# Quantization of the Willmore Energy in Riemannian Manifolds


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# Part 1: Euclidean Theory

# I. Early History: Elasticity Theory

The integral of mean curvature squared

$$\int_{\Sigma} H^2 d\mathrm{vol}_g = \frac{1}{4}\int_{\Sigma}(\kappa_1+\kappa_2)^2 d\mathrm{vol}_g \tag{1}$$

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Euler-Lagrange equation found by Poisson:

$$\begin{aligned} &\mathrm{Z} - p\mathrm{X} - q\mathrm{Y} + (\varepsilon^2 a^2 \pi - 2\Pi)\, k\mathrm{H} + \frac{\varepsilon^2 b^2 k\pi}{2}\cdot \mathrm{GH} \\ &\quad - \frac{\varepsilon^2 b^2 \pi}{8}\left[\frac{1+q^2}{2k}\cdot\frac{d^2\mathrm{H}}{dx^2} - \frac{pq}{k}\cdot\frac{d^2\mathrm{H}}{dx\,dy} + \frac{1+p^2}{2k}\cdot\frac{d^2\mathrm{H}}{dy^2}\right. \\ &\quad \left. - p\mathrm{H}\frac{d\mathrm{H}}{dx} - q\mathrm{H}\frac{d\mathrm{H}}{dy} + k\mathrm{H}^3\right] = 0. \qquad (a). \end{aligned}$$

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$$\begin{aligned} Z - pX - qY + (\varepsilon^2 a^2 \pi - 2\Pi)\, kH + \frac{\varepsilon^2 b^2 k\pi}{2}\cdot GH \\ - \frac{\varepsilon^2 b^2 \pi}{8}\Big[\frac{1+q^2}{2k}\cdot\frac{d^2H}{dx^2} - \frac{pq}{k}\cdot\frac{d^2H}{dx\,dy} + \frac{1+p^2}{2k}\cdot\frac{d^2H}{dy^2} \\ - pH\frac{dH}{dx} - qH\frac{dH}{dy} + kH^3\Big] = 0. \qquad (a). \end{aligned}$$

We can rewrite it (once we neglect the forces) as

$$\Delta_g H + 2H(H^2 - K) = 0. \tag{2}$$

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un coëfficient constant. Cela tient à ce que l'on a identiquement

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His PhD student Olinde Rodrigues identified in 1815 the constant:

S'il s'agit d'une portion quelconque de surface développable, le rayon mobile ne décrira qu'une simple courbe, et l'intégrale sera nulle; ce qui est d'ailleurs évident, puisqu'on a alors $rt - s^2 = 0$. Dans le cas d'une surface fermée et convexe dans toute son étendue, telle qu'un ellipsoïde, on aura

$$\iint \frac{(rt - s^2)\, dx\, dy}{(1 + p^2 + q^2)^{\frac{3}{2}}} = 4\pi;$$

# II. Early History: Classical Results

Theorem (Blaschke-Thomsen, *Konformminimalflächen*, 1923)

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*For all immersion $\vec{\Phi} : \Sigma \to \mathbb{R}^n$ from a closed surface $\Sigma$, we have $W(\vec{\Phi}) \geq 4\pi$, with equality if and only if $\Sigma = S^2$ and $\vec{\Phi}(S^2)$ is a round sphere.*

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It appears in various fields : **Hawking mass** (1972) in **general relativity**, Helfrich energy (1973) to model the **elasticity of cellular membranes**, and in the construction of spectacle lens (patent by Katzman-Rubinstein 2001).

# III. Analytic Challenges: Euler-Lagrange Equation

The Euler-Lagrange equation

$$\Delta_g H + 2H(H^2 - K_g) = 0$$

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A weak formulation should exist for $\vec{\Phi} : \Sigma \to S^3$ is Willmore if and only if its conformal Gauss map $\vec{\psi} : \Sigma \to S^{3,1}$ is **harmonic**

$$-\Delta\vec{\psi} = |\nabla\vec{\psi}|_h^2\,\vec{\psi}, \tag{3}$$

where $\vec{\psi} = (H, H\vec{\Phi} + \vec{n})$ and $\vec{n} : \Sigma \to S^2$ is the unit normal.

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Notice that

$$\begin{aligned} W(\vec{\Phi}) &= \int_\Sigma |\nabla\vec{\psi}|_h^2\, d\mathrm{vol}_g + 2\pi\,\chi(\Sigma) \\ &= \frac{1}{4}\int_\Sigma |d\vec{n}|_g^2 d\mathrm{vol}_g + \pi\,\chi(\Sigma) \end{aligned}$$

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Theorem (Rivière, *Invent. Math.* 2008)

*A weak immersion $\vec{\Phi} : \Sigma \to \mathbb{R}^n$ is Willmore if and only if*

$$d\,\left(*_g \vec{H} - 3 *_g\,\pi_{\vec{n}}\left(d\vec{H}\right) + \star\left(d\vec{n} \wedge \vec{H}\right)\right) = 0. \tag{4}$$

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In $\mathbb{R}^3$ with conformal coordinates, the equation becomes

$$\mathrm{div} \left( 2\, \nabla \vec{H} - 3\, H\, \nabla \vec{n} + \vec{H} \times \nabla^{\perp} \vec{n} \right) = 0, \tag{5}$$

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Metaphysical explanation for the existence of a divergence form: Rivière's theorem on conformally invariant problems (*Invent. Math.* 2006).

