in N$ is the orthogonal projection to $T_p^\perp N$.

A regularity estimate

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Proposition. There exists $\delta > 0$ such that for any smooth solution $u \in H^{1/2}(S^1; N)$ with harmonic extension $u \in H^1(B, \mathbb{R}^n)$ of

$$d\pi_N(u)\partial_r u = f \quad \text{on } \partial B = S^1,$$

with $E(u) \leq \delta^2$ there holds

$$\int_{S^1} |\partial_\phi u|^2 d\phi \leq C\|f\|^2_{L^2(S^1)}.$$

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Proof. We have the orthogonal decomposition

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Extend $\nu(p) := \nabla dist_N(p)$. Note that $dist_N(u) \in H^1_0(B)$ with

$$\|\nabla(dist_N(u))\|^2_{L^2(B)} \leq C\|\nabla u\|^2_{L^2(B)} \leq C\delta^2.$$

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Moreover, $dist_N(u) \in H_0^1(B)$ satisfies the equation

$$\Delta(dist_N(u)) = div(\nu(u) \cdot \nabla u) = \nabla u \cdot d\nu(u) \nabla u \quad \text{in } B \qquad (4)$$

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The divergence theorem gives

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Fourier expansion shows $\|\nabla u\|_{H^{1/2}(B)}^2 = C\|\partial_r u\|_{L^2(S^1)}^2$; thus

$$\|\partial_\phi u\|_{L^2(S^1)}^2 = \|\partial_r u\|_{L^2(S^1)}^2 = \|f\|_{L^2(S^1)}^2 + \|\partial_r(dist_N(u))\|_{L^2(S^1)}^2$$
$$\le \|f\|_{L^2(S^1)}^2 + C\delta\|\nabla u\|_{H^{1/2}(B)}^2 \le \|f\|_{L^2(S^1)}^2 + C\delta\|\partial_r u\|_{L^2(S^1)}^2.$$

A threshold result

Corollary. (i) For any non-constant, smooth solution $u \in H^{1/2}(S^1; N)$ with harmonic extension $u \in H^1(B, \mathbb{R}^n)$ of

$$d\pi_N(u)\partial_r u = 0 \text{ on } \partial B = S^1, \tag{5}$$

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there holds $E(u) \geq \delta^2$, where $\delta = \delta(N) > 0$.

In particular, let $N = \Gamma \subset \mathbb{R}^3$ a closed, smoothly embedded curve, and let $u \in H^1(B, \mathbb{R}^n)$ be smooth, non-constant, harmonic, and satisfying the generalized Plateau condition $u \in H^{1/2}(S^1; N)$.

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$$d\pi_N(u)\partial_r u = 0 \quad \text{on } \partial B = S^1, \tag{5}$$

there holds $E(u) \geq \delta^2$, where $\delta = \delta(N) > 0$.

In particular, let $N = \Gamma \subset \mathbb{R}^3$ a closed, smoothly embedded curve, and let $u \in H^1(B, \mathbb{R}^n)$ be smooth, non-constant, harmonic, and satisfying the generalized Plateau condition $u \in H^{1/2}(S^1; N)$.

(ii) u is conformal iff u solves (5), and then $E(u) \geq \delta(\Gamma)^2 > 0$.

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(ii) u is conformal iff u solves (5), and then $E(u) \geq \delta(\Gamma)^2 > 0$.

Proof. Let $\gamma\colon S^1 \to \Gamma$ be an arc-length reference parametrization of Γ, and let $u = \gamma \circ \xi$ on $\partial B = S^1$ for some smooth $\xi\colon S^1 \to S^1$. Then at any $p \in S^1$ where $|\partial_\theta u| = |\partial_r u| \neq 0$ with some $\lambda \neq 0$ we have

$$|d\pi_\Gamma(u)\partial_r u| = |\gamma'(\xi) \cdot \partial_r u| = \lambda |\partial_\theta u \cdot \partial_r u| = 0,$$

and u is conformal iff u solves (5). The claim then follows from (i).

Non-concentration of energy gives regularity

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Hence the flow (3) can only become singular at time T_0 if energy of size at least $\delta > 0$ concentrates as $t \uparrow T_0$. (Energy concentration can then be analyzed via the usual rescaling procedure.)

