Knotted solutions of the Vortex Filament Equation without self-crossings

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Bird blowing vapor rings

Figure: Red-winged blackbird blowing vapor rings. Photo by Kathrin Swoboda (2019 Audubon Photography Awards).

Experimental vortex filament

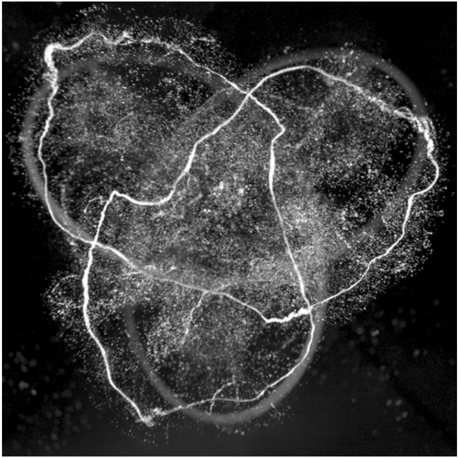


Figure: A trefoil vortex knot in water imaged via light scattering off micro-bubbles. (William Irvine's Lab, U. Chicago, Nature Physics 2013.)

The Vortex Filament Equation

A vortex filament is a thin tubular region of non-zero vorticity surrounded by irrotational fluid.

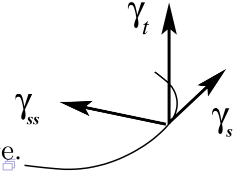
Self induction: For small core radius, with circulation held constant, the filament centerline $\boldsymbol{\gamma}(x,t) \in \mathbb{R}^3$ moves with the fluid so that

$$\boldsymbol{\gamma}_t(x,t) = \mathbf{v}(\boldsymbol{\gamma}(x,t),t),$$

where $\mathbf{v}(\mathbf{P},t)$ is the fluid velocity, and x is the arclength parameter.

Local approximation: As the core radius becomes zero, the dominant diverging term in the local expansion of the velocity field gives (after rescaling) the *binormal motion* (Da Rios, 1904):

$$\text{(VFE)} \qquad \boldsymbol{\gamma}_t = \boldsymbol{\gamma}_x \times \boldsymbol{\gamma}_{xx} = \kappa \mathbf{B}.$$



Here $\mathbf{B}$ is the binormal vector and κ is the curvature.

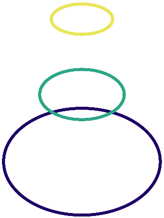
The Hasimoto Map

If a curve $\gamma(x,t)$ evolves according to the VFE, the complex function

$$q(x,t) = \mathcal{H}(\gamma) = \frac{1}{2}\kappa\, e^{i\int^x \tau\, ds} \quad \text{(Hasimoto Map)}$$

of the curvature κ and the torsion τ solves the cubic focusing nonlinear Schrödinger equation (H. Hasimoto, 1972):

$$iq_t + q_{xx} + 2|q|^2 q = 0. \qquad \text{(NLS)}$$



$$\xrightarrow{\mathcal{H}} \qquad q_a(t) = a e^{2ia^2 t}, \ \ a \in \mathbb{R}^+$$

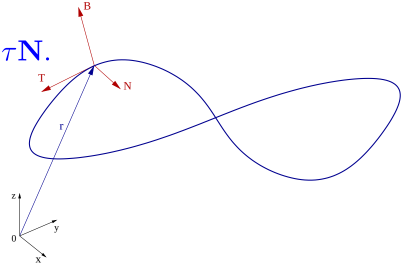
Racing smoke rings

Plane wave potentials

Understanding the Hasimoto Map

Recall the classical Frenet equations:

$$\mathbf{T}_x = \kappa\mathbf{N}, \quad \mathbf{N}_x = -\kappa\mathbf{T} + \tau\mathbf{B}, \quad \mathbf{B}_x = -\tau\mathbf{N}.$$



–Unit tangent $\mathbf{T}$, normal $\mathbf{N}$, binormal $\mathbf{B}$ form an orthonormal frame along arclength-parametrized $\boldsymbol{\gamma}(x)$.

–The curvature κ and torsion τ determine the curve up to rigid motion.