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Theorem (Bernard-Rivière, *Ann. of Math.* 2014)

*Let* $\{\vec{\Phi}_k\}_{k\in\mathbb{N}} : \Sigma \to \mathbb{R}^n$ *be a sequence of Willmore immersions. Assume that*

$$\limsup_{k\to\infty} W(\vec{\Phi}_k) < \infty,$$

*and that the conformal class of* $\vec{\Phi}_k^* g_{\mathbb{R}^n}$ *stays within a compact subset of the moduli space. Then, up to a subsequence, we have*

$$\lim_{k\to\infty} W(\vec{\Phi}_k) = W(\vec{\Phi}_\infty) + \sum_{j=1}^{q} \Big( W(\vec{\Psi}_j) - 4\pi\,\theta_j \Big)\,, \tag{6}$$

*where* $\vec{\Phi}_\infty : \Sigma \to \mathbb{R}^n$ *and* $\vec{\Psi}_j : S^2 \to \mathbb{R}^n$ *are branched Willmore* $\theta_j \in \mathbb{N}$*.*

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Example of loss of compactness by Nicolas Marque (*IMRN*, 2021).

# IV. Proof of the Energy Quantization

Using the boundedness of energy, there are finitely many bubbles. Bubble domain :

$$\int_{B_{\rho_k}(0)} |\nabla \vec{n}_k|^2 dx \geq \varepsilon_0 > 0 \qquad \text{for all } \ k \in \mathbb{N},$$

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**Neck region**: annulus $\Omega_k(\alpha) = B_{\alpha R_k} \setminus \overline{B}_{\alpha^{-1}\rho_k}(0)$. By Rivière's $\varepsilon$-regularity, we need only prove that

$$\lim_{\alpha \to 0} \limsup_{k \to \infty} \int_{\Omega_k(\alpha)} |\nabla \vec{n}_k|^2_{g_k} dx = 0.$$

This is called the **no-neck (energy) property**.

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$$\lim_{\alpha \to 0} \limsup_{k \to \infty} \int_{\Omega_k(\alpha)} H_k^2 d\mathrm{vol}_{g_k} = 0. \tag{7}$$

# V. The “No-Neck Energy” Property

The idea of proof is due to Lin-Rivière in the setting of Ginzburg-Landau vortices (*CPAM*, 2001) and harmonic maps in manifolds (*Duke Math. J.* 2002), then extended by Rivière to Yang-Mills functional (*CAG*, 2002), biharmonic maps (Laurain-Rivière *Adv. Calc. Var.* 2013), and the Willmore energy (Bernard-Rivière, *Ann. of Math.* 2014).

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First, one proves a uniform $L^{2,1}$ estimate:

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Then, one shows an $L^{2,\infty}$ quantization:

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Finally, using the $L^{2,1}/L^{2,\infty}$ duality (17), we deduce that

$$\int_{\Omega_k(\alpha)} e^{2\lambda_k} H_k^2\, dx \leq \left\|e^{\lambda_k} H_k\right\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))} \left\|e^{\lambda_k} H_k\right\|_{\mathrm{L}^{2,\infty}(\Omega_k(\alpha))} \underset{\substack{k\to\infty\\ \alpha\to 0}}{\longrightarrow} 0.$$

# VI. Conservation Laws

Recall the Willmore equation

$$\operatorname{div}\left(2\,\nabla\vec{H}_k - 3\,H_k\,\nabla\vec{n} + \vec{H}_k \times \nabla^{\perp}\vec{n}_k\right) = 0 \qquad \text{in } B(0,1). \tag{10}$$

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By the Poincaré lemma, there exists $\vec{L}_k : B(0,1) \to \mathbb{R}^3$ such that

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We have the following conservation laws:

$$\begin{cases} \nabla^{\perp}\vec{\Phi}_k \times \nabla\vec{L}_k = 0 \\ \nabla^{\perp}\vec{\Phi}_k \times \nabla\vec{L}_k + 2\nabla H_k \cdot \nabla^{\perp}\vec{\Phi}_k = 0 \end{cases} \tag{12}$$

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Therefore, there exists $(S_k, \vec{R}_k) : B(0,1) \to \mathbb{R} \times \mathbb{R}^3$ such that

$$\begin{cases} \nabla S_k = \nabla\vec{\Phi}_k \cdot \vec{L}_k \\ \nabla\vec{R}_k = \nabla\vec{\Phi}_k \times \vec{L}_k + 2H_k\nabla\vec{\Phi}_k. \end{cases} \tag{13}$$

# VI. Conservation Laws

The functions $S_k$ and $\vec{R}_k$ solve the system

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## Lemma (Wente, 1969)

*Let $a, b \in W^{1,2}(\Omega, \mathbb{R})$. Let $u : \Omega \to \mathbb{R}$ be the solution of*

$$\begin{cases} -\Delta u = \langle \nabla a, \nabla^\perp b \rangle & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases}$$

*Then, we have $\nabla u \in L^{2,1}(\Omega)$, and*

$$\|\nabla u\|_{\mathrm{L}^{2,1}(\Omega)} \leq C(\Omega) \|\nabla a\|_{\mathrm{L}^2(\Omega)} \|\nabla b\|_{\mathrm{L}^2(\Omega)} .$$

*In particular, $u \in L^\infty(\Omega)$ by the Sobolev embedding $W^{1,(2,1)}(\Omega) \hookrightarrow C^0(\Omega)$.*

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In particular, $e^{\lambda_k}\vec{L}_k \in L^{2,\infty}(\Omega_k(1/2))$.