H^2-bounds

Lemma. With a constant $C > 0$ depending only on N there holds

$$\frac{d}{dt}\Big(\int_{\partial B}|u_\phi|^2 d\phi\Big) + \int_B |\nabla u_\phi|^2 dz \leq C\int_B |\nabla u|^2 |u_\phi|^2 dz.$$

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$$\frac{d}{dt}\Big(\int_{\partial B}|u_\phi|^2 d\phi\Big) + \int_B |\nabla u_\phi|^2 dz \le C\int_B |\nabla u|^2|u_\phi|^2 dz.$$

Proof. Writing $d\pi_N(u) = 1 - \nu(u) \times \nu(u)$ we compute

$$\frac{1}{2}\frac{d}{dt}\Big(\int_{\partial B}|u_\phi|^2 d\phi\Big) = \int_{\partial B} u_\phi \cdot u_{\phi,t}\, d\phi = -\int_{\partial B} u_{\phi\phi}\cdot u_t\, d\phi$$
$$= \int_{\partial B} u_{\phi\phi}\cdot d\pi_N(u) u_r\, d\phi = -\int_{\partial B}\big(u_\phi\cdot u_{r\phi} - u_\phi\cdot\partial_\phi(\nu(u)\,\nu(u)\cdot u_r)\big)\, d\phi$$
$$= -\frac{1}{2}\int_{\partial B}\partial_r(|u_\phi|^2)\, d\phi - \int_{\partial B} u_\phi\cdot d\nu(u)u_\phi\,\nu(u)\cdot u_r\, d\phi$$
$$= -\int_B |\nabla u_\phi|^2 dz - \int_B \nabla u\cdot\nabla(\nu(u)\, u_\phi\cdot d\nu(u)u_\phi)\, dz.$$

Integral H^2-bounds

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From the previous lemma, with the multiplicative inequality

$$\int_B |\nabla u|^4 \varphi_{z_i,R}^2 dz \leq C\delta \int_{B_R(z_i)} |\nabla^2 u|^2 dz + C\delta R^{-2} \int_{B_R(z_i)} |\nabla u|^2 dz,$$

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Proposition. There exist constants $\delta > 0$ and $C > 0$ such that for any $T < T_0$ there holds

$$\sup_{0<t<T} \int_{\partial B} |u_\phi(t)|^2 d\phi + \int_0^T \int_B |\nabla u_\phi|^2 dx\, dt \le \int_{\partial B} |u_{0,\phi}|^2 d\phi + CTR^{-2} E(u_0),$$

where $R > 0$ is as in (6) .

Uniform H^2- and higher bounds

With similar reasoning we also obtain a uniform H^2-bound.

Proposition. For any smooth $u_0 \in H^{1/2}(S^1; N)$ and any $T < T_0$ as specified above, with a constant $C > 0$ possibly depending also on T there holds

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Higher bounds can then be obtained with two lemmas like the following.

Lemma For any $k \geq 2$, with a constant $C > 0$ depending only on k and Γ, for the solution $u = u(t)$ to (2) for any $0 < t < T_0$ there holds

$$\frac{d}{dt}\left(\|\nabla \partial_\phi^k u\|^2_{L^2(B)}\right) + \|\partial_\phi^k u_r\|^2_{L^2(\partial B)} \leq C \sum_{1 \leq j_i \leq k+1,\, \Sigma_i j_i = k+2} \|\nabla \partial_\phi^k u\|_{L^2(B)} \|\Pi_i \nabla^{j_i} u\|_{L^2(B)}.$$

Local existence, blow-up and asymptotics

• Local existence can be obtained with a contraction-mapping argument for the map $\Phi_\varepsilon : v \to u = \Phi_\varepsilon(v)$, the solution of

$$u_t + (\varepsilon + d\pi(v))\partial_r u = 0, \;\; u_{|_{t=0}} = u_0,$$

using bounds similar to the previous lemma, which are uniform in $0 < \varepsilon \ll 1$.

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Reference: http://arxiv.org/abs/2202.02083

Thank you for your attention