MAT syllabus

Co-ordinate geometry and vectors in the plane. The equations of straight lines and circles. Basic properties of circles. Lengths of arcs of circles.

Revision

- Points in the plane can be described with two co-ordinates (x, y). The x-axis is the line $y = 0$, and the y-axis is the line $x = 0$.
- A vector $\begin{pmatrix} x \\ y \end{pmatrix}$ can store the same information as a pair of co-ordinates. Used in that sense, the vector is called a position vector.
- A vector can also describe the displacement from one point to another, so that $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ could represent the displacement from $(1, 1)$ to $(3, 2)$ for example.
- Vectors can be added by adding the components separately. To show that in a diagram, we might interpret the first vector as a position vector (drawing an arrow starting from the origin) and then interpret the second as a displacement (drawing an arrow starting from the end of the first vector).
- The magnitude of the vector $\begin{pmatrix} x \\ y \end{pmatrix}$ is $\sqrt{x^2 + y^2}$.
- The distance from A to B is the magnitude of the vector displacement from A to B. The distance from (x_1, y_1) to (x_2, y_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
- A vector can be multiplied by a number by multiplying each component by that number. The result is a vector in the same direction but with scaled magnitude.
- A straight line has equation $y = mx + c$, where m is the gradient and c is the y-intercept. Other ways to write the equation of a line are $ax + by + c = 0$ (where that's a different c to the one in the previous expression) or $y - y_1 = m(x - x_1)$. The last expression is useful because that line goes through the point (x_1, y_1) and has gradient m, which might be information that we've been given.
- Two lines are parallel if and only if they have the same gradient. Two lines are perpendicular if and only if their gradients multiply to give -1.
- The equation of a circle with centre (a, b) and radius r is $(x - a)^2 + (y - b)^2 = r^2$.
- The angle in a semicircle is a right angle; if AB is the diameter of a circle, and C is on the circle, then $\angle ACB = 90°$.

- The tangent is at right angles to the radius at any point on a circle's circumference.
- A circle with radius r has area πr^2 and circumference $2\pi r$.
- If two radii of a circle of radius r make an angle of θ (in degrees), then the length of the arc between those radii is $\dfrac{\theta}{360^\circ} 2\pi r$. The area of the sector enclosed by that arc and the radii is $\dfrac{\theta}{360^\circ} \pi r^2$.

Warm-up

1. Draw a diagram to show the three separate position vectors $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.
2. Add the vectors $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$. Show this on your diagram.
3. Find $3\begin{pmatrix} -4 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ -2 \end{pmatrix}$. Show this on your diagram.
4. Find the equation of the line through the points $(1, 5)$ and $(3, -1)$.
5. Find the equation of the line through the point $(3, 5)$ with gradient 2.
6. Find equations of three lines such that the region bounded by the three lines is an equilateral triangle.
7. A circle has centre $(-1, 4)$ and radius 3. Write down an equation for the circle. What's the area of this circle?
8. A circle is given by $x^2 + 9x + y^2 - 3y = 10$. Find the centre and radius of the circle.
9. Points A and B lie on a circle with centre O and radius 2. The angle $\angle AOB$ is 120°. Find the length of the arc between A and B. Find the area enclosed by that arc and the radii OA and OB.
10. A circle is given by $x^2 + y^2 = 4$. The line $x = 1$ splits the circle into two regions. Find the area of each region.
11. Two circles are given by $x^2 + y^2 = 4$ and $(x - 2)^2 + y^2 = 4$. Find the area of the region that's inside both circles.
12. The points $(0, 0)$ and $(1, a)$ and $(0, a + 1/a)$ all lie on the same circle. Find the centre of the circle in terms of a. Hint: angles.

MAT questions

MAT 2013 Q1H

The area bounded by the graphs

$$y = \sqrt{2 - x^2} \quad \text{and} \quad x + \left(\sqrt{2} - 1\right) y = \sqrt{2}$$

equals

(a) $\dfrac{\sin\sqrt{2}}{\sqrt{2}}$, (b) $\dfrac{\pi}{4} - \dfrac{1}{\sqrt{2}}$, (c) $\dfrac{\pi}{2\sqrt{2}}$, (d) $\dfrac{\pi^2}{6}$.

Hint: we've included this in the geometry section and not in the integration section!

MAT 2017 Q1G

For all θ in the range $0 \leqslant \theta < 360°$ the line

$$(y - 1)\cos\theta = (x + 1)\sin\theta$$

divides the disc $x^2 + y^2 \leqslant 4$ into two regions. Let $A(\theta)$ denote the area of the larger region.

Then $A(\theta)$ achieves its maximum value at

(a) one value of θ , (b) two values of θ, (c) three values of θ,
(d) four values of θ, (e) all values of θ .

Hint: don't try to differentiate, draw a picture instead.