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$$e^{\lambda_k}|\vec{L}_k|(x) \leq \frac{C}{|x|}.$$

In particular, $e^{\lambda_k}\vec{L}_k \in L^{2,\infty}(\Omega_k(1/2))$. By the system

$$\begin{cases} \nabla S_k = \nabla \vec{\Phi}_k \cdot \vec{L}_k \\ \nabla \vec{R}_k = \nabla \vec{\Phi}_k \times \vec{L}_k + 2H_k \nabla \vec{\Phi}_k, \end{cases}$$

we deduce that $\nabla S_k, \nabla \vec{R}_k \in L^{2,\infty}(\Omega_k(1/2))$.

# VI. Improved Wente Inequality

**Lemma**

*Let* $a \in W^{1,2}(\Omega, \mathbb{R})$, $b \in W^{1,(2,\infty)}(\Omega, \mathbb{R})$. *Let* $u : \Omega \to \mathbb{R}$ *be the solution of*

$$\begin{cases} -\Delta u = \langle \nabla a, \nabla^{\perp} b \rangle & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases}$$

*Then, we have* $\nabla u \in L^2(\Omega)$, *and*

$$\|\nabla u\|_{\mathrm{L}^2(\Omega)} \leq C_2(\Omega) \|\nabla a\|_{\mathrm{L}^2(\Omega)} \|\nabla b\|_{\mathrm{L}^{2,\infty}(\Omega)}.$$

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Using versions of those inequalities on annuli, we get successively $\nabla S_k, \nabla \vec{R}_k \in L^2(\Omega_k(1/4))$ and $\nabla S_k, \nabla \vec{R}_k \in L^{2,1}(\Omega_k(1/8))$.

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Using versions of those inequalities on annuli, we get successively $\nabla S_k, \nabla \vec{R}_k \in L^2(\Omega_k(1/4))$ and $\nabla S_k, \nabla \vec{R}_k \in L^{2,1}(\Omega_k(1/8))$. By the identity

$$-2\, e^{2\lambda_k} \vec{H}_k = \nabla \vec{R}_k \times \nabla^{\perp} \vec{\Phi}_k + \nabla^{\perp} S_k \cdot \nabla \vec{\Phi}_k,$$

we deduce that

$$\left\| e^{\lambda_k} \vec{H}_k \right\|_{\mathrm{L}^{2,1}(\Omega_k(1/8)))} \leq C.$$

# Part 2: Riemannian Theory

# I. The Case of Riemannian Manifolds

Let $(M^3, h)$ be a closed Riemannian manifold. If $\vec{\Phi} : S^2 \to M^3$ is a smooth immersion, we define

$$W(\vec{\Phi}) = W_{(M^3,h)}(\vec{\Phi}) = \int_{S^2} \left(H^2 + \overline{K}\right) d\mathrm{vol}_g,$$

where $g = \vec{\Phi}^* h$ is the induced metric, $H$ is the mean curvature, and $\overline{K} = K(\vec{\Phi}_* TS^2)$ is the curvature of the 2-plan spanned by $\vec{\Phi}$.

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**Conformal Invariance:** for all Riemannian manifold $(N^3, k)$, if $\Psi : (M^3, h) \to (N^3, k)$ is a conformal diffeomorphism ($\Psi^* k = e^{2u}\, h$ for some smooth function $u : M^3 \to \mathbb{R}$), then

$$W_{(N^3,k)}(\Psi \circ \vec{\Phi}) = W_{(M^3,h)}(\vec{\Phi}). \tag{15}$$

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# II. Main Result

Theorem (M. & Mondino, 2021)

*Let* $\{\vec{\Phi}_k\}_{k\in\mathbb{N}} : S^2 \to M^3$ *be a sequence of Willmore immersions, and assume that*

$$\limsup_{k\to\infty}\Big(W(\vec{\Phi}_k)+\mathrm{Area}(\vec{\Phi}_k)\Big)<\infty.$$

*Then, up to a subsequence, the following energy identity holds*

$$\lim_{k\to\infty} W(\vec{\Phi}_k)=W(\vec{\Phi}_\infty)+\sum_{i=1}^{p}W(\vec{\Psi}_i)+\sum_{j=1}^{q}\left(W_{\mathbb{R}^n}(\vec{\chi}_j)-4\pi\,\theta_j\right),$$

*where* $\vec{\Psi}_i$ *and* $\vec{\chi}_j$ *are Willmore spheres respectively into* $M^3$ *or* $\mathbb{R}^3$ *and* $\theta_j\in\mathbb{N}$.

# II. Comments and Open Problems

(1) The extra hypothesis is natural by conformal invariance: if $\pi : S^3 \setminus \{N\} \to \mathbb{R}^3$ is the stereographic projection, and $\widetilde{\vec{\Phi}} = \pi \circ \vec{\Phi} : S^2 \to \mathbb{R}^3$

$$\int_{S^2} (H^2 + 1) d\mathrm{vol}_g = \int_{S^2} |\widetilde{H}|^2 d\mathrm{vol}_{\widetilde{g}}.$$

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(3) In $\mathbb{R}^n$, the space of Willmore tori of energy $W \leq 8\pi - \delta$ ($\delta > 0$) is compact (Kuwert-Schätzle *Ann. of Math.* 2004 for $n = 3$, Rivière *Invent. Math.* $n \geq 3$).