An alternative is the orthonormal *natural frame*:

$$\mathbf{T}_x = \kappa_1\mathbf{U}_1 + \kappa_2\mathbf{U}_2, \quad (\mathbf{U}_1)_x = -\kappa_1\mathbf{T} + \sigma\mathbf{U}_2, \quad (\mathbf{U}_2)_x = -\kappa_2\mathbf{T} - \sigma\mathbf{U}_1,$$

where σ is a constant twist. The relation with the Frenet system is

$$\mathbf{U}_1 + \mathrm{i}\mathbf{U}_2 = (\mathbf{N} + \mathrm{i}\mathbf{B})\mathrm{e}^{\mathrm{i}\theta}, \quad \kappa_1 + \mathrm{i}\kappa_2 = \kappa\mathrm{e}^{\mathrm{i}\theta}, \quad \theta = \int^x (\tau(u) - \sigma)\,\mathrm{d}s.$$

For $\sigma = 0$, these are *frames of least rotation* (Bishop, 1975) and $q = \frac{1}{2}(\kappa_1 + \mathrm{i}\kappa_2)$ gives the Hasimoto map.

Identifying $\mathbb{R}^3 \cong \mathfrak{su}(2)$ via

$$\mathbf{y} \longrightarrow \sum_{k=1}^{3} y_k E_k, \quad \text{where } E_1 = \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}, \; E_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; E_3 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix},$$

and using transitivity of $SU(2)$ on the space of orthormal triples, write

$$T = \Omega^{-1} E_1 \Omega, \quad U_1 = \Omega^{-1} E_2 \Omega, \quad U_2 = \Omega^{-1} E_3 \Omega \quad \text{for } \Omega \in SU(2).$$

Substituting into the natural frame equations

$$\mathbf{T}_x = \kappa_1 \mathbf{U}_1 + \kappa_2 \mathbf{U}_2, \quad (\mathbf{U}_1)_x = -\kappa_1 \mathbf{T} + \sigma \mathbf{U}_2, \quad (\mathbf{U}_2)_x = -\kappa_2 \mathbf{T} - \sigma \mathbf{U}_1,$$

and setting $\sigma = 2\lambda$ and $q = \frac{1}{2}(\kappa_1 + \mathrm{i}\kappa_2) = \frac{1}{2}\kappa e^{\mathrm{i}\theta}$, we obtain

$$\boxed{\Omega_x = \begin{pmatrix} -\mathrm{i}\lambda & \mathrm{i}q \\ \mathrm{i}\bar{q} & \mathrm{i}\lambda \end{pmatrix} \Omega \quad \Longleftrightarrow \quad \left[-E_1 \frac{\mathrm{d}}{\mathrm{d}x} + \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix} \right] \Omega = \lambda \Omega}$$

the spatial part of the AKNS system[1]. Thus, the AKNS eigenvalue problem arises as the 2λ-natural Frenet system for a space curve of curvature κ and torsion τ.

[1] after Ablowitz, Kaup, Newell, and Segur (1974).

Inverting the Hasimoto Map

The Lax pair for the NLS equation is given by the *AKNS system*

$$\boldsymbol{\phi}_x = \begin{pmatrix} -\mathrm{i}\lambda & \mathrm{i}q \\ \mathrm{i}\bar{q} & \mathrm{i}\lambda \end{pmatrix} \boldsymbol{\phi}, \qquad x\boldsymbol{\phi}_t = \begin{pmatrix} \mathrm{i}(|q|^2 - 2\lambda^2) & 2\mathrm{i}\lambda q - q_x \\ 2\mathrm{i}\lambda\bar{q} - \bar{q}_x & \mathrm{i}(2\lambda^2 - |q|^2) \end{pmatrix} \boldsymbol{\phi}, \quad (1)$$

for a $\mathbb{C}^2$-valued eigenfunction $\boldsymbol{\phi}(x, t; \lambda)$ and *complex potential* q.

Given q, we can recover the curve and its natural frame from solving the AKNS system.

A fundamental matrix solution Ω of (1) leads to an "inverse" of the Hasimoto map via a formula due to Sym and Pohlmeyer:

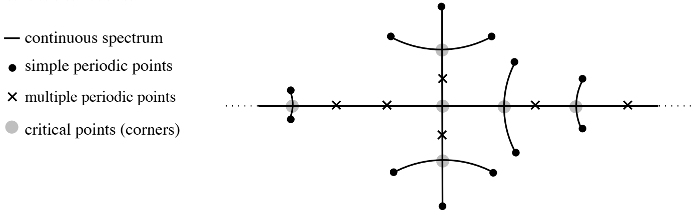
$$\Omega^{-1} \left. \frac{\mathrm{d}\Omega}{\mathrm{d}\lambda} \right|_{\lambda=\Lambda_0} = \begin{pmatrix} -\mathrm{i}\gamma_1 & \gamma_2 + \mathrm{i}\gamma_3 \\ -\gamma_2 + \mathrm{i}\gamma_3 & \mathrm{i}\gamma_1 \end{pmatrix} \simeq \boldsymbol{\gamma}, \qquad \Lambda_0 \in \mathbb{R}.$$

Such $\boldsymbol{\gamma}$ solves the VFE (plus a tangential drift) and has curvature $\kappa = |q|$ and torsion $\tau = \frac{\mathrm{d}}{\mathrm{d}x}\arg(q) - 2\Lambda_0$.