MAT 2016 Q1I

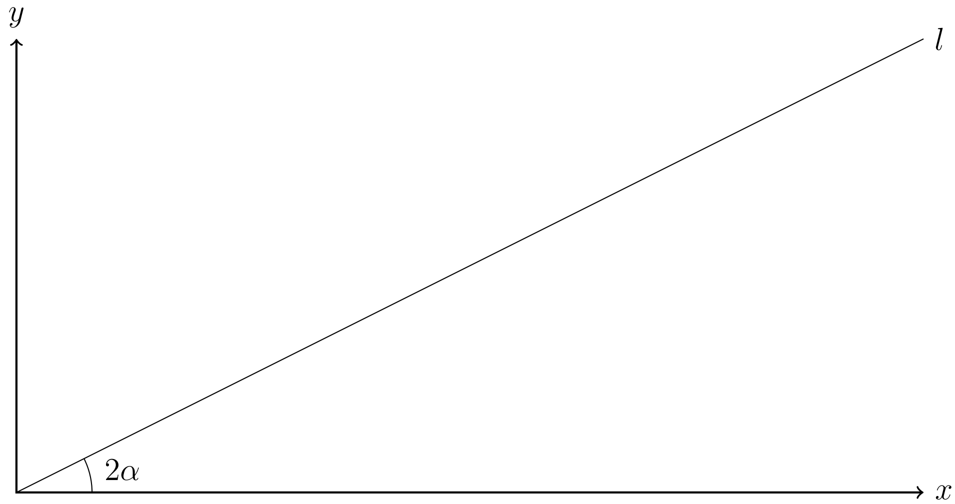
Let a and b be positive real numbers. If $x^2 + y^2 \leqslant 1$ then the largest that $ax + by$ can equal is

(a) $\dfrac{1}{a} + \dfrac{1}{b}$, (b) $\max(a, b)$, (c) $\sqrt{a^2 + b^2}$, (d) $a + b$, (e) $a^2 + ab + b^2$.

Hint: where does $ax + by$ take a particular value? How does the value relate to your answer to that question? How can we maximise this value?

MAT 2016 Q4

The line l passes through the origin at angle 2α above the x-axis, where $2\alpha < 90°$.



Circles C_1 of radius 1 and C_2 of radius 3 are drawn between l and the x-axis, just touching both lines.

(i) What is the centre of circle C_1?

(ii) What is the equation of circle C_1?

(iii) For what value of α do circles C_1 and C_2 touch?

(iv) For this value of α (for which the circles C_1 and C_2 touch) a third circle, C_3, larger than C_2, is to be drawn between l and the x-axis. C_3 just touches both lines and also touches C_2. What is the radius of this circle C_3?

(v) For the same value of α, what is the area of the region bounded by the x-axis and the circles C_1 and C_2?

Hints: Draw a large diagram. Make sure that circle C_1 is touching both lines. Draw in any relevant radii and mark any right angles. It's interesting that we've been told that the angle is 2α rather than α. Maybe sketch the angle bisector just to see what happens?

For part (iii) we're asked to make the circles touch. We could convert that to an algebra problem (something about having a unique solution rather than two or zero solutions), but it's more straightforward to stick to ideas from geometry. Draw a diagram where the circles just touch. Can you tell me anything about the distance between the centres?

For part (v) note that circle C_3 doesn't matter any more; this part of the question is just about the first two circles C_1 and C_2 again.

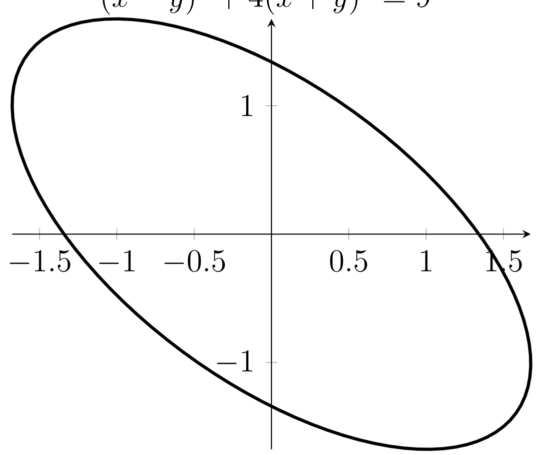
Extension

The following material is included for your interest only, and not for MAT preparation.

Here's some content on ellipses. Just to be clear, ellipses are not on the MAT syllabus!
The first time you meet ellipses, they're just stretched circles

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = c^2.$$

That gives you ellipses that are lined up with the x-axis and y-axis (in the sense that the longest distance across the ellipse and the shortest distance across the ellipse are each parallel to either the x-axis or the y-axis). But what if we see something like $(x-y)^2+4(x+y)^2 = 9$? That looks a bit like the equation for an ellipse above, and if we sketch it (for example, by typing it into a curve-sketching website like Desmos) then it certainly looks like an ellipse!



In fact, it is an ellipse, but it's been rotated. Rotations like this are not on the MAT syllabus, but you might learn about them later on in A-level Maths or Further Maths or equivalent, especially if you learn about matrices.

More generally, anything of the form $(ax+by)^2 + (cx+dy)^2 = r^2$ is an ellipse.

If we're given something like $Ax^2 + Bxy + Cy^2 = r^2$, then we're faced with quite a difficult problem to work out whether it's in the form above (sort of like completing two squares at once!). At Oxford on our mathematics degree, we teach people how to deal with these expressions, which are called "quadratic forms". I won't explain the logic here, but it turns out that the expression there represents an ellipse if and only if $B^2 < 4AC$ (does that expression seem familiar?)

Otherwise, if $B^2 > 4AC$ then the equation $Ax^2 + Bxy + Cy^2 = r^2$ represents a hyperbola (a shape like $xy = 1$), or if $B^2 = 4AC$ then the equation represents a pair of straight lines (in this special case, can you solve for the equations of the lines?)