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(4) Application to the **Willmore flow** (Kuwert-Schätzle *Ann. of Math.* [2001, 2004], Palmurella-Rivière *Adv. Math.* 2022).

# III. Analytic Difficulties

Critical points satisfy the Euler-Lagrange equation

$$\Delta_g H + 2H(H^2 - K_g) + \mathrm{Ric}(\vec{n},\vec{n})H + \langle (\nabla_{\vec{n}} R)(\vec{e}_1,\vec{e}_2)\vec{e}_2,\vec{e}_1 \rangle = 0. \quad (16)$$

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It requires that $H \in L^3(S^2)$ though $\vec{\Phi} \in W^{2,2}(S^2)$.

Using Mondino-Rivière's result in Riemannian manifolds (*Adv. Math.* 2013), the equation (16) becomes ($\vec{H} = H\vec{n}$)

$$\begin{aligned} &\mathrm{Re}\,\left(\nabla_{\bar{z}}\left(\nabla_z \vec{H} - 3\nabla_z^N \vec{H} - i\, \star_h \left(\nabla_z \vec{n} \wedge \vec{H}\right)\right)\right) \\ &= \frac{1}{2} e^{2\lambda}\left(\mathrm{Ric}(\vec{n}, \vec{n})H - 2\,\overline{K}\,H + \langle (\nabla_{\vec{n}} R)(\vec{e}_1, \vec{e}_2)\vec{e}_2, \vec{e}_1 \rangle\right)\vec{n} \\ &+ \frac{1}{2} e^{2\lambda}\left(\mathscr{R}_2(d\vec{\Phi}) - 8\,\mathrm{Re}\,\left(\langle R(\vec{e}_{\bar{z}}, \vec{e}_z)\vec{e}_z, \vec{H}\rangle \vec{e}_{\bar{z}}\right)\right), \end{aligned}$$

where $e^{2\lambda} = 2|\partial_z \vec{\Phi}|^2$, $\vec{e}_z = \partial_z \vec{\Phi}$, and $\vec{n} : S^2 \to S^2$ is the unit normal.

# IV. Lorentz and Orlicz Spaces

Let $\varphi : (0,\infty) \to (0,\infty)$ be a concave function such that $\varphi(0) = 0$, and $\Omega \subset \mathbb{R}^n$. For all measurable $f : \Omega \to \mathbb{R}^m$, define the norm

$$\|f\|_{N(\varphi)} = \int_0^\infty \varphi\left(\lambda_f(t)\right) dt,$$

where $\lambda_f(t) = \mathscr{L}^n\left(\Omega \cap \{x : |f(x)| > t\}\right)$.

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Theorem (Steigerwalt-White, 1971)

*The functional $\|\cdot\|_{N(\varphi)}$ is a norm and $(N(\varphi), \|\cdot\|_{N(\varphi)})$ is a Banach space.*

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### Theorem (Steigerwalt-White, 1971)

*The functional $\|\cdot\|_{N(\varphi)}$ is a norm and $(N(\varphi), \|\cdot\|_{N(\varphi)})$ is a Banach space.*

**Dual Spaces**. Define the decreasing rearrangement $f_* : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{\infty\}$ of $f$ by $f_*(t) = \inf \mathbb{R}_+ \cap \{s : \lambda_f(s) \leq t\}$, and

$$\|f\|_{M(\varphi)} = \sup_{t>0} \frac{1}{\varphi(t)} \int_0^t f_*(s)ds.$$

# IV. Lorentz and Orlicz Spaces

### Theorem (Steigerwalt-White, 1971)

*(1) Assume that $\varphi(t) = o(t)$ as $t \to \infty$. Then $M(\varphi)$ is a norm and $(M(\varphi), \|\cdot\|_{M(\varphi)})$ is a Banach space.*

*(2) For all $(f, g) \in N(\varphi) \times M(\varphi)$, we have $f \cdot g \in L^1(X, \mu)$ and*

$$\left| \int_X f \cdot g \, d\mu \right| \leq \|f\|_{N(\varphi)} \|g\|_{M(\varphi)} . \tag{17}$$

*In particular, we have $N(\varphi)^* = M(\varphi)$.*

### Remark

If $L^{p,1} = N(t^{\frac{1}{p}})$ and $L^{p,\infty} = M(t^{1-\frac{1}{p}})$, we recover that $(L^{p,1})^* = L^{p',\infty}$ for all $1 < p < \infty$.

# V. Proof of the Main Theorem

(1) Prove that $e^{\lambda_k}$ is bounded in $L^p$ for some $p > 2$ independent of $k$. It follows from our assumption on the boundedness of area.

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(2) Using Bernard-Rivière's approach, get a uniform Harnack inequality

$$\|\lambda_k - d_k \log |z| - A_k\|_{\mathrm{L}^\infty(\Omega_k(\alpha))} \leq C,$$

where $d_k \underset{k \to \infty}{\longrightarrow} d > -1$.

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$$\|\lambda_k - d_k \log|z| - A_k\|_{\mathrm{L}^\infty(\Omega_k(\alpha))} \leq C,$$

where $d_k \underset{k\to\infty}{\longrightarrow} d > -1$.