Constructing Closed Vortex Filaments

Floquet spectrum

For periodic NLS potentials $q(x+L,t) = q(x,t)$, the *Floquet spectrum* is the set $\sigma(q)$ of λ's such that the eigenfunctions of the AKNS system are *bounded in* x*:*



Given $\Phi(x;\lambda)$ a fundamental matrix solution of the AKNS system with $\Phi(0;\lambda) = \mathrm{I}$, we define the *Floquet discriminant*

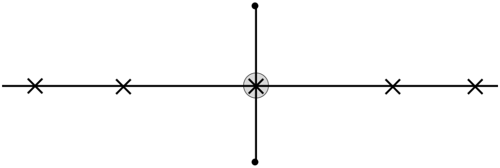
$$\Delta(q;\lambda) = \operatorname{tr}\Phi(L;\lambda),$$

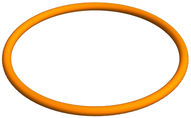
and represent the Floquet spectrum as

$$\sigma(q) = \{\lambda \in \mathbb{C} \,|\, \Delta(q;\lambda) \in \mathbb{R}, -2 \le \Delta \le 2\},$$

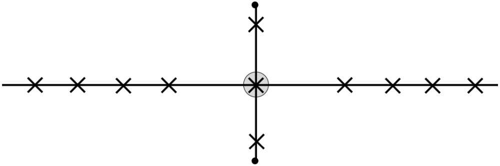
Key property: $\sigma(q)$ is conserved under the NLS evolution.

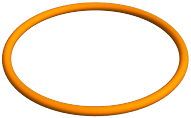
Examples of Floquet Spectra



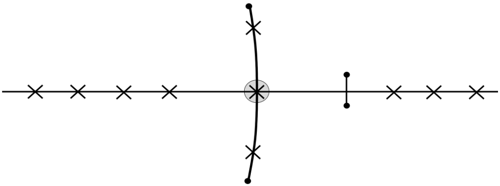


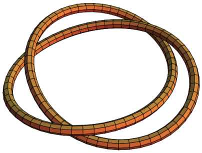
$$L = 2\pi, q(x,t) \equiv 1 = \frac{k}{2}$$





$$L = 4\pi, q(x,t) \equiv 1$$

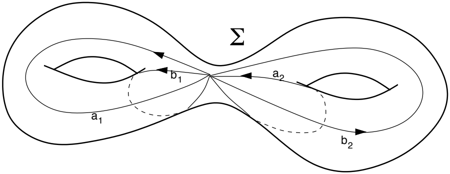




Finite genus potentials

Finite genus potentials q have $2g+2$ simple points $\lambda_j \in \sigma$ and $g+1$ critical points α_k, where g is the genus of a hyperelliptic Riemann surface with branch points λ_j.

$$y^2 = \prod_{i=1}^{2g+2} (\lambda - \lambda_i)$$



Such potentials can be written in terms of Riemann θ-functions:

$$q(x,t) = Ae^{-iEx+\mathrm{i}Nt}\frac{\theta(\mathrm{i}\mathbf{V}x + \mathrm{i}\mathbf{W}t + \mathbf{r})}{\theta(\mathrm{i}\mathbf{V}x + \mathrm{i}\mathbf{W}t)},$$

with $A, E, N \in \mathbb{R}$ and vectors $\mathbf{V}, \mathbf{W}, \mathbf{r} \in \mathbb{R}^g$ determined by period integrals on Σ.

The Baker-Akhiezer Eigenfunction

The Baker-Akhiezer eigenfunction (Its, Krichever, Previato) is key for constructing finite genus NLS potentials and associated filaments.