(3) Construct by convolution a function $\vec{U}_k$ such that

$$\partial_z \vec{U}_k = i\left(\nabla_z \vec{H}_k - 3\nabla_z^N \vec{H}_k - i \star_h \left(\nabla_z \vec{n}_k \wedge \vec{H}_k\right)\right) = \vec{Y}_k, \qquad (18)$$

and satisfying the estimates

$$|\vec{U}_k| \leq \frac{C}{|z|}\left(1 + \log_+\left(\frac{R_k}{|z|}\right)\right),$$
$$\mathrm{Im}\,(\vec{U}_k) \in W^{1,(2,\infty)}(B(0,R_k)).$$

# V. Proof of the Main Theorem

An estimate

$$|u(z)| \leq \frac{C}{|z|}\left(\log\left(\frac{R}{|z|}\right)+1\right) \qquad \text{for all } z \in B(0,R)$$

implies that $u \in M(\varphi) = L^{2,\infty}_{\log^\beta}(B(0,R))$ (with $\beta = 1$), where for all $0 \leq \beta \leq 1$

$$\varphi_\beta(t) = \sqrt{t}\left(1+\log_+^\beta\left(R\sqrt{\frac{\pi}{t}}\right)\right).$$

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$$\varphi_{\beta}(t) = \sqrt{t}\left(1+\log_{+}^{\beta}\left(R\sqrt{\frac{\pi}{t}}\right)\right).$$

By a standard decomposition, one need only consider holomorphic maps.

## Lemma

*Let $u : B(0,R) \to \mathbb{C}$ be a holomorphic function and fix some $0 \leq \alpha < 1$ and $0 \leq \beta \leq 1$. If $u \in L^{2,\infty}_{\log^{\beta}}(B(0,R))$, then $u \in W^{1,1} \cap L^{2,1}(B(0,\alpha R))$ and*

$$\|u\|_{\mathrm{L}^{2,1}(B(0,\alpha R))} + \|\nabla u\|_{\mathrm{L}^{1}(B(0,\alpha R))} \leq \frac{C\alpha}{(1-\sqrt{\alpha})^{\frac{5}{2}}}\log^{\beta}\left(\frac{2}{1-\sqrt{\alpha}}\right)\|u\|_{\mathrm{L}^{2,\infty}_{\log^{\beta}}(B(0,R))}.$$

# V. Proof of the Main Theorem

By a fixed point argument (cf. Mondino-Rivière), $\exists \alpha_0 > 0$ and $\vec{L}_k$ such that

$$\nabla_z \vec{L}_k = \vec{Y}_k$$

$$e^{\lambda_k} |\vec{L}_k| \leq \frac{C}{|z|} + \psi_k \qquad \text{in } \Omega_k(\alpha_0)$$

where $\psi_k \in L^p(B(0, \alpha_0 R_k))$ for some $p > 2$ independent of $k$.

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where $\psi_k \in L^p(B(0, \alpha_0 R_k))$ for some $p > 2$ independent of $k$.

By a generalisation of Rivière's **conservation laws** (Mondino-Rivière), we construct $S_k, \vec{R}_k \in W^{1,(2,\infty)}(B(0, \alpha_0 R_k))$ such that

$$\begin{cases} \partial_z S_k = \langle \partial_z \vec{\Phi}_k, \overline{\vec{L}_k} \rangle & \text{in } B(0, \alpha_0 R_k) \\ \mathrm{Im}\,(S_k) = 0 & \text{on } \partial B(0, \alpha_0 R_k), \end{cases} \tag{19}$$

$$\begin{cases} \nabla_z \vec{R}_k = \partial_z \vec{\Phi}_k \wedge \vec{L}_k - 2i\, \partial_z \vec{\Phi}_k \wedge \vec{H}_k & \text{in } B(0, \alpha_0 R_k) \\ \mathrm{Im}\,(\vec{R}_k) = 0 & \text{on } \partial B(0, \alpha_0 R_k). \end{cases} \tag{20}$$

# V. Proof of the Main Theorem

Using Calderón-Zygmund estimates, we prove that $\mathrm{Im}\,(S_k), \mathrm{Im}\,(\vec{R}_k)$ are bounded in $W^{2,q}(B(0, \alpha_0 R_k))$ for some $q > 1$.

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The coupled system

$$\begin{cases} \nabla_z \vec{R}_k = i\left(\star_h\left(\vec{n}_k \llcorner \nabla_z \vec{R}_k\right) + (\partial_z S_k)\star_h \vec{n}_k\right) & \text{in } B(0,\alpha_0 R_k) \\ \partial_z S_k = -i\langle \nabla_z \vec{R}_k, \star_h \vec{n}_k\rangle & \text{in } B(0,\alpha_0 R_k) \end{cases}, \quad (21)$$

permits to make Jacobian equations appear for $\mathrm{Re}\,(\vec{R}_k)$ and $\mathrm{Re}\,(S_k)$, and using estimates inspired by the Wente inequality, and averaging methods, one finds that $\nabla \vec{R}_k, \nabla S_k \in L^{2,1}(\Omega_k(\alpha_0/2))$.

# VI. Averaging Lemma

## Lemma (Bernard-Rivière, *Annals of Math.* 2014)

*Let* $k, m \in \mathbb{N}$, $u \in W^{1,1}(B(0,1),\mathbb{C})$, $f \in L^2(B(0,1),\mathbb{C})$, $\vec{v} \in W^{1,(2,\infty)}(B(0,1),\Lambda^k\mathbb{C}^m)$, $\vec{w} \in W^{1,2}\cap L^\infty(B(0,1),\Lambda^k\mathbb{R}^m)$ *such that*

$$\partial_z u = -i\left(\langle \partial_z \vec{v}, \vec{w}\rangle + f\right).$$

*Let* $0 < r < R < \infty$ *and* $\Omega = B_R \setminus \overline{B}_r(0)$. *Assume that* $\mathrm{Im}\,(\vec{v}) \in W^{1,2}(\Omega)$ *and*

$$|\nabla \mathrm{Re}\,(\vec{v})(z)| \leq \frac{C_0}{|z|} \qquad \text{for all } r \leq |z| \leq R.$$

*Then, we have*

$$\left(\int_r^R \left|\frac{d}{d\rho}\mathrm{Re}\,(u)_\rho\right|^2 \rho\, d\rho\right)^{\frac{1}{2}} \leq \sqrt{2\pi}\binom{n}{k} C_0 \,\|\nabla \vec{w}\|_{\mathrm{L}^2(\Omega)} + \frac{1}{\sqrt{2\pi}}\|\vec{w}\|_{\mathrm{L}^\infty(\Omega)}\|\nabla \mathrm{Im}\,(\vec{v})\|_{\mathrm{L}^2(\Omega)} + \frac{1}{\sqrt{2\pi}}\|f\|_{\mathrm{L}^2(\Omega)},$$

*where* $\varphi_\rho = \dfrac{1}{2\pi\rho}\displaystyle\int_{\partial B(0,\rho)} \varphi\, d\mathscr{H}^1$ *is the average.*

# VII. Jacobians and the Wente Inequality

Lemma (Laurain-Rivière, *Anal. PDE*, 2014)

*Let* $0 < 4r < R < \infty$, $\Omega = B(0,R) \setminus \overline{B}(0,r) \to \mathbb{R}$, $a, b : \Omega \to \mathbb{R}$ *such that* $\nabla a \in \mathrm{L}^{2,\infty}(\Omega)$ *and* $\nabla b \in \mathrm{L}^2(\Omega)$, *and* $u \in W^{1,(2,\infty)}(\Omega)$ *be a solution of*

$$\Delta u = \nabla a \cdot \nabla^{\perp} b \qquad \text{in } \ \Omega.$$

*Assume that* $\nabla u_\rho \in L^2(\Omega)$. *Then* $\nabla u \in \mathrm{L}^2(\Omega)$, *and there exists* $C_0 < \infty$ *independent of* $0 < 4r < R$ *such that for all* $\left(\dfrac{r}{R}\right)^{\frac{1}{2}} < \alpha < \dfrac{1}{2}$

$$\|\nabla u\|_{\mathrm{L}^2(B_{\alpha R} \setminus \overline{B}_{\alpha^{-1} r})} \leq C_0 \left( \|\nabla a\|_{\mathrm{L}^{2,\infty}(\Omega)} \|\nabla b\|_{\mathrm{L}^2(\Omega)} + \|\nabla u_\rho\|_{\mathrm{L}^2(\Omega)} + \|\nabla u\|_{\mathrm{L}^{2,\infty}(\Omega)} \right).$$

# VII. Jacobians and the Wente Inequality

From the conservation laws and the system (21), we obtain quasi-Jacobian systems:

$$\begin{cases} \Delta\left(\mathrm{Re}\,(\vec{R}_k)\right) = -\star_h\left(\nabla\vec{n}_k \,\llcorner\, \nabla^{\perp}\mathrm{Re}\,(\vec{R}_k)\right) - \star_h\left(\nabla\vec{n}_k \,\llcorner\, \nabla^{\perp}(\mathrm{Re}\,(S_k))\right) \\ \qquad\qquad\qquad + \vec{G}_{1,k} \\ \Delta\,(\mathrm{Re}\,(S_k)) = \langle\nabla(\star_h\vec{n}_k), \nabla^{\perp}\mathrm{Re}\,(\vec{R}_k)\rangle + G_{2,k} \end{cases}$$

for some $\vec{G}_{1,k}$ and $G_{2,k}$ which are bounded in $\mathrm{L}^q(B(0,\alpha_0 R_k))$ for all $1 \leq q < 2$.

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for some $\vec{G}_{1,k}$ and $G_{2,k}$ which are bounded in $\mathrm{L}^q(B(0,\alpha_0 R_k))$ for all $1 \leq q < 2$.

The previous averaging lemma and the improved Wente inequality show that $\nabla S_k, \nabla\vec{R}_k \in L^2(\Omega_k(2\alpha_0/3))$. Another averaging argument shows that $S_k, \vec{R}_k \in L^\infty(\Omega_k(2\alpha_0/3))$.