Given a non-special divisor $\mathcal{D}$ of degree $g+1$, there is a unique $\mathbb{C}^2$-valued function $\boldsymbol{\psi}(x,t,P)$, $P=(\lambda,y)\in\Sigma$ such that:

– $\boldsymbol{\psi}$ is meromorphic on $\Sigma\setminus\infty_\pm$ with poles in $\mathcal{D}$;
– $\boldsymbol{\psi}$ has prescribed essential singularities at ∞_- and ∞_+, respectively:

$$\boldsymbol{\psi}\sim e^{-\mathrm{i}(\lambda x+2\lambda^2 t)}\left[\begin{pmatrix}1\\0\end{pmatrix}+O(\lambda^{-1})\right],\qquad \boldsymbol{\psi}\sim e^{\mathrm{i}(\lambda x+2\lambda^2 t)}\left[\begin{pmatrix}0\\1\end{pmatrix}+O(\lambda^{-1})\right]$$

$$\boldsymbol{\psi}=\frac{\mathrm{e}^{\mathrm{i}(\Omega_1(P)+\frac{1}{2}E)x+\mathrm{i}(\Omega_2(P)-\frac{1}{2}N)t}\theta(\mathbf{D})}{\theta(\mathrm{i}x\mathbf{V}+\mathrm{i}t\mathbf{W}-\mathbf{D})\theta(\boldsymbol{\mathcal{A}}(P)-D)}\begin{bmatrix}\mathrm{e}^{-\mathrm{i}Ex+\mathrm{i}Nt}\theta(\mathrm{i}x\mathbf{V}+\mathrm{i}t\mathbf{W}+\boldsymbol{\mathcal{A}}(P)-\mathbf{D})\\-\mathrm{i}\mathrm{e}^{\Omega_3(P)}\theta(\mathrm{i}x\mathbf{V}+\mathrm{i}t\mathbf{W}+\boldsymbol{\mathcal{A}}(P)-\mathbf{D}-\mathbf{r})\end{bmatrix},$$

where $E,N\in\mathbb{R}$ and $\mathbf{V},\mathbf{W},\mathbf{r}\in\mathbb{R}^g$ are determined by period integrals on Σ, Ω_i's are normalized Abelian integrals, $\mathbf{D}$ depends on the choice of divisor $\mathcal{D}$, and $\boldsymbol{\mathcal{A}}$ is Abel map of Σ.

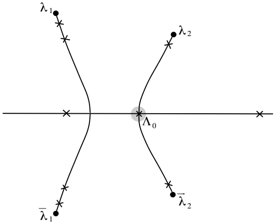
Finite genus filaments

A VFE solution $\boldsymbol{\gamma}$ is given by the Sym-Pohlmeyer formula $\Phi^{-1}\frac{\mathrm{d}\Phi}{\mathrm{d}\lambda}\big|_{\lambda=\Lambda_0}$, with $\Phi(x,t;\lambda) = [\boldsymbol{\psi}(P^+), \boldsymbol{\psi}(P^-)]$ constructed from the Baker-Akhiezer eigenfunction at points $P^+, P^- \in \Sigma$ projecting to $\lambda \in \mathbb{R}$.

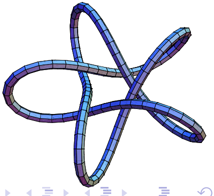
For example, the expression for the E_1-component of $\boldsymbol{\gamma}$ is

$$\gamma_1 = \left(\mathrm{i}(x\Omega_1' + t\Omega_2') + \nabla \log \theta(\mathrm{i}\mathbf{V}x + \mathrm{i}\mathbf{W}t - \mathbf{D}) \cdot \frac{\mathrm{d}\boldsymbol{\mathcal{A}}(P)}{\mathrm{d}\lambda} \right)\Bigg|_{\lambda=\Lambda_0}.$$

Example: $g = 1$, cnoidal potential.



$$q = Ae^{-iEx+\mathrm{i}Nt}\,\mathrm{cn}(vx+wt)$$



Closure Conditions

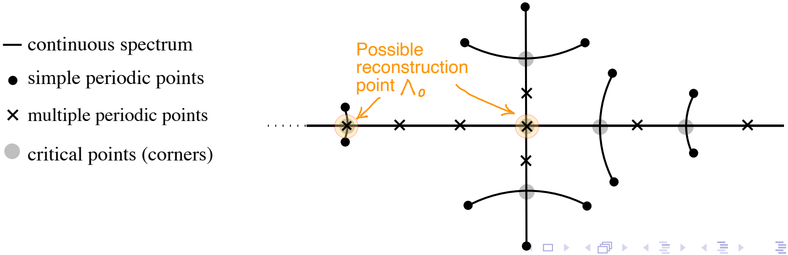
Linear growth in x is controlled by the *quasimomentum differential*

$$\Omega_1'(\lambda)\mathrm{d}\lambda = \frac{\lambda^{g+1} + c_0\lambda^g + \ldots + c_g}{\sqrt{\prod_i(\lambda - \lambda_i)}}\,\mathrm{d}\lambda.$$

Thus, a necessary condition for γ to be smoothly closed is $\Omega_1'(\Lambda_0) = 0$.