# VII. Jacobians and the Wente Inequality

Lemma (Laurain-Rivière, *Anal. PDE*, 2014, M.-Rivière 2019)

*Let* $0 < 16r < R < \infty$, $\Omega = B(0,R) \setminus \overline{B}(0,r) \to \mathbb{R}$, $a, b \in W^{1,2}(\Omega)$, *and* $u : \Omega \to \mathbb{R}$ *be a solution of*

$$\Delta u = \nabla a \cdot \nabla^{\perp} b \quad \textit{in} \;\; \Omega.$$

*Assume that* $\|u\|_{\mathrm{L}^\infty(\partial\Omega)} < \infty$*. Then there exists a constant* $C_1 < \infty$ *such that for all* $\left(\frac{r}{R}\right)^{\frac{1}{2}} < \alpha < \frac{1}{4}$,

$$\begin{aligned}
&\|u\|_{\mathrm{L}^\infty(\Omega)} + \|\nabla u\|_{\mathrm{L}^{2,1}(B_{\alpha R}\setminus \overline{B}_{\alpha^{-1}r}(0))} + \left\|\nabla^2 u\right\|_{\mathrm{L}^1(B_{\alpha R}\setminus \overline{B}_{\alpha^{-1}r}(0))} \\
&\leq C_1 \left(\|\nabla a\|_{\mathrm{L}^2(\Omega)} \|\nabla b\|_{\mathrm{L}^2(\Omega)} + \|u\|_{\mathrm{L}^\infty(\partial\Omega)}\right).
\end{aligned}$$

# VIII. Final Argument

Finally, we get $S_k, \vec{R}_k \in W^{2,1} \cap W^{1,(2,1)}$, and the identity

$$\begin{aligned} e^{\lambda_k} \vec{H}_k = & -\mathrm{Im}\left(\nabla_z \vec{R}_k \llcorner e^{-\lambda_k} \partial_{\overline{z}} \vec{\Phi}_k\right) - \frac{1}{2} e^{\lambda_k} \mathrm{Im}\left(\vec{L}_k\right) \\ & + \mathrm{Im}\left(e^{-\lambda_k} \partial_{\overline{z}} \vec{\Phi}_k \, \partial_z S_k\right) + \mathrm{Re}\left(\left\langle \partial_z \vec{\Phi}_k, \mathrm{Im}\left(\vec{L}_k\right)\right\rangle e^{-\lambda_k} \partial_{\overline{z}} \vec{\Phi}_k\right) \end{aligned}$$

yields the $L^{2,1}$ estimate for the mean curvature, which concludes the proof of the theorem.

# III. Weak $L^2$ Energy Quantization

# I. Improved Wente Inequality

## Lemma (Bernard-Rivière, *Ann. Math.* 2014)

*Let* $a, b \in W^{1,1}(\mathbb{D})$ *and* $u$ *be the solution of the Dirichlet problem*

$$\begin{cases} -\Delta u = \nabla a \cdot \nabla^{\perp} b & \text{in } \mathbb{D} \\ \quad u = 0 & \text{in } \partial\mathbb{D}. \end{cases}$$

*Assume that* $\nabla a \in L^{2,\infty}$ *and* $\nabla b \in L^{p,q}$ *for some* $1 < p < \infty$ *and* $1 \leq q \leq \infty$. *Then, we have*

$$\|\nabla u\|_{\mathrm{L}^{p,q}(\mathbb{D})} \leq C_{p,q} \|\nabla a\|_{\mathrm{L}^{2,\infty}(\mathbb{D})} \|\nabla b\|_{\mathrm{L}^{p,q}(\mathbb{D})} .$$

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## Theorem (Bernard-Rivière, *Ann. Math.* 2014)

*Under the previous hypothesis, there exists $\varepsilon_0 > 0$ with the following properties. If*

$$\sup_{\rho_k < r < R_k/2} \int_{B_{2r} \setminus \overline{B}_r(0)} |\nabla \vec{n}_k|^2 dx \leq \varepsilon_0,$$

*then*

$$\lim_{\alpha \to 0} \limsup_{k \to \infty} \|\nabla \vec{n}_k\|_{\mathrm{L}^{2,\infty}(\Omega_k(\alpha))} = 0. \tag{22}$$

# II. Proof

By Rivière's $\varepsilon$-regularity, we get

$$|\nabla \vec{n}_k(x)| \leq C \left( \fint_{B_{2|x|} \setminus \overline{B}_{|x|}(0)} |\nabla \vec{n}_k|^2 dx \right) \leq \frac{C\sqrt{\varepsilon_0}}{|x|}. \tag{23}$$

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In particular, we have $\|\nabla \vec{n}_k\|_{\mathrm{L}^{2,\infty}(\Omega_k(1/2))} \leq C\sqrt{\varepsilon_0}$. By contradiction, assume that $|x_k||\nabla \vec{n}_k(x_k)| \geq \varepsilon_1 > 0$ for some $x_k \in \Omega_k(1/2)$ such that

$$\log\left|\frac{|x_k|}{\rho_k}\right| \underset{k\to\infty}{\longrightarrow} \qquad \text{and} \quad \log\left|\frac{R_k}{|x_k|}\right| \underset{k\to\infty}{\longrightarrow} \infty.$$

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Then, by the previous analysis, we deduce that

$$\left\|e^{\lambda_k}\vec{L}_k\right\|_{\mathrm{L}^{2,\infty}(\Omega_k(1/2))} + \|\nabla S_k\|_{\mathrm{L}^{2,1}(\Omega_k(1/2))} + \left\|\nabla \vec{R}_k\right\|_{\mathrm{L}^{2,1}(\Omega_k(1/2))} \leq C. \tag{24}$$

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Consider the following map:
$\vec{\Psi}_k(y) = e^{-\lambda_k(x_k)-\log|x_k|}\left(\vec{\Phi}_k(|x_k|y) - \vec{\Phi}_k(x_k)\right)$.

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By Rivière's $\varepsilon$-regularity, we get

$$|\nabla\vec{n}_k(x)| \leq C\left(\fint_{B_{2|x|}\setminus\overline{B}_{|x|}(0)} |\nabla\vec{n}_k|^2 dx\right) \leq \frac{C\sqrt{\varepsilon_0}}{|x|}. \tag{23}$$

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$\vec{\Psi}_k(y) = e^{-\lambda_k(x_k)-\log|x_k|}\left(\vec{\Phi}_k(|x_k|y) - \vec{\Phi}_k(x_k)\right)$.Then, we have
$e^{\widetilde{\lambda}_k}\widetilde{\vec{L}}_k(y) = e^{\lambda_k(|x_k|y)}\vec{L}_k(|x_k|y)$, $\widetilde{S}_k(y) = S_k(|x_k|y)$ and $\widetilde{\vec{R}}_k(y) = \vec{R}_k(|x_k|y)$.

# II. Proof

By Rivière's $\varepsilon$-regularity, we get

$$|\nabla \vec{n}_k(x)| \leq C \left( \fint_{B_{2|x|}\setminus \overline{B}_{|x|}(0)} |\nabla \vec{n}_k|^2 dx \right) \leq \frac{C\sqrt{\varepsilon_0}}{|x|}. \tag{23}$$

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$e^{\widetilde{\lambda}_k} \widetilde{\vec{L}}_k(y) = e^{\lambda_k(|x_k|y)} \vec{L}_k(|x_k|y)$, $\widetilde{S}_k(y) = S_k(|x_k|y)$ and $\widetilde{\vec{R}}_k(y) = \vec{R}_k(|x_k|y)$.
Therefore, (24) holds for $e^{\widetilde{\lambda}_k} \widetilde{\vec{L}}_k, \widetilde{S}_k$, and $\widetilde{\vec{R}}_k$.

# II. Proof

We obtain by Rivière's compactness result and (23) a Willmore immersion $\vec{\Psi}_\infty : \mathbb{C} \to \mathbb{R}^3$ such that

$$\int_{B_2 \setminus \overline{B}_1(0)} |\nabla \vec{n}_\infty|^2 dx \geq \frac{\varepsilon_1^2}{C} > 0, \tag{25}$$

and

$$\begin{cases} \Delta \widetilde{S}_\infty = \nabla \vec{n}_\infty \cdot \nabla^\perp \widetilde{\vec{R}}_\infty \\ \Delta \widetilde{\vec{R}}_\infty = \nabla \widetilde{\vec{R}}_\infty \times \nabla^\perp \vec{n}_\infty - \nabla S_\infty \cdot \nabla^\perp \vec{n}_\infty \end{cases}$$

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Therefore, the previous Wente inequality shows that

$$\begin{aligned} \left\|\nabla \widetilde{S}_\infty\right\|_{\mathrm{L}^{2,1}(\mathbb{C})} + \left\|\nabla \widetilde{\vec{R}}_\infty\right\|_{\mathrm{L}^{2,1}(\mathbb{C})} &\leq C\left\|\nabla \vec{n}_\infty\right\|_{\mathrm{L}^{2,\infty}(\mathbb{C})}\left(\left\|\nabla \widetilde{S}_\infty\right\|_{\mathrm{L}^{2,1}(\mathbb{C})} + \left\|\nabla \widetilde{\vec{R}}_\infty\right\|_{\mathrm{L}^{2,1}(\mathbb{C})}\right) \\ &\leq C\sqrt{\varepsilon_0}\left(\left\|\nabla \widetilde{S}_\infty\right\|_{\mathrm{L}^{2,1}(\mathbb{C})} + \left\|\nabla \widetilde{\vec{R}}_\infty\right\|_{\mathrm{L}^{2,1}(\mathbb{C})}\right). \end{aligned}$$

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For $\varepsilon_0 > 0$ small enough, we get $\vec{S}_\infty = 0$ and $\vec{R}_\infty = 0$, which implies that $\vec{H}_\infty = -\frac{1}{2} e^{-2\widetilde{\lambda}_\infty} \left(\nabla \widetilde{\vec{R}}_\infty \times \nabla^\perp \vec{\Psi}_\infty + \nabla^\perp \widetilde{S}_\infty \cdot \nabla \vec{\Psi}_\infty\right) = 0$.

# II. Proof

Using Hélein's moving frame methods, one constructs a moving frame $(\vec{e}_1, \vec{e}_2) : \mathbb{C} \to S^2 \times S^2$ such that

$$\begin{cases} \vec{n}_\infty = \vec{e}_1 \times \vec{e}_2, \\ \displaystyle\int_{\mathbb{C}} \left(|\nabla \vec{e}_1|^2 + |\nabla \vec{e}_2|^2\right) dx \leq C \int_{\mathbb{C}} |\nabla \vec{n}_\infty|^2 dx < \infty. \end{cases} \tag{26}$$

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The Liouville equation gives

$$e^{2\widetilde{\lambda}_\infty} K_{g_\infty} = -\nabla^\perp \vec{e}_1 \cdot \nabla \vec{e}_2 \tag{27}$$

and

$$\int_{\mathbb{C}} |\nabla \vec{n}_\infty|^2 = -2 \int_{\mathbb{C}} K_{g_\infty} d\mathrm{vol}_{g_\infty} = 2 \int_{\mathbb{C}} \operatorname{div}\left(\nabla^\perp \vec{e}_1 \cdot \vec{e}_2\right) dx = 0,$$

a contradiction by (25).