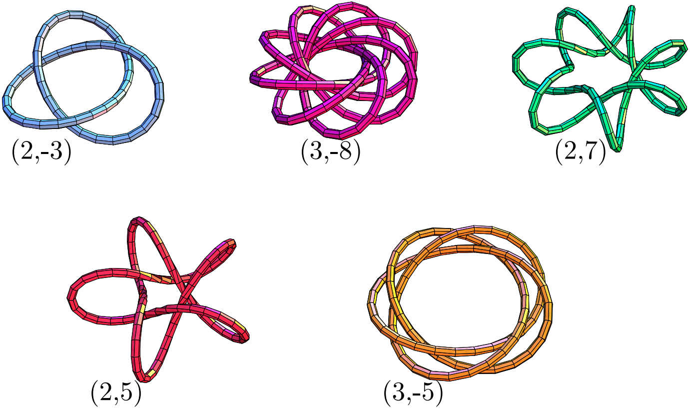
Closure Conditions (Grinevich & Schmidt, C-Ivey)

A filament obtained by the Sym-Pohlmeyer formula from an L-periodic q is smoothly closed of length L iff the reconstruction point $\Lambda_0 \in \mathbb{R}$ is:
1. *a real critical point,* *and* 2. *a double point of $\sigma(q)$.*



Kida Filaments

General genus-1 solutions of the VFE evolve by rigid motion



and realize all (n, m) torus knots, $n < |m|$ (Ivey & Singer, 1999).

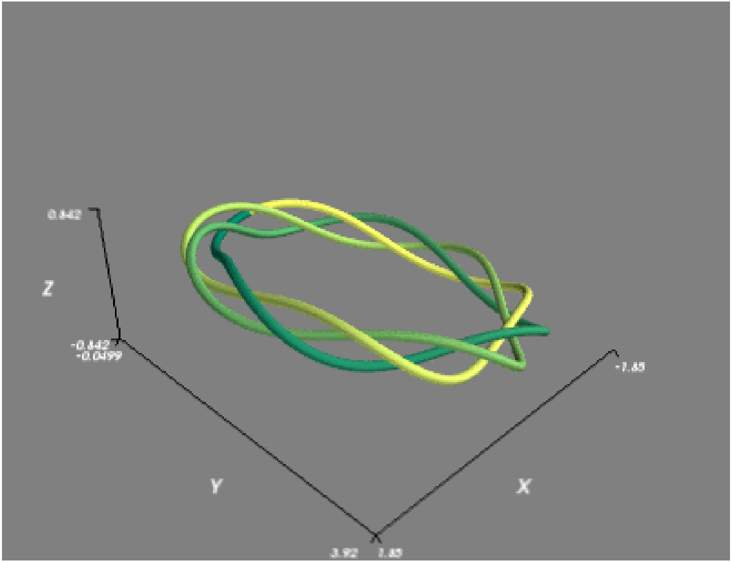
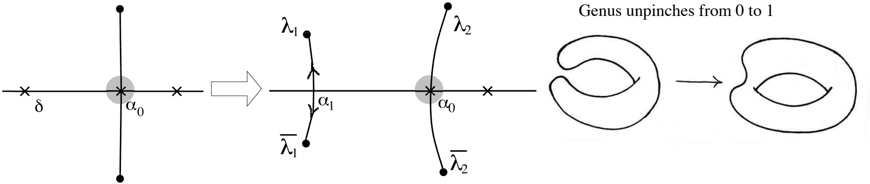


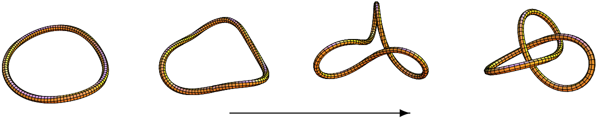
Figure: VFE evolution of a (2, 10)-torus knot (by Carter Rhea).

Filaments of Constant Knot Type determined by the spectrum

Small-amplitude filaments of arbitrary genus

Higher genus filaments can be constructed effectively in a neighborhood of the (unphysical) multiply covered circle, by deforming its spectrum while maintaining periodicity and closure.





increasing deformation parameter

Isoperiodic Deformations

Deforming spectral data while maintaining periodicity and closure.

For given genus g, let $\lambda_1, \ldots, \lambda_{2g+2}$ be the branch points, and $\alpha_0, \ldots, \alpha_g$ be the zeros of the quasimomentum differential $\mathrm{d}\Omega_1$.

For arbitrary real 'controls' $c_0(\xi), \ldots, c_g(\xi)$, the deformation

$$\frac{d\lambda_j}{d\xi} = -\sum_{k=0}^{g} \frac{c_k}{\lambda_j - \alpha_k} \qquad \frac{d\alpha_k}{d\xi} = \sum_{\ell \neq k} \frac{c_k + c_\ell}{\alpha_\ell - \alpha_k} - \frac{1}{2} \sum_{j=1}^{2g+2} \frac{c_k}{\lambda_j - \alpha_k}$$

preserves the frequencies $V_1, \ldots, V_g$ (Grinevich & Schmidt 1995, after Krichever 1994). Also,

Closure

If the Sym-Pohlmeyer formula at $\Lambda_0 = \alpha_k$ produces a closed curve, then deformations with $c_k = 0$ preserve closure.

Determining Frequencies

Suppose we start with a genus g solution, closed at α_0.

• pinch g pairs of branch points by running g successive deformations, first using

$$c_0 = c_1 = \ldots = c_{g-1} = 0, \quad c_g = -1,$$

until one pair of branch points collides on the real axis, then using

$$c_0 = c_1 = \ldots = c_{g-2} = c_g = 0, \quad c_{g-1} = -1,$$

until the next pair collides, and so on until only one pair remains.

• let $\delta_1, \ldots \delta_g \in \mathbb{R}$ be limiting values of $\alpha_1, \ldots, \alpha_g$ when this is over; then a residue calculation gives

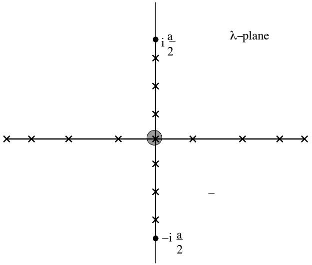
$$V_k = -2\,\mathrm{sign}(\delta_k)\sqrt{\delta_k^2 + 1}$$

Idea: Reverse this process, starting in genus zero and selecting the δ_k to give desired frequencies.

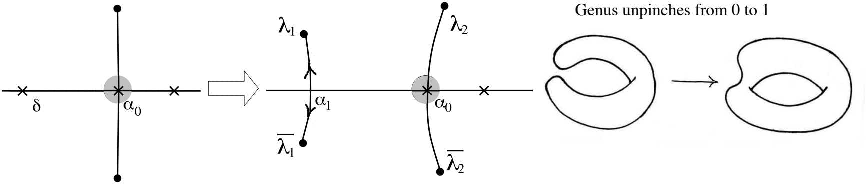
Unpinching

– Start in genus $g = 0$ with a plane wave potential/circular filament.

– Perform a sequence of closure-preserving isoperiodic deformations.



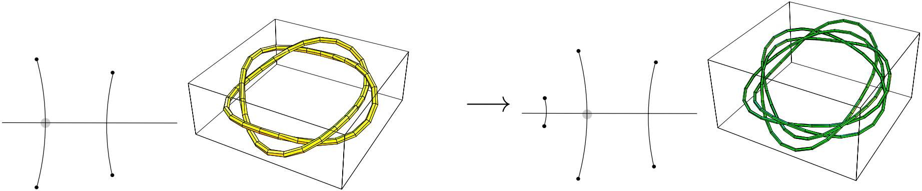
Each deformation 'opens up' a real double point δ to two simple points λ's and a critical point α, thus increasing the genus by 1 at each step.



Iterated torus knots

Theorem (Cabling Theorem, C-Ivey)

Opening up additional real double points results in cable knots.

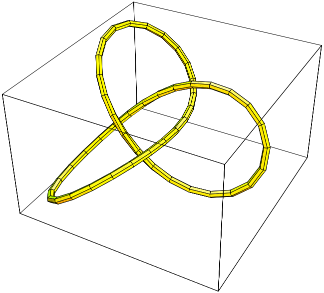


Deformation Schemes

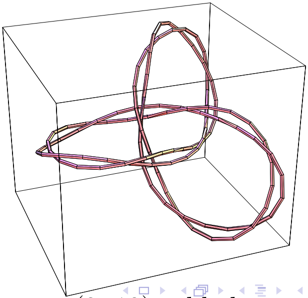
The notation $[n; m_1, \ldots, m_g]$, $n < |m_k|$ and $\gcd(n, m_1, \ldots, m_g) = 1$, describes a sequence of deformations that:

- begins with the n-times covered circle, with $\alpha_0 = 0$ and selected double points located at $\delta_k = -\operatorname{sign}(m_k)\sqrt{(m_k/n)^2 - 1}$
- opens up $\delta_1 = -\operatorname{sign}(m_1)\sqrt{(m_1/n)^2 - 1}$, then
- opens up $\delta_2 \approx -\operatorname{sign}(m_2)\sqrt{(m_2/n)^2 - 1}$, and so on.

Example: Deformation $[4; -6, -13]$

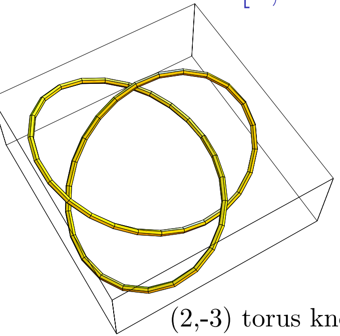


(2,-3) torus knot

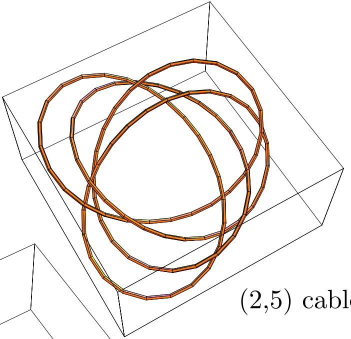


(2,-13) cable knot

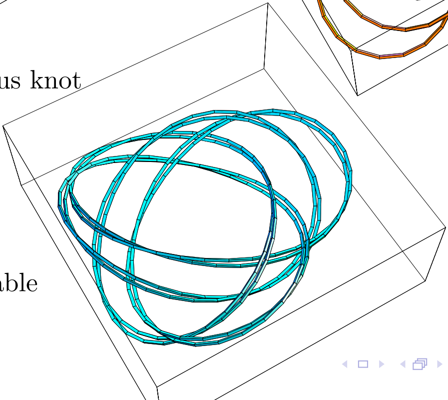
Deformation $[8; -12, 10, -17]$



(2,-3) torus knot



(2,5) cable



(2,-17) cable

Cabling Theorem

Each deformation step is a cabling operation.

Theorem (C-Ivey)

The scheme $[n; m_1, \ldots, m_g]$ produces a sequence of filaments $\boldsymbol{\gamma}^{(k)}$, beginning with a circle, such that

- $\boldsymbol{\gamma}^{(k)}$ is a closed genus k filament of length $2n\pi/\ell_k$, where

$$\ell_k = \gcd(n, m_1, \ldots, m_k), \qquad 0 \leq k \leq g.$$

- At any time t, $\boldsymbol{\gamma}^{(k)}$ is a $\left(\dfrac{\ell_{k-1}}{\ell_k}, \dfrac{m_k}{\ell_k}\right)$ cable about $\boldsymbol{\gamma}^{(k-1)}$.

For example, the deformation $[4; -6, -13]$ gives first a $(2, -3)$ torus knot of length 4π, and then a $(2, -13)$ cable of length 8π on the trefoil.

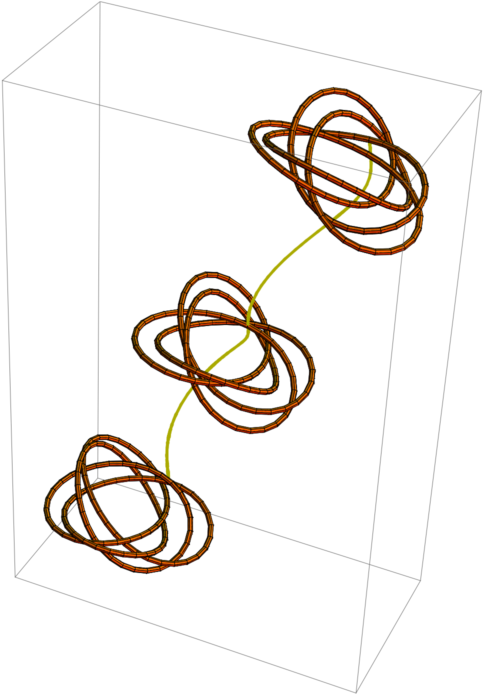
Invariance of knot type during the evolution

The following is an important consequence of the Cabling Theorem:

Corollary (Invariance of knot type)

The knot type of $\boldsymbol{\gamma}^{(k)}(x,t;\epsilon_k)$ *is fixed for all time.*

Remark. The VFE vector field is local $\implies$ topological changes can occur. Nonetheless, we construct a neighborhood of n-covered circles, within the class of finite-gap VFE solutions, consisting of filaments whose knot type is preserved.



Still frames of the evolution of a (2,5)-cable on a trefoil knot.

The dark yellow curve is the modified trajectory of a single point on the curve.

Thank You!

Outline of Proof: Analiticity

1. At kth step, branch points $\lambda_j(\epsilon)$ are analytic in ϵ.

   Finite-gap potentials are determined by the Dirichlet spectrum and branch points by trace formulas (Ablowitz-Ma), e.g.

$$\Im(q(x,t)) = -\frac{1}{2}\sum_{k=0}^{g}\left(\lambda_{2k} + \lambda_{2k-1} - 2\mu_k(x,t)\right).$$

   Dynamic Dirichlet eigenvalues $\mu_j(x,t), \nu_j(x,t)$ are analytic in ϵ, $\implies q(x,t)$ is analytic in ϵ.

Outline of Proof: Deformations of q and γ

2. Let $q = q_0 + \epsilon q_1 + O(\epsilon^2)$, and expand q_1 in terms of a biorthogonal basis of *squared eigenfunctions* for q_0:

$$\left\{(\psi_1)^2\right\}\Big|_{\text{simple pts } \sigma} \quad \left\{\psi_1^+\psi_1^-\right\}\Big|_{\text{crit. pts } \kappa} \quad \left\{u(\psi_1^+)^2 + \overline{u(\psi_2^+)^2}\right\}\Big|_{\text{double pts } \delta}$$

At k step all simple and critical points, and all double points except δ_k are *stationary* at order ϵ, so

$$q_1 = u(\psi_1^+)^2 + \overline{u(\psi_2^+)^2}\Big|_{\delta_k}, \qquad u \in \mathbb{C}$$

3. The Sym formula gives $\gamma = \gamma_0 + \epsilon\gamma_1 + O(\epsilon^2)$ with:

$$\gamma_1 = \frac{-1}{2(\Lambda_0 - \delta_k)^2}\left((2\,\mathrm{Im}\, u\psi_1^+\psi_2^+)T + (\mathrm{Re}\, q_1)\,U + (\mathrm{Im}\, q_1)\,V\right),$$

where (T, U, V) is the *natural frame* of constant twist Λ_0 along γ_0.

Outline of Proof: How γ_1 winds about γ_0

4. At step k, since q_0 is close to the plane wave solution (γ_0 close to the circle)

   $\arg(q_1)$ *is monotone in* x, *and winds* m_k *times around counterclockwise as* x *goes from* 0 *to* $n\pi$.

5. Because γ_0 is length $L = n\pi/\ell_{k-1}$, then γ_1 winds around m_k times for every ℓ_{k-1} circuits of γ_0.

   $\Longrightarrow$ for sufficiently small ϵ, γ is a $\left(\dfrac{\ell_{k-1}}{\ell_k}, \dfrac{m_k}{\ell_k}\right)$ cable about γ_0.

6. The natural frame is unlinked with γ_0. This follows from continuity and White's formula.

Extra detail: Natural frames unlinked

Show that $\gamma_0 + \epsilon U$ is unlinked with γ_0, using *White's formula*:

$$\begin{aligned}\mathrm{Lk}\left(\gamma_0, \gamma_0 + \epsilon U_1\right) &= \mathrm{Wr}\left(\gamma_0\right) + \frac{1}{2\pi}\int \left(T \times U\right) \mathrm{d}U \\ &= \mathrm{Wr}\left(\gamma_0\right) + \frac{1}{2\pi} L\Lambda_0\end{aligned}$$

If γ_0 is close to the circle, its self-linking number (given by *Pohl's formula*) is zero:

$$0 = \mathrm{SL}(\gamma_0) = \mathrm{Wr}(\gamma_0) + \frac{1}{2\pi}\int \tau \, dx$$

Thus,

$$\mathrm{Lk}\left(\gamma_0, \gamma_0 + \epsilon U\right) = \frac{1}{2\pi}\int \left(\Lambda_0 - \tau\right) dx = \frac{-1}{2\pi}\int \mathsf{Im}\left(\frac{d \log q_0}{dx}\right) dx = 0$$